Brief Contents

Contents

Preface

While the paper-setting pattern and assessment methodology have been revised many times over and newer criteria devised to help develop more aspirant-friendly engineering entrance tests, the need to standardize the selection processes and their outcomes at the national level has always been felt. A combined national level engineering entrance examination has finally been proposed by the Ministry of Human Resource Development, Government of India. The Joint Entrance Examination (JEE) to India's prestigious engineering institutions (IITs, IIITs, NITs, ISM, IISERs, and other engineering colleges) aims to serve as a common national-level engineering entrance test, thereby eliminating the need for aspiring engineers to sit through multiple entrance tests.

While the methodology and scope of an engineering entrance test are prone to change, there are two basic objectives that any test needs to serve:

1. The objective to test an aspirant's caliber, aptitude, and attitude for the engineering field and profession.
2. The need to test an aspirant's grasp and understanding of the concepts of the subjects of study and their applicability at the grass-root level.

Students appearing for various engineering entrance examinations cannot bank solely on conventional shortcut measures to crack the entrance examination. Conventional techniques alone are not enough as most of the questions asked in the examination are based on concepts rather than on just formulae. Hence, it is necessary for students appearing for joint entrance examination to not only gain a thorough knowledge and understanding of the concepts but also develop problem-solving skills to be able to relate their understanding of the subject to real-life applications based on these concepts.

This series of books is designed to help students to get an all-round grasp of the subject so as to be able to make its useful application in all its contexts. It uses a right mix of fundamental principles and concepts, illustrations which highlight the application of these concepts, and exercises for practice. The objective of each book in this series is to help students develop their problem-solving skills/accuracy, the ability to reach the crux of the matter, and the speed to get answers in limited time. These books feature all types of problems asked in the examination—be it MCQs (one or more than one correct), assertion-reason type, matching column type, comprehension type, or integer type questions. These problems have skillfully been set to help students develop a sound problem-solving methodology.

Not discounting the need for skilled and guided practice, the material in the books has been enriched with a number of fully solved concept application exercises so that every step in learning is ensured for the understanding and application of the subject. This whole series of books adopts a multi-faceted approach to mastering concepts by including a variety of exercises asked in the examination. A mix of questions helps stimulate and strengthen multi-dimensional problem-solving skills in an aspirant.

It is imperative to note that this book would be as profound and useful as you want it to be. Therefore, in order to get maximum benefit from this book, we recommend the following study plan for each chapter.

Step 1: Go through the entire opening discussion about the fundamentals and concepts.

Step 2: After learning the theory/concept, follow the illustrative examples to get an understanding of the theory/concept.

Overall the whole content of the book is an amalgamation of the theme of mathematics with ahead-of-time problems, which equips the students with the knowledge of the field and paves a confident path for them to accomplish success in the JEE.

With best wishes!

Ghanshyam Tewani

CHAPTER

1

Introduction to Vectors

- Coordinate Axes and Coordinate Planes in Three-Dimensional Space
- Evolution of Vector Concept
- Types of Vectors
- Addition of Vectors
- Components of a Vector
- Multiplication of a Vector by a Scalar
- Vector Joining Two Points
- Section Formula
- Vector Along the Bisector of Given Two Vectors
- Linear Combination, Linear Independence and Linear Dependence

COORDINATE AXES AND COORDINATE PLANES IN THREE-DIMENSIONAL SPACE

Consider three planes intersecting at a point O such that these three planes are mutually perpendicular to each other as shown in the following figure.

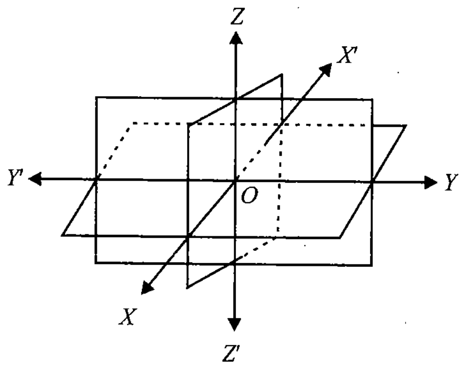


Fig. 1.1

These three planes intersect along the lines $X'OX$, $Y'OY$ and $Z'OZ$, called the x-, y- and z-axes, respectively. We may note that these lines are mutually perpendicular to each other. These lines constitute the *rectangular coordinate system*. The planes XOY, YOZ and ZOX, called respectively, the XY-plane, the YZ-plane and the ZX-plane, are known as the three coordinate planes. We take the XOY plane as the plane of the paper and the line $Z'OZ$ as perpendicular to the plane XOY. If the plane of the paper is considered to be horizontal, then the line $Z'OZ$ will be vertical. The distances measured from XY-plane upwards in the direction of OZ are taken as positive and those measured downwards in the direction of OZ are taken as negative. Similarly, the distances measured to the right of ZX-plane along OY are taken as positive, to the left of ZX-plane and along OY' as negative, in front of the YZ-plane along OX as positive and to the back of it along OX' as negative. The point O is called the *origin* of the coordinate system. The three coordinate planes divide the space into eight parts known as *octants*. These octants can be named as $XOYZ$, $X'OYZ$, $X'OY'Z$, $XOY'Z$, $XOYZ'$, $X'OYZ'$, $X'OY'Z'$ and $XOY'Z'$ and are denoted by I, II, III, ..., VIII, respectively.

Coordinates of a Point in Space

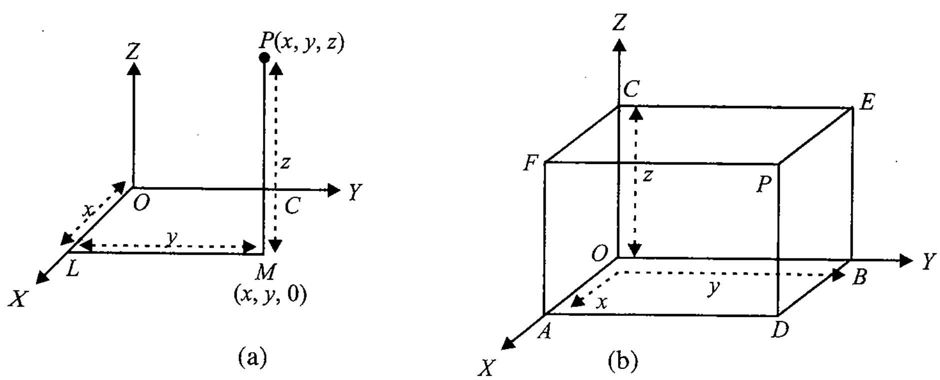


Fig. 1.2

Consider a point P in space, we drop a perpendicular PM on the XY-plane with M as the foot of this perpendicular. Then, from point M, we draw a perpendicular ML to the x-axis, meeting it at L. Let OL be x, LM be y and MP be z. Then x, y and z are called the x-, y- and z-coordinates, respectively, of point P in the space. In Fig. 1.2, we may note that the point $P(x, y, z)$ lies in the octant $XOYZ$ and so all x, y, z are positive. If P was in any other octant, the signs of x, y and z would change accordingly. Thus, to each point P in the space, there corresponds an ordered triplet (x, y, z) of real numbers.

We observe that if P (x, y, z) is any point in the space, then x, y and z are perpendicular distances from YZ, ZX and XY planes, respectively.

Note: *The coordinates of the origin O are* (0, 0, 0). *The coordinates of any point on the x-axis will be* $(x, 0, 0)$ *and the coordinates of any point in the YZ-plane will be* $(0, y, z)$.

The sign of the coordinates of a point determines the octant in which the point lies. The following table shows the signs of the coordinates in the eight octants:

Octant / Coordinates	I	II	III	IV	V	VI	VII	VIII
x	+	–	–	+	+	–	–	+
y	+	+	–	–	+	+	–	–
z	+	+	+	+	–	–	–	–

Distance between Two Points

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points referred to a system of rectangular axes OX, OY and OZ. Through the points P and Q draw planes parallel to the coordinate planes so as to form a rectangular parallelopiped with one diagonal PQ.

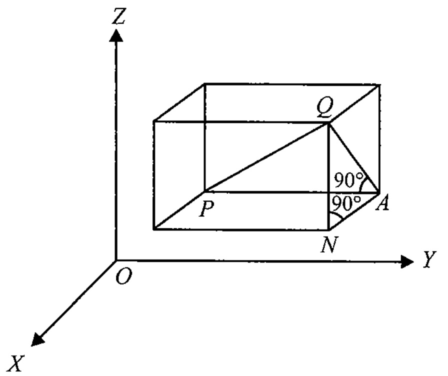


Fig. 1.3

Now, since $\angle PAQ$ is a right angle, it follows that in triangle PAQ,

$$PQ^2 = PA^2 + AQ^2 \quad \text{(i)}$$

Also, triangle ANQ is right-angled with $\angle ANQ$ being the right angle. Therefore,

$$AQ^2 = AN^2 + NQ^2 \quad \text{(ii)}$$

From (i) and (ii), we have

$PQ^2 = PA^2 + AN^2 + NQ^2$

Now $PA = y_2 - y_1$, $AN = x_2 - x_1$ and $NQ = z_2 - z_1$

Hence $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

Therefore, $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

This gives us the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2).

In particular, if $x_1 = y_1 = z_1 = 0$, i.e., point P is origin O, then $OQ = \sqrt{x_2^2 + y_2^2 + z_2^2}$, which gives the distance between the origin O and any point Q (x_2, y_2, z_2).

Section Formula

Let the two given points be $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. Let point R (x, y, z) divide PQ in the given ratio $m : n$ internally. Draw PL, QM and RN perpendicular to the XY-plane. Obviously $PL \parallel RN \parallel OM$ and feet of these perpendiculars lie in the XY-plane. Through point R draw a line ST parallel to line LM. Line ST will intersect line LP externally at point S and line MQ at T, as shown in the following figure.

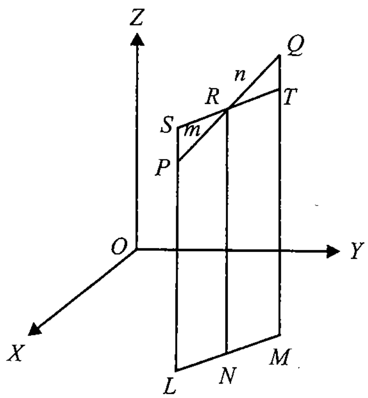


Fig. 1.4

Also note that quadrilaterals $LNRS$ and $NMTR$ are parallelograms.

The triangles PSR and QTR are similar. Therefore,

$$\frac{m}{n} = \frac{PR}{QR} = \frac{SP}{QT} = \frac{SL - PL}{QM - TM} = \frac{NR - PL}{QM - NR} = \frac{z - z_1}{z_2 - z}$$

This implies $z = \dfrac{mz_2 + nz_1}{m + n}$

Hence, the coordinates of the point R which divides the line segment joining two points $P(x_1, y_1, z_1)$ and Q (x_2, y_2, z_2) internally in the ratio $m : n$ are $\dfrac{mx_2 + nx_1}{m+n}$, $\dfrac{my_2 + ny_1}{m+n}$ and $\dfrac{mz_2 + nz_1}{m+n}$.

If point R divides PQ externally in the ratio $m : n$, then its coordinates are obtained by replacing n with $-n$ so that the coordinates become $\frac{mx_2 - nx_1}{m-n}$, $\frac{my_2 - ny_1}{m-n}$ and $\frac{mz_2 - nz_1}{m-n}$.

Notes:

1. *If R is the midpoint of PQ, then $m : n = 1:1$; so* $x = \frac{x_1 + x_2}{2}$, $y = \frac{y_1 + y_2}{2}$, $z = \frac{z_1 + z_2}{2}$.
   These are the coordinates of the midpoint of the segment joining P (x_1, y_1, z_1) and Q (x_2, y_2, z_2)
2. *The coordinates of the point R which divides PQ in the ratio $k : 1$ are obtained by taking* $k = \frac{m}{n}$, *which are given by* $\left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}, \frac{kz_2 + z_1}{k+1}\right)$
3. *If vertices of triangle are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$, and $AB = c$, $BC = a$, $AC = b$, then centroid of the triangle is* $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$ *and its incenter is*
   $\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$

EVOLUTION OF VECTOR CONCEPT

In our day-to-day life, we come across many queries such as 'What is your height?' and 'How should a football player hit the ball to give a pass to another player of his team?' Observe that a possible answer to the first query may be 1.5 m, a quantity that involves only one value (magnitude) which is a real number. Such quantities are called *scalars*. However, an answer to the second query is a quantity (called force) which involves muscular strength (magnitude) and direction (in which another player is positioned). Such quantities are called *vectors*. In mathematics, physics and engineering, we frequently come across with both types of quantities, namely scalar quantities such as length, mass, time, distance, speed, area, volume, temperature, work, money, voltage, density and resistance and vector quantities such as displacement, velocity, acceleration, force, momentum and electric field intensity.

Let 'l' be a straight line in plane or three-dimensional space. This line can be given two directions by means of arrowheads. A line with one of these directions prescribed is called a directed line (Fig.1.5 (i), (ii)).

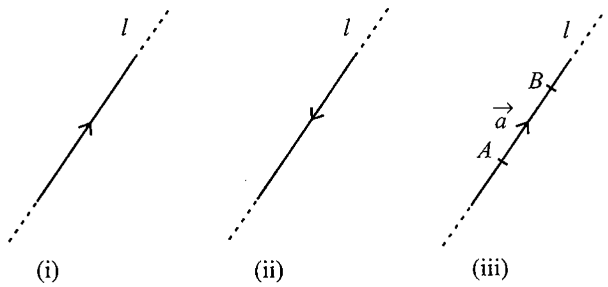


Fig. 1.5

Now observe that if we restrict the line l to the line segment AB, then a magnitude is prescribed on line (i) with one of the two directions, so that we obtain a directed line segment, Fig. 1.5 (iii). Thus, a directed line segment has magnitude as well as direction.

Definition

A quantity that has magnitude as well as direction is called a vector.

Notice that a directed line segment is a vector (Fig. 1.5 (iii), denoted as $\overrightarrow{AB}$ or simply as $\vec{a}$, and read as 'vector $\overrightarrow{AB}$' or 'vector $\vec{a}$'.

Point A from where vector $\overrightarrow{AB}$ starts is called its initial point, and point B where it ends is called its terminal point. The distance between initial and terminal points of a vector is called the magnitude (or length) of the vector, denoted as $|\overrightarrow{AB}|$ or $|\vec{a}|$ or a. The arrow indicates the direction of the vector.

Position Vector

Consider a point P in space, having coordinates (x, y, z) with respect to the origin O (0, 0, 0). Then, the vector $\overrightarrow{OP}$ having O and P as its initial and terminal points, respectively, is called the position vector of the point P with respect to O. Using distance formula, the magnitude of $\overrightarrow{OP}$ (or $\vec{r}$) is given by $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$.

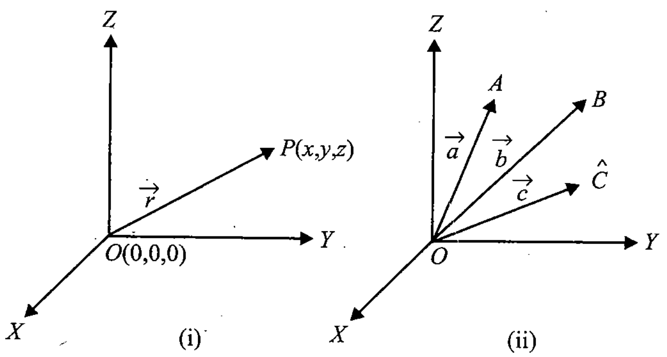


Fig. 1.6

In practice, the position vectors of points A, B, C, etc., with respect to origin O are denoted by $\vec{a}, \vec{b}, \vec{c}$, etc., respectively (Fig.1.6 (ii)).

Direction Cosines

Consider the position vector $\overrightarrow{OP}$ (or $\vec{r}$) of a point P (x, y, z). The angles α, β and γ made by the vector $\vec{r}$ with the positive directions of x-, y- and z-axes, respectively, are called its direction angles. The cosine values of these angles, i.e., $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called direction cosines of the vector $\vec{r}$ and are usually denoted by l, m and n, respectively.

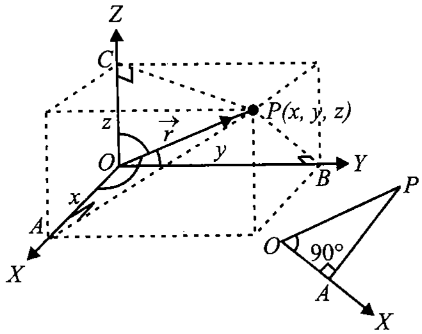


Fig. 1.7

From the figure, one may note that triangle OAP is right angled, and in it, we have $\cos\alpha = x/r$ (r stands for $|\vec{r}|$). Similarly, from the right-angled triangles OBP and OCP, we may write $\cos\beta = y/r$ and $\cos\gamma = z/r$. Thus, the coordinates of point P may also be expressed as (lr, mr, nr). The numbers lr, mr and nr, proportional to the direction cosines, are called the direction ratios of vector $\vec{r}$ and denoted as a, b and c, respectively (see this topic in detail in Chapter 3).

TYPES OF VECTORS

Zero Vector

A vector whose initial and terminal points coincide is called a zero vector (or null vector) and is denoted as $\vec{0}$. A zero vector cannot be assigned a definite direction as it has zero magnitude or, alternatively, it may be regarded as having any direction. The vectors $\overrightarrow{AA}$, $\overrightarrow{BB}$ represent the zero vector.

Unit Vector

A vector of unit magnitude is called a unit vector. Unit vectors are denoted by small letters with a cap on them.

Thus, $\hat{a}$ is unit vector of $\vec{a}$, where $|\hat{a}| = 1$, i.e., if vector $\vec{a}$ is divided by magnitude $|\vec{a}|$, then we get a unit vector in the direction of $\vec{a}$. Thus, $\dfrac{\vec{a}}{|\vec{a}|} = \hat{a} \Leftrightarrow \vec{a} = |\vec{a}|\ \hat{a}$, where $\hat{a}$ is the unit vector in the direction of $\vec{a}$.

Coinitial Vectors

Two or more vectors having the same initial point are called coinitial vectors.

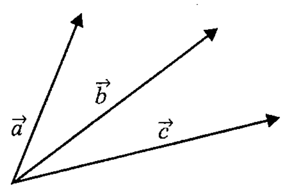


Fig. 1.8

Equal Vectors

Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal if they have the same magnitude and direction regardless of the positions of their initial points. They are written as $\vec{a} = \vec{b}$.

Negative of a Vector

A vector whose magnitude is the same as that of a given vector (say, $\overrightarrow{AB}$), but the direction is opposite to that of it, is called negative of the given vector. For example, vector $\overrightarrow{BA}$ is negative of vector $\overrightarrow{AB}$ and is written as $\overrightarrow{BA} = -\overrightarrow{AB}$.

Free Vectors

Vectors whose initial points are not specified are called free vectors.

Localised Vectors

A vector drawn parallel to a given vector but through a specified point as the initial point is called a localised vector.

Parallel Vectors

Two or more vectors are said to be parallel if they have the same support or parallel support.

Parallel vectors may have equal or unequal magnitudes and their directions may be same or opposite as shown in Fig. 1.9.

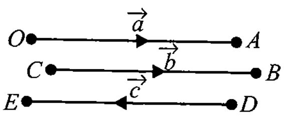


Fig. 1.9

Like and Unlike Vectors

Two parallel vectors having the same direction are called **like vectors** (see Fig. 1.10 (a)).

Two parallel vectors having opposite directions are called **unlike vectors** (see Fig. 1.10 (b)).

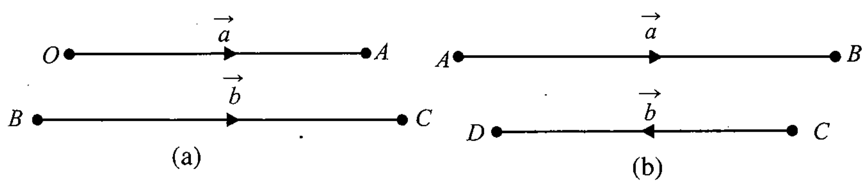


Fig. 1.10

Collinear Vectors

Vectors $\vec{a}$ and $\vec{b}$ are collinear if they have same direction or are parallel or anti-parallel. Since their magnitudes are different, we can find some scalar λ for which $\vec{a} = \lambda \vec{b}$. If $\lambda > 0$, $\vec{a}$ and $\vec{b}$ are in the same direction; if $\lambda < 0$, $\vec{a}$ and $\vec{b}$ are in opposite directions. Collinear vectors are often called dependent vectors.

Non-Collinear Vectors

Two vectors acting in different directions are called non-collinear vectors. Non-collinear vectors are often called independent vectors. Here we cannot write vector $\vec{a}$ in terms of $\vec{b}$, though they have same magnitude. However we can find component of one vector in the direction of the other. Two non-collinear vectors describe plane.

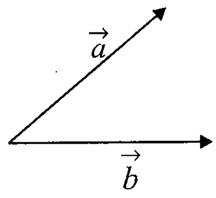


Fig. 1.11

Coplanar Vectors

Two parallel vectors or non-collinear vectors are always coplanar or two vectors $\vec{a}$ and $\vec{b}$ in different directions determine unique plane in space. Now if vector $\vec{c}$ lies in the plane of $\vec{a}$ and $\vec{b}$, vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors. Generally more than two vectors are coplanar if all are in the same plane.

Three non-coplanar vectors describe space.

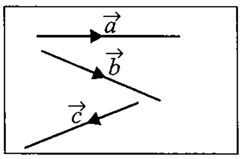


Fig. 1.12

ADDITION OF VECTORS

A vector $\overrightarrow{AB}$ simply means the displacement from point A to point B. Now consider a situation where a boy moves from A to B and then from B to C. The net displacement made by the boy from point A to point C is given by vector $\overrightarrow{AC}$ and expressed as

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

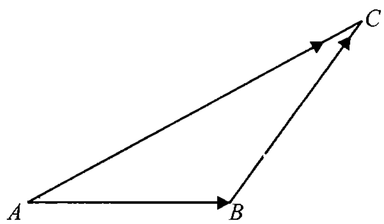


Fig. 1.13

This is known as the triangle law of vector addition.

In general, if we have two vectors $\vec{a}$ and $\vec{b}$ (Fig. 1.14 (i)), then to add them, they are positioned such that the initial point of one coincides with the terminal point of the other (Fig. 1.14 (ii)).

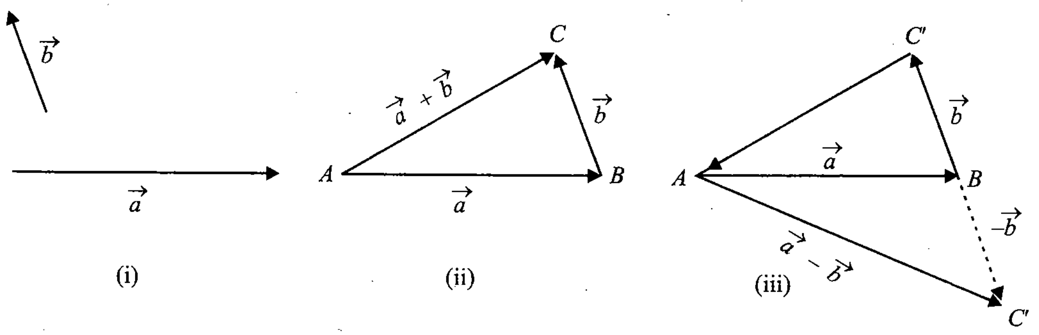


Fig. 1.14

For example, in Fig. 1.14 (ii), we have shifted vector $\vec{b}$ without changing its magnitude and direction, so that its initial point coincides with the terminal point of $\vec{a}$. Then the vector $\vec{a}+\vec{b}$, represented by the third side AC of the triangle ABC, gives us the sum (or resultant) of the vectors $\vec{a}$ and $\vec{b}$, i.e., in triangle ABC (Fig. 1.14 (ii)), we have

$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$$

Since $\overrightarrow{AC}=-\overrightarrow{CA}$, from the above equation, we have

$$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\overrightarrow{AA}=\vec{0}$$

This means that when the sides of a triangle are taken in order, it leads to zero resultant as the initial and terminal points get coincided (Fig. 1.14 (iii)).

Now, construct a vector $\overrightarrow{BC}$ so that its magnitude is same as that of vector $\overrightarrow{BC}$, but the direction is opposite to that of $\overrightarrow{BC}$ (Fig. 1.14 (iii)), i.e.,

$$\overrightarrow{BC}=-\overrightarrow{BC'}$$

Then, on applying triangle law from Fig. 1.14 (iii), we have

$$\overrightarrow{AC'}=\overrightarrow{AB}+\overrightarrow{BC'}=\overrightarrow{AB}+(-\overrightarrow{BC})=\vec{a}-\vec{b}$$

Vector $\overrightarrow{AC'}$ is said to represent the difference of $\vec{a}$ and $\vec{b}$

Now, consider a boat going from one bank of a river to the other in a direction perpendicular to the flow of the river. Then, it is acted upon by two velocity vectors—one is the velocity imparted to the boat by its engine and the other one is the velocity of the flow of river water. Under the simultaneous influence of these two velocities, the boat actually starts travelling with a different velocity. To have a precise idea about the effective speed and direction (i.e., the resultant velocity) of the boat, we have the following law of vector addition.

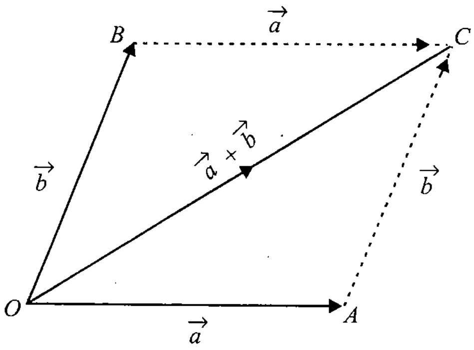


Fig. 1.15

If we have two vectors $\vec{a}$ and $\vec{b}$ represented by the two adjacent sides of a parallelogram in magnitude and direction (Fig. 1.15), then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by the diagonal of the parallelogram through their common point. This is known as the parallelogram law of vector addition.

Notes:

1. *From figure using the triangle law, one may note that*

$$\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$$

or $$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC} \qquad (\text{since } \overrightarrow{AC} = \overrightarrow{OB})$$

which is parallelogram law. Thus, we may say that the two laws of vector addition are equivalent to each other.

2. *If* $\overrightarrow{OA}$ *and* $\overrightarrow{AC}$ *are collinear, their sum is still* $\overrightarrow{OC}$. *Although in this case we do not have a triangle or a parallelogram in their usual sense.*

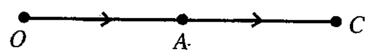


Fig. 1.16

3. *As from the figure:*

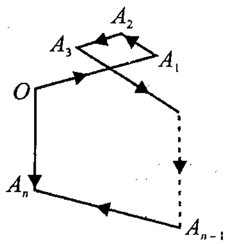


Fig. 1.17

$\overrightarrow{OA} + \overrightarrow{A_1A_2} + \cdots + \overrightarrow{A_{n-1}A_n} = \overrightarrow{OA_n}$ *by the polygon law of addition.*

Properties of Vector Addition

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (commutative property)

2. $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$ (associative property)
3. $\vec{a}+\vec{0} = \vec{a}$ (additive identity)
4. $\vec{a}+(-\vec{a}) = \vec{0}$ (additive inverse)
5. $|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$ and $|\vec{a}-\vec{b}| \ge |\vec{a}|-|\vec{b}|$

Example 1.1 **If vector $\vec{a}+\vec{b}$ bisects the angle between $\vec{a}$ and $\vec{b}$, then prove that $|\vec{a}| = |\vec{b}|$.**

Sol.

We know that vector $\vec{a}+\vec{b}$ is along the diagonal of the parallelogram whose adjacent sides are vectors $\vec{a}$ and $\vec{b}$. Now if $\vec{a}+\vec{b}$ bisects the angle between vectors $\vec{a}$ and $\vec{b}$, then the parallelogram must be a rhombus, hence $|\vec{a}| = |\vec{b}|$.

Example 1.2 **If $\overrightarrow{AO}+\overrightarrow{OB}=\overrightarrow{BO}+\overrightarrow{OC}$, then prove that B is the midpoint of AC.**

Sol.

$\overrightarrow{AO}+\overrightarrow{OB}=\overrightarrow{BO}+\overrightarrow{OC}$

$\Rightarrow$ $\overrightarrow{AB}=\overrightarrow{BC}$

$\Rightarrow$ Vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are collinear

$\Rightarrow$ Points A, B, C are collinear

Also $|\overrightarrow{AB}|=|\overrightarrow{BC}|$

$\Rightarrow$ B is the midpoint of AC

Example 1.3 **$ABCDE$ is a pentagon. Prove that the resultant of forces $\overrightarrow{AB}$, $\overrightarrow{AE}$, $\overrightarrow{BC}$, $\overrightarrow{DC}$, $\overrightarrow{ED}$ and $\overrightarrow{AC}$ is $3\overrightarrow{AC}$.**

Sol.

$$\vec{R} = \overrightarrow{AB}+\overrightarrow{AE}+\overrightarrow{BC}+\overrightarrow{DC}+\overrightarrow{ED}+\overrightarrow{AC}$$

$$= (\overrightarrow{AB}+\overrightarrow{BC}) + (\overrightarrow{AE}+\overrightarrow{ED}+\overrightarrow{DC}) + \overrightarrow{AC}$$

$$= \overrightarrow{AC}+\overrightarrow{AC}+\overrightarrow{AC} = 3\overrightarrow{AC}$$

Example 1.4 **Prove that the resultant of two forces acting at point O and represented by $\overrightarrow{OB}$ and $\overrightarrow{OC}$ is given by $2\overrightarrow{OD}$, where D is the midpoint of BC.**

Sol.

$$\vec{R} = \overrightarrow{OB}+\overrightarrow{OC}$$

$$= (\overrightarrow{OD}+\overrightarrow{DB})+(\overrightarrow{OD}+\overrightarrow{DC})$$

$$= 2\overrightarrow{OD}+(\overrightarrow{DB}+\overrightarrow{DC}) = 2\overrightarrow{OD}+\vec{0} = 2\overrightarrow{OD}$$

(Since D is the midpoint of BC, we have $\overrightarrow{DB} = -\overrightarrow{DC}$)

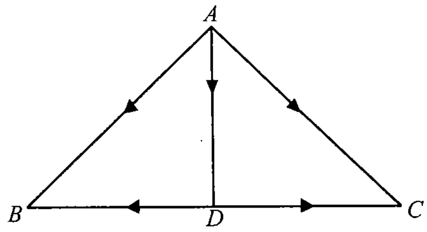


Fig. 1.18

Example 1.5 **Prove that the sum of three vectors determined by the medians of a triangle directed from the vertices is zero.**

Sol.

$\overrightarrow{AB}+\overrightarrow{AC}=2\overrightarrow{AD}$

$\overrightarrow{BC}+\overrightarrow{BA}=2\overrightarrow{BE}$

$\overrightarrow{CA}+\overrightarrow{CB}=2\overrightarrow{CF}$

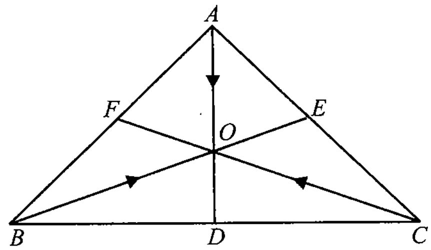


Fig. 1.19

Adding, we get

$(\overrightarrow{AB}+\overrightarrow{BA})+(\overrightarrow{AC}+\overrightarrow{CA})+(\overrightarrow{BC}+\overrightarrow{CB})=2(\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF})$

or $\vec{0}+\vec{0}+\vec{0}=2(\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF})$

or $\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}=\vec{0}$

Example 1.6 ***ABC* is a triangle and *P* any point on *BC*. If $\overrightarrow{PQ}$ is the sum of $\overrightarrow{AP}$, $\overrightarrow{PB}$ and $\overrightarrow{PC}$, show that *ABQC* is a parallelogram and *Q*, therefore, is a fixed point.**

Sol.

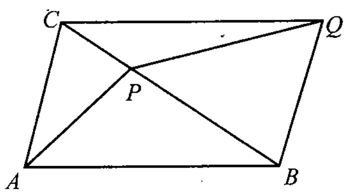


Fig. 1.20

Here $\overrightarrow{PQ} = \overrightarrow{AP} + \overrightarrow{PB} + \overrightarrow{PC}$

$\overrightarrow{PQ} - \overrightarrow{PC} = \overrightarrow{AP} + \overrightarrow{PB}$

$\overrightarrow{PQ} + \overrightarrow{CP} = \overrightarrow{AP} + \overrightarrow{PB}$

$\overrightarrow{CQ} = \overrightarrow{AB} \Rightarrow CQ = AB$ and $CQ \parallel AB$

$\therefore$ $ABQC$ is a parallelogram.

But A, B and C are given to be fixed points and $ABQC$ is a parallelogram

Therefore, Q is a fixed point.

Example 1.7 **Two forces $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are acting at the vertex A of a quadrilateral $ABCD$ and two forces $\overrightarrow{CB}$ and $\overrightarrow{CD}$ at C. Prove that their resultant is given by $4\overrightarrow{EF}$, where E and F are the midpoints of AC and BD, respectively.**

Sol. $\overrightarrow{AB} + \overrightarrow{AD} = 2\overrightarrow{AF}$, where F is the midpoint of BD.

$\overrightarrow{CB} + \overrightarrow{CD} = 2\overrightarrow{CF}$

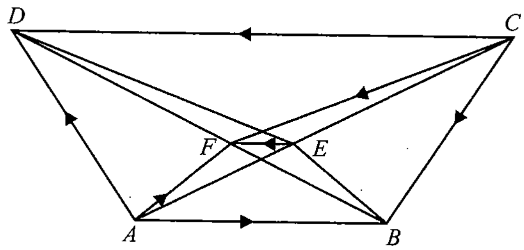


Fig. 1.21

$$\therefore \quad \overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} = 2(\overrightarrow{AF} + \overrightarrow{CF})$$

$$= -2(\overrightarrow{FA} + \overrightarrow{FC})$$

$$= -2[2\overrightarrow{FE}]\text{, where } E \text{ is the midpoint of } AC$$

$$= 4\overrightarrow{EF}$$

Example 1.8 **If $O(\vec{0})$ is the circumcentre and O' the orthocentre of a triangle ABC, then prove that**

i. $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OO'}$

ii. $\overrightarrow{O'A} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2\overrightarrow{O'O}$

iii. $\overrightarrow{AO'} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2\overrightarrow{AO} = \overrightarrow{AP}$

where AP is the diameter through A of the circumcircle.

Sol.

O is the circumcentre, which is the intersection of the right bisectors of the sides of the triangle, and O' is the orthocenter, which is the point of intersection of altitudes drawn from the vertices. Also, from geometry, we know that

$2OD = AO'$. Therefore,

$2\overrightarrow{OD} = \overrightarrow{AO'}$ (i)

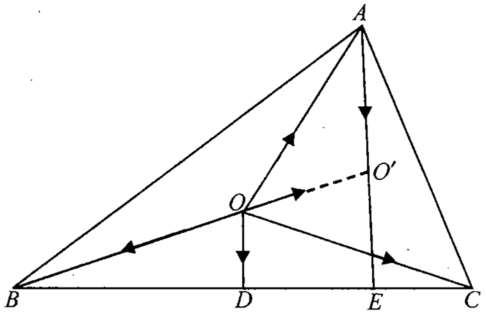


Fig. 1.22

i. To prove: $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OO'}$

Now $\overrightarrow{OB} + \overrightarrow{OC} = 2\overrightarrow{OD} = \overrightarrow{AO'}$

$\Rightarrow \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AO'} = \overrightarrow{OO'}$ (by (i))

ii. To prove: $\overrightarrow{O'A} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2\overrightarrow{OO'}$

L.H.S. $= 2\overrightarrow{DO} + 2\overrightarrow{O'D}$ (by (i))

$= 2(\overrightarrow{O'D} + \overrightarrow{DO}) = 2\overrightarrow{O'O}$

iii. To prove: $\overrightarrow{AO'} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2\overrightarrow{AO} = \overrightarrow{AP}$

L.H.S. $= 2\overrightarrow{AO'} - \overrightarrow{AO'} + \overrightarrow{O'B} + \overrightarrow{O'C}$

$= 2\overrightarrow{AO'} + (\overrightarrow{O'A} + \overrightarrow{O'B} + \overrightarrow{O'C}$

$= 2\overrightarrow{AO'} + 2\overrightarrow{O'O} = 2\overrightarrow{AO}$

$= \overrightarrow{AP}$ (where AP is the diameter through A of the circumcircle).

COMPONENTS OF A VECTOR

Let us take the points A (1, 0, 0), B (0, 1, 0) and C (0, 0, 1) on the x-axis, y-axis and z-axis, respectively.

Then, clearly $|\overrightarrow{OA}| = 1$, $|\overrightarrow{OB}| = 1$ and $|\overrightarrow{OB}| = 1$

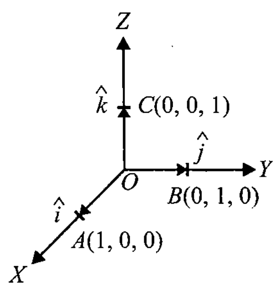


Fig. 1.23

The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$, each having magnitude 1, are called unit vectors along the axes OX, OY and OZ, respectively, and are denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$, respectively.

Now, consider the position vector $\overrightarrow{OP}$ of a point $P(x, y, z)$ as shown in the following figure. Let P_1 be the foot of the perpendicular from P on the plane XOY.

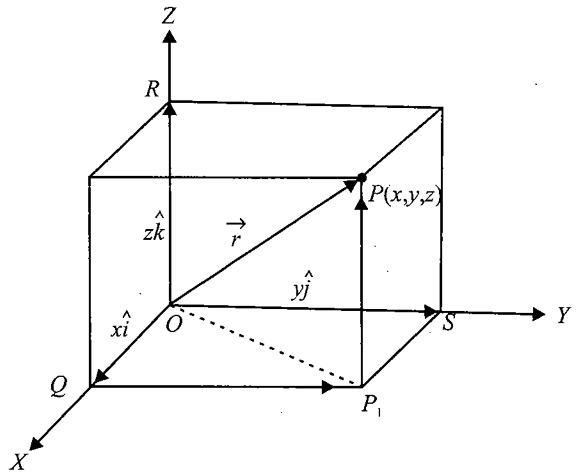


Fig. 1.24

We, thus, see that P_1P is parallel to z-axis. As $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors along the x-, y- and z-axes, respectively, and by the definition of the coordinates of P, we have $\overrightarrow{P_1P} = \overrightarrow{OR} = z\hat{k}$. Similarly, $\overrightarrow{OS} = y\hat{j}$ and $\overrightarrow{OQ} = x\hat{i}$.

Therefore, it follows that $\overrightarrow{OP_1} = \overrightarrow{OQ} + \overrightarrow{OP_1} = x\hat{i} + y\hat{j}$ and $\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P} = x\hat{i} + y\hat{j} + z\hat{k}$

Hence, the position vector of P with reference to O is given by

$|\overrightarrow{OP}|$ (or $\vec{r}$) $= x\hat{i} + y\hat{j} + z\hat{k}$

This form of any vector is called its component form. Here, x, y and z are called the scalar components of $\vec{r}$, and $x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ are called the vector components of $\vec{r}$ along the respective axes. Sometimes x, y and z are also called rectangular components.

The length of any vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is readily determined by applying the Pythagoras theorem twice. We note that in the right-angled triangle OQP_1,

$$|\overrightarrow{OP_1}| = \sqrt{|\overrightarrow{OQ}|^2 + |\overrightarrow{QP_1}|^2} = \sqrt{x^2 + y^2}$$

And in the right-angled triangle OP_1P, we have

$$|\overrightarrow{OP}| = \sqrt{|\overrightarrow{OP_1}|^2 + |\overrightarrow{P_1P}|^2} = \sqrt{(x^2 + y^2) + z^2}$$

Hence, the length of any vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is given by

$$|\vec{r}| = |x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

Notes:

If $\vec{a}$ *and* $\vec{b}$ *are any two vectors given in the component form* $a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ *and* $b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, *respectively, then*

1. *The sum (or resultant) of vectors* $\vec{a}$ *and* $\vec{b}$ *is given by*

$$\vec{a}+\vec{b} = (a_1+b_1)\,\hat{i} + (a_2+b_2)\,\hat{j} + (a_3+b_3)\,\hat{k}$$

2. *The difference between vectors* $\vec{a}$ *and* $\vec{b}$ *is given by*

$$\vec{a}-\vec{b} = (a_1-b_2)\,\hat{i} + (a_2-b_2)\,\hat{j} + (a_3-b_3)\,\hat{k}$$

3. *Vectors* $\vec{a}$ *and* $\vec{b}$ *are equal if and only if*

$$\vec{b} = \lambda\vec{a} = (\lambda a_1)\,\hat{i} + (\lambda a_2)\,\hat{j} + (\lambda a_3)\,\hat{k}$$

The addition of vectors and the multiplication of a vector by a scalar together give the following distributive laws:

Let $\vec{a}$ *and* $\vec{b}$ *be any two vectors, and k and m be any scalars. Then*

i. $k\vec{a}+m\vec{a} = (k+m)\,\vec{a}$

ii. $k\,(m\vec{a}) = (km)\,\vec{a}$

iii. $k\,(\vec{a}+\vec{b}) = k\,\vec{a} + k\,\vec{b}$

Remarks

i. One may observe that whatever be the value of λ, vector $\lambda\vec{a}$ is always collinear to vector $\vec{a}$. In fact, two vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if there exists a non-zero scalar λ such that $\vec{b} = \lambda\vec{a}$. If the vectors $\vec{a}$ and $\vec{b}$ are given in the component form, i.e., $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then

$$b_1\hat{i}+b_2\hat{j}+b_3\hat{k} = \lambda\,(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})$$

$$\Leftrightarrow b_1\hat{i}+b_2\hat{j}+b_3\hat{k} = (\lambda a_1)\,\hat{i}+(\lambda a_2)\,\hat{j}+(\lambda a_3)\,\hat{k}$$

$$\Leftrightarrow b_1 = \lambda a_1,\ b_2 = \lambda a_2,\ b_3 = \lambda a_3$$

$$\Leftrightarrow \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$$

ii. If $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, then a_1, a_2, a_3 are also called direction ratios of $\vec{a}$.

iii. In case it is given that l, m, n are direction cosines of a vector, then $l\hat{i}+m\hat{j}+n\hat{k}$ $= (\cos\alpha)\,\hat{i}+(\cos\beta)\,\hat{j}+(\cos\gamma)\,\hat{k}$ is the unit vector in the direction of that vector where α, β and γ are the angles which the vector makes with the x-, y- and z-axes, respectively.

Example 1.9 **A unit vector of modulus 2 is equally inclined to x- and y-axes at an angle $\frac{\pi}{3}$. Find the length of projection of the vector on z-axis.**

Sol.

Given that the vector is inclined at an angle $\frac{\pi}{3}$ with both x- and y-axes.

$\Rightarrow \quad \cos\alpha = \cos\beta = \frac{1}{2}$

Also we know that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \quad \cos^2\gamma = \frac{1}{2}$

$\Rightarrow \quad \cos\gamma = \pm\frac{1}{\sqrt{2}}$

$\Rightarrow$ Given vector is $2(\cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k})$

$$= 2\left(\frac{\hat{i}}{2} + \frac{\hat{j}}{2} \pm \frac{\hat{k}}{\sqrt{2}}\right) = \hat{i} + \hat{j} \pm \sqrt{2}\hat{k}$$

$\Rightarrow$ Length of projection of vector on z-axis is $\sqrt{2}$ units.

Example 1.10 **If the projections of vector $\vec{a}$ on x-, y- and z-axes are 2, 1 and 2 units, respectively, find the angle at which vector $\vec{a}$ is inclined to z-axis.**

Sol.

Since projections of vector $\vec{a}$ on x-, y- and z-axes are 2, 1 and 2 units, respectively,

Vector $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$

$|\vec{a}| = \sqrt{2^2 + 1^2 + 2^2} = 3$

Then $\cos\gamma = \frac{2}{3}$ (where γ is the angle of vector $\vec{a}$ with z-axis)

$\Rightarrow \quad \gamma = \cos^{-1}\frac{2}{3}$

MULTIPLICATION OF A VECTOR BY A SCALAR

Let $\vec{a}$ be a vector and λ a scalar. Then the product of vector $\vec{a}$ by scalar λ, denoted as $\lambda\vec{a}$, is called the multiplication of vector $\vec{a}$ by the scalar λ. Note that $\lambda\vec{a}$ is also a vector, collinear to vector $\vec{a}$. Vector $\lambda\vec{a}$ has the direction same (or opposite) as that of vector $\vec{a}$ if the value of λ is positive (or negative). Also, the magnitude of vector $\lambda\vec{a}$ is $|\lambda|$ times the magnitude of vector $\vec{a}$, i.e.,

$$|\lambda\vec{a}| = |\lambda||\vec{a}|$$

A geometric visualization of multiplication of a vector by a scalar is given in the following figure.

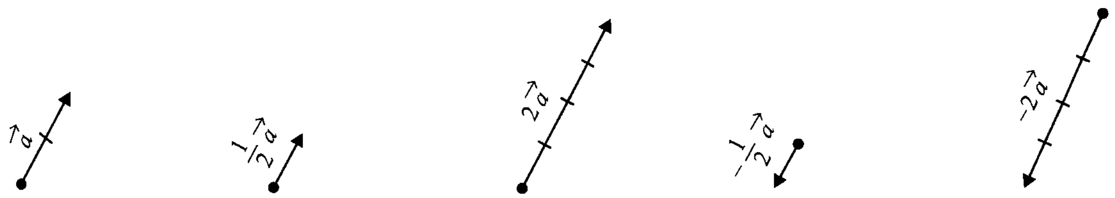


Fig. 1.25

When $\lambda = -1$, $\lambda\vec{a} = -\vec{a}$, which is a vector having magnitude equal to the magnitude of $\vec{a}$ and direction opposite to that of the direction of $\vec{a}$.

Vector $-\vec{a}$ is called the negative (or additive inverse) of vector $\vec{a}$ and we always have $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$.

Also, if $\lambda = \dfrac{1}{|\vec{a}|}$, provided $\vec{a} \neq 0$, i.e., $\vec{a}$ is not a null vector, then

$$|\lambda\vec{a}| = |\lambda|\,|\vec{a}| = \frac{1}{|\vec{a}|}|\vec{a}| = 1$$

Example 1.11 **Find the vector of magnitude 9 units in the direction of vector $\vec{a} = \hat{i} + 2\hat{j} + 2\hat{k}$.**

Sol.

Given vector $\vec{a} = \hat{i} + 2\hat{j} + 2\hat{k}$

Unit vector in the direction of $\vec{a}$ is $\hat{a} = \dfrac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \dfrac{\hat{i} + 2\hat{j} + 2\hat{k}}{3}$

Now vector of magnitude 9 in the direction of $\vec{a}$ is $9\dfrac{\hat{i} + 2\hat{j} + 2\hat{k}}{3} = 3(\hat{i} + 2\hat{j} + 2\hat{k})$

VECTOR JOINING TWO POINTS

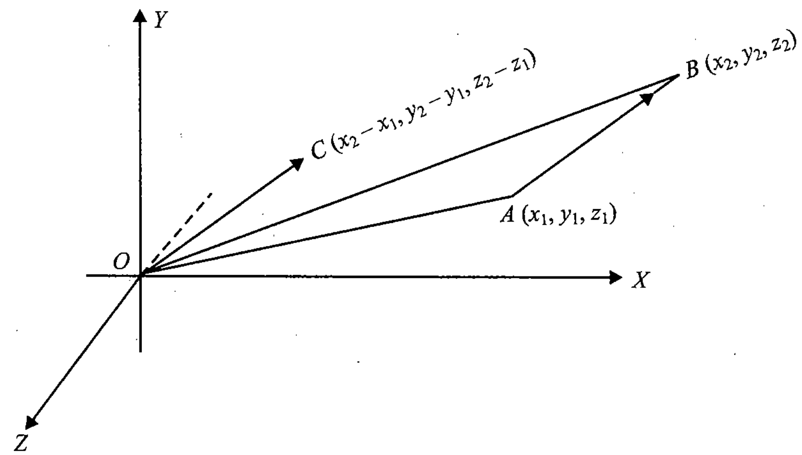


Fig. 1.26

In the figure, vector $\vec{AB}$ is shifted without rotation and placed at origin.

Now vector $\vec{AB} = \vec{OC}$

Since $|\vec{AB}| = |\vec{OC}|$, coordinates of point C are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$

Hence vector $\vec{OC} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

Thus $\vec{AB} = \vec{OC} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

$$= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$

$$= \vec{OB} - \vec{OA}$$

= Position vector of B – position vector of A

Also from above, we have $\vec{OB} = \vec{OA} + \vec{AB}$ which describes triangle rule of vector addition.

Further $\vec{OB} = \vec{OA} + \vec{AB} = \vec{OA} + \vec{OC}$ ($\because \vec{OC} = \vec{AB}$), which describes parallelogram rule of vector addition.

Example 1.12 **If $2\overrightarrow{AC} = 3\overrightarrow{CB}$, then prove that $2\overrightarrow{OA} + 3\overrightarrow{OB} = 5\overrightarrow{OC}$, where O is the origin.**

Sol.

$$2\overrightarrow{AC} = 3\overrightarrow{CB} \Rightarrow 2(\overrightarrow{OC} - \overrightarrow{OA}) = 3(\overrightarrow{OB} - \overrightarrow{OC})$$

$$\Rightarrow \quad 2\overrightarrow{OA} + 3\overrightarrow{OB} = 5\overrightarrow{OC}$$

Example 1.13 **Prove that points $\hat{i} + 2\hat{j} - 3\hat{k}$, $2\hat{i} - \hat{j} + \hat{k}$ and $2\hat{i} + 5\hat{j} - \hat{k}$ form a triangle in space.**

Sol.

Given points are $A(\hat{i} + 2\hat{j} - 3\hat{k}), B(2\hat{i} - \hat{j} + \hat{k}), C(2\hat{i} + 5\hat{j} - \hat{k})$

Vectors $\vec{AB} = \hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{AC} = \hat{i} + 3\hat{j} + 2\hat{k}$

Clearly vectors $\vec{AB}$ and $\vec{AC}$ are non-collinear as there does not exist any real λ for which $\vec{AB} = \lambda\vec{AC}$.

Hence, vectors $\vec{AB}$ and $\vec{AC}$ or given three points form a triangle.

SECTION FORMULA

Internal Division

Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$, respectively, and C be a point dividing AB internally in the ratio $m : n$. Then the position vector of C is given by $\overrightarrow{OC} = \frac{m\vec{b} + n\vec{a}}{m+n}$.

Proof:

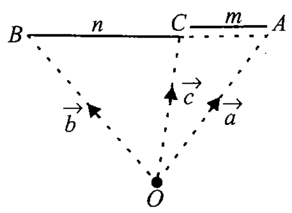


Fig. 1.27

Let O be the origin. Then $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$. Let $\vec{c}$ be the position vector of C which divides AB internally in the ratio $m : n$. Then $\frac{AC}{CB} = \frac{m}{n}$.

$\Rightarrow \quad n\overrightarrow{AC} = m\overrightarrow{CB}$

$\Rightarrow \quad n(\text{P.V. of } C - \text{P.V. of } A) = m(\text{P.V. of } B - \text{P.V. of } C)$

$\Rightarrow \quad n(\vec{c} - \vec{a}) = m(\vec{b} - \vec{c})$

$\Rightarrow \quad n\vec{c} - n\vec{a} = m\vec{b} - m\vec{c}$

$\Rightarrow \quad \vec{c}(n+m) = m\vec{b} + n\vec{a}$

$\Rightarrow \quad \vec{c} = \frac{m\vec{b} + n\vec{a}}{m+n}$ or $\overrightarrow{OC} = \frac{m\vec{b} + n\vec{a}}{m+n}$

External Division

Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$, respectively, and C be a point dividing $\overrightarrow{AB}$ externally in the ratio $m : n$. Then the position vector of $\vec{C}$ is given by $\overrightarrow{OC} = \frac{m\vec{b} - n\vec{a}}{m-n}$.

Proof:

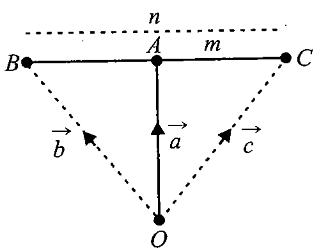


Fig. 1.28

Let O be the origin. Then $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$. Let $\vec{c}$ be the position vector of point C dividing AB externally in the ratio $m : n$.

Then, $\frac{AC}{BC} = \frac{m}{n}$

$\Rightarrow \quad nAC = mBC$

$\Rightarrow \quad n\overrightarrow{AC} = m\overrightarrow{BC}$

$\Rightarrow \quad n$(P.V. of C – P.V. of A) = m(P.V. of C – P.V. of B)

$\Rightarrow \quad n(\vec{c} - \vec{a}) = m(\vec{c} - \vec{b})$

$\Rightarrow \quad n\vec{c} - n\vec{a} = m\vec{c} - m\vec{b}$

$\Rightarrow \quad \vec{c}(m - n) = m\vec{b} - n\vec{a}$

$\Rightarrow \quad \vec{c} = \frac{m\vec{b} - n\vec{a}}{m - n}$ or $\overrightarrow{OC} = \frac{m\vec{b} - n\vec{a}}{m - n}$

Notes:

1. *If C is the midpoint of AB, then it divides AB in the ratio* 1 : 1.

   Therefore, the *P.V. of C is* $\frac{1 \cdot \vec{a} + 1 \cdot \vec{b}}{1+1} = \frac{\vec{a} + \vec{b}}{2}$. *Thus, the position vector of the midpoint of AB is* $\frac{1}{2}(\vec{a} + \vec{b})$.

2. *We have* $\vec{c} = \frac{m\vec{b} + n\vec{a}}{m+n} = \frac{m}{m+n}\vec{b} + \frac{n}{m+n}\vec{a}$. *Therefore,*

   $\vec{c} = \lambda\vec{a} + \mu\vec{b}$, *where* $\lambda = \frac{n}{m+n}$ *and* $\mu = \frac{m}{m+n}$

   Thus, position vector of any point C on $\overrightarrow{AB}$ *can always be taken as* $\vec{c} = \lambda\vec{a} + \mu\vec{b}$, *where* $\lambda + m = 1$.

3. *We have* $\vec{c} = \frac{m\vec{b} + n\vec{a}}{m+n}$. *Therefore,*

   $(m + n)\vec{c} = m\vec{b} + n\vec{a}$

   $n\overrightarrow{OA} + m\overrightarrow{OB} = (m+n)\overrightarrow{OC}$, *where* $\vec{C}$ *is a point on* $\overrightarrow{AB}$ *dividing it in the ratio m : n.*

   In ΔABC, *having vertices* $A(\vec{a}), B(\vec{b})$ *and* $C(\vec{a})$

   Centroid is $\frac{\vec{a} + \vec{b} + \vec{c}}{3}$

   Incentre is $\frac{BC\,\vec{a} + AC\,\vec{b} + AB\,\vec{c}}{AB + AC + AB}$

   Orthocentre is $\frac{\tan A\,\vec{a} + \tan B\,\vec{b} + \tan C\,\vec{c}}{\tan A + \tan B + \tan C}$

   Circumcentre is $\frac{\sin 2A\,\vec{a} + \sin 2B\,\vec{b} + \sin 2C\,\vec{c}}{\sin 2A + \sin 2B + \sin 2C}$

Example 1.14 **If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be the position vectors of points *A*, *B*, *C* and *D*, respectively referred to same origin *O* such that no three of these points are collinear and $\vec{a} + \vec{c} = \vec{b} + \vec{d}$, then prove that the quadrilateral *ABCD* is a parallelogram.**

Sol.

Since $\vec{a} + \vec{c} = \vec{b} + \vec{d}$

$\Rightarrow \quad \dfrac{\vec{a} + \vec{c}}{2} = \dfrac{\vec{b} + \vec{d}}{2}$

$\Rightarrow$ Midpoint of *AC* and *BD* coincide

$\Rightarrow$ Quadrilateral *ABCD* is a parallelogram

Example 1.15 **Find the point of intersection of *AB* and *CD*, where *A*(6, –7, 0), *B*(16, –19, –4), *C*(0, 3, –6) and *D*(2, –5, 10).**

Sol.

Let *AB* and *CD* intersect at *P*.

Let *P* divides *AB* in ratio λ:1 and *CD* in ratio μ:1

Then coordinates of *P* are $\left(\dfrac{16\lambda + 6}{\lambda + 1}, \dfrac{-19\lambda - 7}{\lambda + 1}, \dfrac{-4\lambda}{\lambda + 1}\right)$ or $\left(\dfrac{2\mu}{\mu + 1}, \dfrac{-5\mu + 3}{\mu + 1}, \dfrac{10\mu - 6}{\mu + 1}\right)$

Comparing we have $\lambda = -\dfrac{1}{3}$ or $\mu = 1$.

Using these values, we get point of intersection as (1, –1, 2)

Here it is also proved that lines *AB* and *CD* intersect or points *A*, *B*, *C* and *D* are coplanar.

Example 1.16 **Find the angle of vector $\vec{a} = 6\hat{i} + 2\hat{j} - 3\hat{k}$ with *x*-axis.**

Sol.

$\vec{a} = 6\hat{i} + 2\hat{j} - 3\hat{k}$

$\Rightarrow \quad |\vec{a}| = \sqrt{(6)^2 + (2)^2 + (-3)^2} = 7$

$\Rightarrow$ Angle of vector with *x*-axis is $\cos^{-1} \dfrac{6}{7}$

Example 1.17

a. Show that the lines joining the vertices of a tetrahedron to the centroids of opposite faces are concurrent.

b. Show that the joins of the midpoints of the opposite edges of a tetrahedron intersect and bisect each other.

Sol.

a. G_1, the centroid of ΔBCD, is $\dfrac{\vec{b} + \vec{c} + \vec{d}}{3}$ and *A* is $\vec{a}$. The position vector of point *G* which divides AG_1 in the ratio 3 : 1 is

$$\frac{3\cdot\frac{\vec{b}+\vec{c}+\vec{d}}{3}+1\cdot\vec{a}}{3+1}=\frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$$

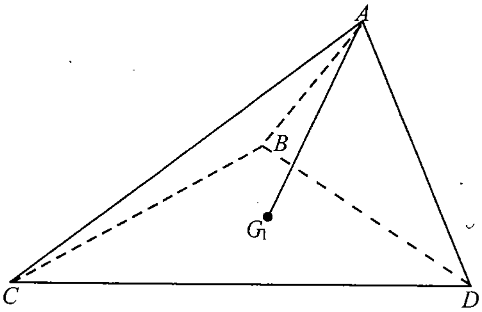


Fig. 1.29

The symmetry of the result shows that this point will also lie on BG_2, CG_3 and DG_4 (where G_2, G_3, G_4 are centroids of faces ACD, ABD and ABC, respectively). Hence, these four lines are concurrent at point $\frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$, which is called the centroid of the tetrahedron.

b. The midpoint of DA is $\frac{\vec{a}+\vec{d}}{2}$ and that of BC is $\frac{\vec{b}+\vec{c}}{2}$ and the midpoint of these midpoints is $\frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$ and symmetry of the result proves the fact.

Example 1.18 **The midpoints of two opposite sides of a quadrilateral and the midpoints of the diagonals are the vertices of a parallelogram. Prove this using vectors.**

Sol.

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ be the position vectors of vertices A, B, C and D, respectively.

Let E, F, G and H be the midpoints of AB, CD, AC and BD, respectively.

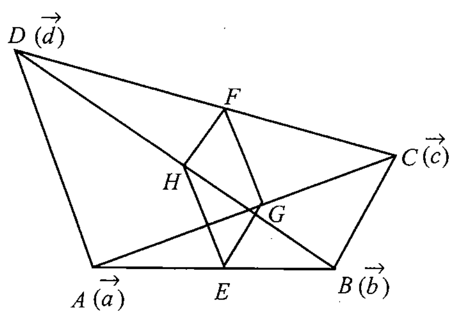


Fig. 1.30

P.V. of $E = \frac{\vec{a}+\vec{b}}{2}$

P.V. of $F = \dfrac{\vec{c}+\vec{d}}{2}$

P.V. of $G = \dfrac{\vec{a}+\vec{c}}{2}$

P.V. of $H = \dfrac{\vec{b}+\vec{d}}{2}$

$\overrightarrow{EG}$ = P.V. of G − P.V. of $E = \dfrac{\vec{a}+\vec{c}}{2} - \dfrac{\vec{a}+\vec{b}}{2} = \dfrac{\vec{c}-\vec{b}}{2}$

$\overrightarrow{HF}$ = P.V. of F − P.V. of $H = \dfrac{\vec{c}+\vec{d}}{2} - \dfrac{\vec{b}+\vec{d}}{2} = \dfrac{\vec{c}-\vec{b}}{2}$

$\therefore \quad \overrightarrow{EG} = \overrightarrow{HF} \Rightarrow EG \parallel HF$ and $EG = HF$

$\Rightarrow$ *EGHF* is a parallelogram.

Some More Solved Examples

Example 1.19 **Check whether the three vectors $2\hat{i} + 2\hat{j} + 3\hat{k}, -3\hat{i} + 3\hat{j} + 2\hat{k}$ and $3\hat{i} + 4\hat{k}$ form a triangle or not.**

Sol.

If vectors $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}, \vec{b} = -3\hat{i} + 3\hat{j} + 2\hat{k}$ and $\vec{c} = 3\hat{i} + 4\hat{k}$ form a triangle, then we must have $\vec{a} + \vec{b} + \vec{c} = 0$.

But for given vectors, $\vec{a} + \vec{b} + \vec{c} \neq 0$. Hence these vectors do not form a triangle.

Example 1.20 **Find the resultant of vectors $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 4\hat{k}$. Find the unit vector in the direction of the resultant vector.**

Sol.

The resultant vector of $\vec{a}$ and $\vec{b}$ is $\vec{a} + \vec{b} = 2\hat{i} + \hat{j} - 2\hat{k} = \vec{c}$ (let)

Now unit vector in the direction of $\vec{c}$ is $\hat{c} = \dfrac{2\hat{i} + \hat{j} - 2\hat{k}}{\sqrt{(2)^2 + (1)^2 + (-2)^2}} = \dfrac{1}{3}(2\hat{i} + \hat{j} - 2\hat{k})$

Example 1.21 **If in parallelogram ABCD, diagonal vectors are $\overrightarrow{AC} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\overrightarrow{BD} = -6\hat{i} + 7\hat{j} - 2\hat{k}$, then find the adjacent side vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$.**

Sol.

Let $\vec{AB} = \vec{a}$ and $\vec{AD} = \vec{b}$

Then $\vec{AC} = \vec{a} + \vec{b}$ and $\vec{BD} = \vec{b} - \vec{a}$

$\Rightarrow \quad \vec{b} = \dfrac{\overrightarrow{AC} + \overrightarrow{BD}}{2}$ and $\vec{a} = \dfrac{\overrightarrow{AC} - \overrightarrow{BD}}{2}$

$\Rightarrow \quad \vec{AB} = -2\hat{i} + 35 + \hat{k}$ and $\vec{AD} = 4\hat{i} - 2\hat{j} + 3\hat{k}$

Example 1.22 **If two sides of a triangle are $\hat{i} + 2\hat{j}$ and $\hat{i} + \hat{k}$, then find the length of the third side.**

Sol.

Given sides of the triangle are $\vec{a} = \hat{i} + 2\hat{j}$ and $\vec{b} = \hat{i} + \hat{k}$

If vector along the third side is $\vec{c}$, then we must have $\vec{a} + \vec{b} + \vec{c} = 0$

Then $\vec{c} = -(\hat{i} + 2\hat{j}) - (\hat{i} + \hat{k}) = -2\hat{i} - 2\hat{j} - \hat{k}$

Therefore, the length of the third side $|\vec{c}|$ is $\sqrt{(-2)^2 + (-2)^2 + (-1)^2} = 3$

Example 1.23 **Three coinitial vectors of magnitudes *a*, 2*a* and 3*a* meet at a point and their directions are along the diagonals of three adjacent faces of a cube. Determine their resultant *R*. Also prove that sum of the three vectors determined by the diagonals of three adjacent faces of a cube passing through the same corner, the vectors being directed from the corner, is twice the vector determined by the diagonal of the cube.**

Sol.

Let the length of an edge of the cube be taken as unity and the vectors represented by *OA*, *OB* and *OC* (let the three coterminous edges of unit be $\hat{i}$, $\hat{j}$ and $\hat{k}$, respectively). *OR*, *OS* and *OT* are the three diagonals of the three adjacent faces of the cube along which act the forces of magnitudes a, $2a$ and $3a$, respectively. To find the vectors representing these forces, we shall first find unit vectors in these directions and then multiply them by the corresponding given magnitudes of these forces.

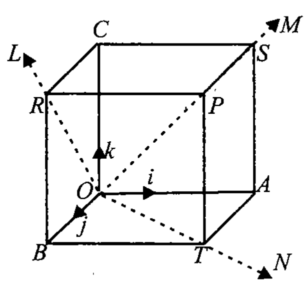


Fig. 1.31

Since $\overrightarrow{OR} = \hat{j} + \hat{k}$, the unit vector along OR is $\frac{1}{\sqrt{2}}(\hat{j}+\hat{k})$.

Hence, force $\vec{F}_1$ of magnitude a along OR is given by

$$\vec{F}_1 = \frac{a}{\sqrt{2}}(\hat{i}+\hat{k})$$

Similarly, force $\vec{F}_1$ of magnitude $2a$ along OS is $\frac{2a}{\sqrt{2}}(\hat{k}+\hat{i})$ and force $\vec{F}_3$ of magnitude $3a$ along OT is $\frac{3a}{\sqrt{2}}(\hat{i}+\hat{j})$.

If $\vec{R}$ be their resultant, then $\vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$

$$= \frac{a}{\sqrt{2}}(\hat{j}+\hat{k}) + \frac{2a}{\sqrt{2}}(\hat{k}+\hat{i}) + \frac{3a}{\sqrt{2}}(\hat{i}+\hat{j})$$

$$= \frac{5a}{\sqrt{2}}\hat{i} + \frac{4a}{\sqrt{2}}\hat{j} + \frac{3a}{\sqrt{2}}\hat{k}$$

Again, $\overrightarrow{OR} + \overrightarrow{OS} + \overrightarrow{OT} = \hat{j}+\hat{k}+\hat{i}+\hat{k}+\hat{i}+\hat{j}$

$$= 2(\hat{i}+\hat{j}+\hat{k})$$

Also $\overrightarrow{OP} = \overrightarrow{OT} + \overrightarrow{TP} = (\hat{i}+\hat{j}+\hat{k})$ $(\because \overrightarrow{OT} = \hat{i}+\hat{j}$ and $\overrightarrow{TP} = \overrightarrow{OC} = \hat{k})$

$\overrightarrow{OR} + \overrightarrow{OS} + \overrightarrow{OT} = 2\overrightarrow{OP}$

Example 1.24 **The axes of coordinates are rotated about the z-axis through an angle of $\pi/4$ in the anticlockwise direction and the components of a vector are $2\sqrt{2}$, $3\sqrt{2}$, 4. Prove that the components of the same vector in the original system are –1, 5, 4.**

Sol.

If $\hat{i}, \hat{j}, \hat{k}$ are the new unit vectors along the coordinate axes, then

$$\vec{a} = 2\sqrt{2}\hat{i} + 2\sqrt{2}\hat{j} + 4\hat{k} \quad \text{(i)}$$

$\hat{i}, \hat{j}, \hat{k}$ are obtained by rotating by 45° about the z-axis.

Then $\hat{i}$ is replaced by $\hat{i}\cos 45° + \hat{j}\sin 45° = \frac{\hat{i}+\hat{j}}{\sqrt{2}}$

and

$\hat{j}$ is replaced by $-\hat{i}\cos 45° + \hat{j}\sin 45° = \frac{-\hat{i}+\hat{j}}{\sqrt{2}}$

$\hat{k} = \hat{k}$,

$$\vec{a} = 2\sqrt{2}\left[\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right] + 3\sqrt{2}\left[\frac{-\hat{i}+\hat{j}}{\sqrt{2}}\right] + 4\hat{k}$$

$$\vec{a} = (2-3)\hat{i} + (2+3)\hat{j} + 4\hat{k}$$

$$\vec{a} = -\hat{i} + 5\hat{j} + 4\hat{k}$$

Example 1.25 **If the resultant of two forces is equal in magnitude to one of the components and perpendicular to it in direction, find the other component using the vector method.**

Sol.

Let P be horizontal in the direction of unit vector $\hat{i}$. The resultant is also P but perpendicular to it in the direction of unit vector $\hat{j}$. If Q be the other force making an angle θ (obtuse) as the resultant is perpendicular to P, then the two forces are $P\hat{i}$ and $Q\cos\theta\,\hat{i} + Q\sin\theta\,\hat{j}$. Their resultant is $P\hat{j}$.

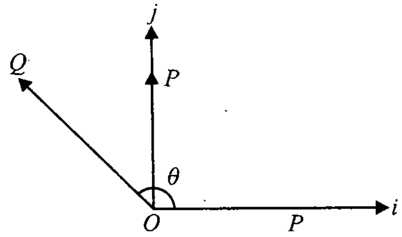


Fig. 1.32

$\therefore P\hat{j} = P\hat{i} + (Q\cos\theta\,\hat{i} + Q\sin\theta\,\hat{j})$

Comparing the coefficients of $\hat{i}$ and $\hat{j}$, we get

$P + Q\cos\theta = 0$ and $Q\sin\theta = P$

or $Q\cos\theta = -P$ and $Q\sin\theta = P$

Squaring and adding $Q = P\sqrt{2}$ and dividing

$\tan\theta = -1$

$\theta = 135°$

Example 1.26 **A man travelling towards east at 8 km/h finds that the wind seems to blow directly from the north. On doubling the speed, he finds that it appears to come from the north-east. Find the velocity of the wind.**

Sol.

The velocity of wind relative to man = Actual velocity of wind – Actual velocity of man (i)

Let $\hat{i}$ and $\hat{j}$ represent unit vectors along east and north. Let the actual velocity of wind be given by $x\hat{i} + y\hat{j}$.

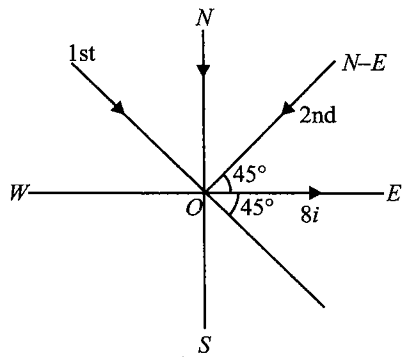


Fig. 1.33

In the first case the man's velocity is $8\hat{i}$ and that of the wind blowing from the north relative to the man is $-p\hat{j}$. Therefore,

$-p\hat{j} = (x\hat{i} + y\hat{j}) - 8\hat{i}$ [from Eq. (i)]

Comparing coefficients, $x - 8 = 0,\ y = -p$ (ii)

In the second case when the man doubles his speed, wind seems to come from the north–east direction

$-q(\hat{i} + \hat{j}) = (x\hat{i} + y\hat{j}) - 16\hat{i}$

$\therefore \quad x - 16 = -q,\ y = -q$ (iii)

Putting $x = 8$, we get $q = 8$

$y = -8$

Hence, the velocity of wind is $x\hat{i} + y\hat{j} = 8\ (\hat{i} - \hat{j})$

Its magnitude is $\sqrt{(8^2 + 8^2)} = 8\sqrt{2}$ and $\tan\theta = -1$

$\theta = -45°$.

Hence, its direction is from the north–west.

Concept Application Exercise 1.1

1. If $ABCD$ is a rhombus whose diagonals cut at the origin O, then prove that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = \vec{O}$.
2. Let D, E and F be the middle points of the sides BC, CA and AB, respectively, of a triangle ABC. Then prove that $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \vec{0}$.
3. Let $ABCD$ be a parallelogram whose diagonals intersect at P and let O be the origin. Then prove that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 4\overrightarrow{OP}$.
4. If A, B, C and D be any four points and E and F be the middle points of AC and BD, respectively, then prove that $\overrightarrow{CB} + \overrightarrow{CD} + \overrightarrow{AD} + \overrightarrow{AB} = 4\overrightarrow{EF}$.
5. If $\overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{BO} + \overrightarrow{OC}$, then A, B and C are (where O is the origin)
   a. coplanar **b.** collinear **c.** non-collinear **d.** none of these
6. If the sides of an angle are given by vectors $\vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$, then find the internal bisector of the angle.
7. $ABCD$ is a parallelogram. If L and M be the middle points of BC and CD, respectively, express $\overrightarrow{AL}$ and $\overrightarrow{AM}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AD}$. Also show that $\overrightarrow{AL} + \overrightarrow{AM} = (3/2)\overrightarrow{AC}$.

8. $ABCD$ is a quadrilateral and E the point of intersection of the lines joining the middle points of opposite sides. Show that the resultant of $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ and $\overrightarrow{OD}$ is equal to $4\overrightarrow{OE}$, where O is any point.

9. What is the unit vector parallel to $\vec{a} = 3\hat{i} + 4\hat{j} - 2\hat{k}$? What vector should be added to $\vec{a}$ so that the resultant is the unit vector $\hat{i}$?

10. The position vectors of points A and B w.r.t. an origin are $\vec{a} = \hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{b} = 3\hat{i} + \hat{j} - 2\hat{k}$, respectively. Determine vector $\overrightarrow{OP}$ which bisects angle AOB, where P is a point on AB.

11. If $\vec{r}_1, \vec{r}_2, \vec{r}_3$ are the position vectors of three collinear points and scalars p and q exist such that $\vec{r}_3 = p \cdot \vec{r}_1 + q\vec{r}_2$, then show that $p + q = 1$.

VECTOR ALONG THE BISECTOR OF GIVEN TWO VECTORS

We know that the diagonal in a parallelogram is not necessarily the bisector of the angle formed by two adjacent sides. However, the diagonal in a rhombus bisects the angle between two adjacent sides.

Consider vectors $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$ forming a parallelogram $ABCD$ as shown in the following figure.

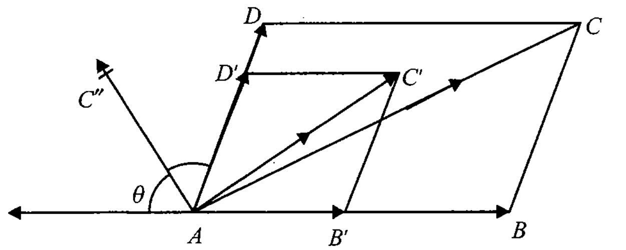


Fig. 1.34

Consider the two unit vectors along the given vectors, which form a rhombus $AB'C'D'$.

Now $\overrightarrow{AB'} = \dfrac{\vec{a}}{|\vec{a}|}$ and $\overrightarrow{AD'} = \dfrac{\vec{b}}{|\vec{b}|}$. Therefore,

$$\overrightarrow{AC'} = \frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}$$

So any vector along the bisector is $\lambda\left(\dfrac{\vec{a}}{|\vec{a}|} + \dfrac{\vec{b}}{|\vec{b}|}\right)$.

Similarly, any vector along the external bisector is $\overrightarrow{AC''} = \lambda\left(\dfrac{\vec{a}}{|\vec{a}|} - \dfrac{\vec{b}}{|\vec{b}|}\right)$

Example 1.27 **If $\vec{a} = 7\hat{i} - 4\hat{j} - 4\hat{k}$ and $\vec{b} = -2\hat{i} - \hat{j} + 2\hat{k}$, determine vector $\vec{c}$ along the internal bisector of the angle between vectors $\vec{a}$ and $\vec{b}$, such that $|\vec{c}| = 5\sqrt{6}$.**

Sol.

$$\hat{a} = \frac{1}{9}(7\hat{i} - 4\hat{j} - 4\hat{k})$$

$$\hat{b} = \frac{1}{3}(-2\hat{i} - \hat{j} + 2\hat{k})$$

$$\vec{c} = \lambda[\hat{a} + \hat{b}] = \lambda\frac{1}{9}(\hat{i} - 7\hat{j} + 2\hat{k}) \qquad \text{(i)}$$

$$|\vec{c}| = 5\sqrt{6}$$

$$\Rightarrow \frac{\lambda^2}{81}(1 + 49 + 4) = 25 \times 6$$

$$\lambda^2 = \frac{25 \times 6 \times 81}{54} = 225$$

$\lambda = \pm 15$

Putting the value of λ in (i), we get

$$\vec{c} = \pm\frac{5}{3}(\hat{i} - 7\hat{j} + 2\hat{k})$$

Example 1.28 **Find a unit vector $\vec{c}$ if $-\hat{i} + \hat{j} - \hat{k}$ bisects the angle between vectors $\vec{c}$ and $3\hat{i} + 4\hat{j}$.**

Sol.

Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$, where $x^2 + y^2 + z^2 = 1$. (i)

Unit vector along $3\hat{i} + 4\hat{j}$ is $\frac{3\hat{i} + 4\hat{j}}{5}$.

The bisector of these two is $-\hat{i} + \hat{j} - \hat{k}$ (given). Therefore,

$$-\hat{i} + \hat{j} - \hat{k} = \lambda\left(x\hat{i} + y\hat{j} + z\hat{k} + \frac{3\hat{i} + 4\hat{j}}{5}\right)$$

$$-\hat{i} + \hat{j} - \hat{k} = \frac{1}{5}\lambda[(5x + 3)\hat{i} + (5y + 4)\hat{j} + 5z\hat{k}] \qquad \text{(ii)}$$

$$\frac{\lambda}{5}(5x + 3) = -1, \quad \frac{\lambda}{5}(5y + 4) = 1, \quad \frac{\lambda}{5}5z = -1$$

$$x = -\frac{5 + 3\lambda}{5\lambda}, \quad y = \frac{5 - 4\lambda}{5\lambda}, \quad z = -\frac{1}{\lambda}$$

Putting these values in (i), i.e., $x^2 + y^2 + z^2 = 1$, we get

$(5 + 3\lambda)^2 + (5 - 4\lambda)^2 + 25 = 25\lambda^2$

$25\lambda^2 - 10\lambda + 75 = 25\lambda^2$

$\lambda = 15/2$

$\therefore \quad \vec{c} = \frac{1}{15}\,(-11\hat{i}+10\hat{j}-2\hat{k})$

LINEAR COMBINATION, LINEAR INDEPENDENCE AND LINEAR DEPENDENCE

Linear Combination

A vector $\vec{r}$ is said to be a linear combination of vectors $\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n$ if there exist scalars $m_1, m_2, \cdots, m_n$ such that $\vec{r} = m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots + m_n\vec{a}_n$.

Linearly Independent

A system of vectors $\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n$ is said to be linearly independent if

$m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots + m_n\vec{a}_n = \vec{0} \Rightarrow m_1 = m_2 = \cdots = m_n = 0$

It can be easily verified that

i. A pair of non-collinear vectors is linearly independent.

Proof:

Let $\vec{a}_1$ and $\vec{a}_2$ are non-collinear vectors such that $m_1\vec{a}_1 + m_2\vec{a}_2 = \vec{O}$

Let $m_1, m_2 \neq 0$

$\Rightarrow \vec{a}_1 = -\frac{m_2}{m_1}\vec{a}_2$

$\Rightarrow \vec{a}_1$ and $\vec{a}_2$ are collinear, which contradicts the given fact.

Hence $m_1, m_2 = 0$

ii. A triad of non-coplanar vector is linearly independent.

Linearly Dependent

A set of vectors $\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n$ is said to be linearly dependent if there exist scalars $m_1, m_2, \cdots, m_n$, not all zero, such that $m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots + m_n\vec{a}_n = \vec{0}$.

It can be easily verified that

i. A pair of collinear vectors is linearly dependent.

ii. A triad of coplanar vectors is linearly dependent.

Theorem 1.1

If $\vec{a}$ and $\vec{b}$ be two non-collinear vectors, then every vector $\vec{r}$ coplanar with $\vec{a}$ and $\vec{b}$ can be expressed in one and only one way as a linear combination $x\vec{a} + y\vec{b}$; x and y being scalars.

Proof:

i.

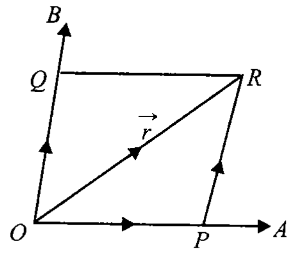


Fig. 1.35

Let O be any point such that $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.

As $\vec{r}$ is coplanar with $\vec{a}$ and $\vec{b}$, the lines OA, OB and OR are coplanar.

Through R, draw lines parallel to OA and OB, meeting them at P and Q, respectively.

Clearly, $\overrightarrow{OP} = x\,\overrightarrow{OA} = x\vec{a}$ ($\because \overrightarrow{OP}$ and $\overrightarrow{OA}$ are collinear vectors)

Also $\overrightarrow{OQ} = y\,\overrightarrow{OB} = y\vec{b}$ ($\because \overrightarrow{OQ}$ and $\overrightarrow{OB}$ are collinear vectors)

$\vec{r} = \overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OP} + \overrightarrow{OQ}$ ($\because \overrightarrow{OQ}$ and $\overrightarrow{PR}$ are equal)

$$= x\vec{a} + y\vec{b} \qquad \text{(i)}$$

Thus, $\vec{r}$ can be expressed in one way as a linear combination $x\vec{a} + y\vec{b}$.

ii. To prove that this resolution is unique, let $\vec{r} = x'\vec{a} + y'\vec{b}$ be another representation of $\vec{r}$ as a linear combination of $\vec{a}$ and $\vec{b}$.

Then, $x\vec{a} + y\vec{b} = x'\vec{a} + y'\vec{b}$

or $(x - x')\vec{a} + (y - y')\vec{b} = \vec{0}$

Since $\vec{a}$ and $\vec{b}$ are non-collinear vectors, we must have

$x - x' = 0,\ y - y' = 0$

i.e., $x = x',\ y = y'$

Thus the representation is unique.

Note:

If OA and OB are perpendicular, then these two lines can be taken as the x- and the y-axes, respectively. Let $\hat{i}$ be the unit vector along the x-axis and $\hat{j}$ be the unit vector along the y-axis. Therefore, we have

$\vec{r} = x\vec{i} + y\vec{j}$

Also $r = \sqrt{x^2 + y^2}$

Theorem 1.2

If $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar vectors, then any vector $\vec{r}$ can be uniquely expressed as a linear combination $x\vec{a} + y\vec{b} + z\vec{c}$; x, y and z being scalars.

Proof:

i.

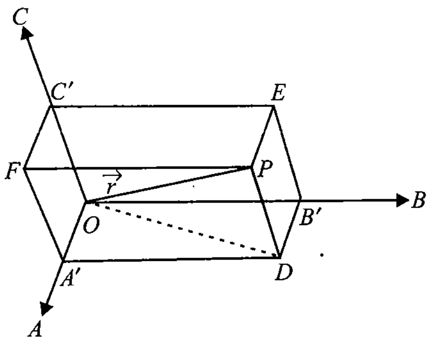


Fig. 1.36

Take any point O so that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$ and $\overrightarrow{OP} = \vec{r}$.

On OP as diagonal, construct a parallelpiped having edges OA', OB' and OC' along OA, OB and OC, respectively. Then there exist three scalars x, y and z such that

$$\overrightarrow{OA'} = x\,\overrightarrow{OA} = x\vec{a},\ \overrightarrow{OB'} = y\,\overrightarrow{OB} = y\vec{b},\ \overrightarrow{OC'} = z\,\overrightarrow{OC} = z\vec{c}$$

$\therefore\ \vec{r} = \overrightarrow{OP}$

$= \overrightarrow{OA'} + \overrightarrow{A'P}$

$= \overrightarrow{OA'} + \overrightarrow{A'D} + \overrightarrow{DP}$ (by definition of addition of vectors)

$= \overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'}$

$= x\vec{a} + y\vec{b} + z\vec{c}$ (i)

Thus $\vec{r}$ can be represented as a linear combination of $\vec{a}, \vec{b}$ and $\vec{c}$.

ii To prove that this representation is unique, let, if possible, $\vec{r} = x'\vec{a} + y'\vec{b} + z'\vec{c}$ be another representation of $\vec{r}$ as a linear combination of $\vec{a}, \vec{b}$ and $\vec{c}$. (ii)

Then from (i) and (ii), we have

$x\vec{a} + y\vec{b} + z\vec{c} = \vec{r} = x'\vec{a} + y'\vec{b} + z'\vec{c}$

or

$(x - x')\vec{a} + (y - y')\vec{b} + (z - z')\vec{c} = \vec{0}$

Since $\vec{a}, \vec{b}$ and $\vec{c}$ are independent, $x - x' = 0$, $y - y' = 0$ and $z - z' = 0$, or $x = x'$, $y = y'$ and $z = z'$. Hence proved.

Theorem 1.3

If vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ are coplanar, then

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

Proof:

If vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, then there exist scalars λ and μ such that $\vec{c} = \lambda\vec{a} + \mu\vec{b}$. Hence,

$$c_1\hat{i} + c_2\hat{j} + c_3\hat{k} = \lambda(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) + \mu(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

Now $\hat{i}, \hat{j}$ and $\hat{k}$ are non-coplanar and hence independent. Then,

$c_1 = \lambda a_1 + \mu b_1$, $c_2 = \lambda a_2 + \mu b_2$ and $c_3 = \lambda a_3 + \mu b_3$

The above system of equations in terms of λ and μ is consistent.

$$\Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Similarly, if vectors $x_1\vec{a} + y_1\vec{b} + z_1\vec{c}$, $x_2\vec{a} + y_2\vec{b} + z_2\vec{c}$ and $x_3\vec{a} + y_3\vec{b} + z_3\vec{c}$ are coplanar (where $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar). Then $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$ can be proved with the same arguments.

To prove that four points $A(\vec{a}), B(\vec{b}), C(\vec{c})$ and $D(\vec{d})$ are coplanar, it is just sufficient to prove that vectors $\overrightarrow{AB}, \overrightarrow{BD}$ and $\overrightarrow{CD}$ are coplanar.

Notes:

1. *Two collinear vectors are always linearly dependent.*
2. *Two non-collinear non-zero vectors are always linearly independent.*
3. *Three coplanar vectors are always linearly dependent.*
4. *Three non-coplanar non-zero vectors are always linearly independent.*
5. *More than three vectors are always linearly dependent.*
6. *Three points with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are collinear if and only if there exist scalars x, y and z not all zero such that (i) $x\vec{a} + y\vec{b} + z\vec{c} = 0$ and (ii) $x + y + z = 0$.*

   Proof:

   Let us suppose that points A, B and C are collinear and their position vectors are $\vec{a}, \vec{b}$ and $\vec{c}$, respectively. Let C divide the join of $\vec{a}$ and $\vec{b}$ in the ratio y : x. Then,

$\vec{c} = \dfrac{x\vec{a} + y\vec{b}}{x+y}$

or $x\vec{a} + y\vec{b} - (x+y)\vec{c} = \vec{0}$

or $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, *where* $z = -(x+y)$

Also, $x+y+z = x+y-(x+y) = 0.$

Conversely, let $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, *where* $x+y+z=0$. Therefore,

$x\vec{a} + y\vec{b} = -z\vec{c} = (x+y)\vec{c}, \ \because \ x+y=-z$

or $\vec{c} = \dfrac{x\vec{a} + y\vec{b}}{x+y}$

This relation shows that $\vec{c}$ *divides the join of* $\vec{a}$ *and* $\vec{b}$ *in the ratio* $y : x$. *Hence the three points A, B and C are collinear.*

7. *Four points with position vectors* $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ *are coplanar if there exist scalars x, y, z and w (sum of any two is not zero) such that* $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = \vec{0}$ *with* $x + y + z + w = 0$.

Proof:

$x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = \vec{0}$

$\Rightarrow \ x\vec{a} + y\vec{b} = -(z\vec{c} + w\vec{d})$ (i)

$x + y + z + w = 0$

$\Rightarrow \ x + y = -(w+z)$ (ii)

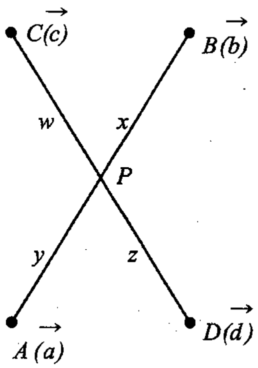


Fig. 1.37

From (i) and (ii), we have $\dfrac{x\vec{a} + y\vec{b}}{x+y} = \dfrac{z\vec{c} + w\vec{d}}{z+w}$

Thus there is point P

$\Rightarrow \ \dfrac{x\vec{a} + y\vec{b}}{x+y} = \dfrac{z\vec{c} + w\vec{d}}{z+w}$ (iii)

$\dfrac{x\vec{a} + y\vec{b}}{x+y}$ *is the position vector of a point on AB which divides it in the ratio* $y : x$.

$\dfrac{z\vec{c}+w\vec{d}}{z+w}$ *is the position vector of a point on CD which divides it in the ratio w : z.*

From (iii), these points are coincident; hence the points are coplanar.

Example 1.29 **The vectors $2\hat{i}+3\hat{j}$, $5\hat{i}+6\hat{j}$ and $8\hat{i}+\lambda\hat{j}$ have their initial points at (1, 1). Find the value of λ so that the vectors terminate on one straight line**

Sol.

Since the vectors $2\hat{i}+3\hat{j}$ and $5\hat{i}+6\hat{j}$ have (1, 1) as the initial point, therefore their terminal points are (3, 4) and (6, 7), respectively. The equation of the line joining these two points is $x - y + 1 = 0$. The terminal point of $8\hat{i}+\lambda\hat{j}$ is $(9, \lambda + 1)$. Since the vectors terminate on the same straight line, $(9, \lambda + 1)$ lies on $x - y + 1 = 0$. Therefore,

$9 - \lambda - 1 + 1 = 0$

$\Rightarrow \lambda = 9$

Example 1.30 **If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-zero vectors, no two of which are collinear, $\vec{a}+2\vec{b}$ is collinear with $\vec{c}$ and $\vec{b}+3\vec{c}$ is collinear with $\vec{a}$, then find the value of $|\vec{a}+2\vec{b}+6\vec{c}|$.**

Sol.

Given $\vec{a}+2\vec{b}=\lambda\vec{c}$ (i)

and $\vec{b}+3\vec{c}=\mu\vec{a}$, (ii)

where no two of $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear vectors.

Eliminating $\vec{b}$ from the above relations, we have

$\vec{a}-6\vec{c}=\lambda\vec{c}-2\mu\vec{a}$

$\vec{a}(1+2\mu)=(\lambda+6)\vec{c}$

$\Rightarrow \mu=-\dfrac{1}{2}$ and $\lambda=-6$ as $\vec{a}$ and $\vec{c}$ are non-collinear.

Putting $\mu=-\dfrac{1}{2}$ in (ii) or $\lambda=-6$ in (i), we get

$\vec{a}+2\vec{b}+3\vec{c}=\vec{0}$

$\Rightarrow |\vec{a}+2\vec{b}+3\vec{c}|=0$

Example 1.31 **a. Prove that the points $\vec{a}-2\vec{b}+3\vec{c}$, $2\vec{a}+3\vec{b}-4\vec{c}$ and $-7\vec{b}+10\vec{c}$ are collinear, where $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar.**

b. Prove that the points $A(1, 2, 3)$, $B(3, 4, 7)$ and $C(-3, -2, -5)$ are collinear. Find the ratio in which point C divides AB.

Sol.

a. Let the given points be A, B and C. Therefore,

$\overrightarrow{AB}$ = P.V. of B – P.V. of A

$= (2\vec{a}+3\vec{b}-4\vec{c})-(\vec{a}-2\vec{b}+3\vec{c})$

$= \vec{a}+5\vec{b}-7\vec{c}$

$\overrightarrow{AC}$ = P.V. of C – P.V. of A

$= (-7\vec{b} + 10\vec{c}) - (\vec{a} - 2\vec{b} + 3\vec{c})$

$= -\vec{a} - 5\vec{b} + 7\vec{c} = -\overrightarrow{AB}$

Since $\overrightarrow{AC} = -\overrightarrow{AB}$, it follows that the points A, B and C are collinear.

b. Let C divide AB in the ratio $k : 1$; then $C(-3, -2, -5) \equiv \left(\frac{3k+1}{k+1}, \frac{4k+2}{k+1}, \frac{7k+3}{k+1}\right)$

$\Rightarrow \frac{3k+1}{k+1} = -3, \frac{4k+2}{k+1} = -2$ and $\frac{7k+3}{k+1} = -5$

$\Rightarrow k = -\frac{2}{3}$ from all relations

Hence, C divides AB externally in the ratio 2:3.

Example 1.32 **Check whether the given three vectors are coplanar or non-coplanar:** $-2\hat{i} - 2\hat{j} + 4\hat{k}, -2\hat{i} + 4\hat{j} - 2\hat{k}, 4\hat{i} - 2\hat{j} - 2\hat{k}$

Sol.

Given vectors are $-2\hat{i} - 2\hat{j} + 4\hat{k}, -2\hat{i} + 4\hat{j} - 2\hat{k}, 4\hat{i} - 2\hat{j} - 2\hat{k}$

$$\Rightarrow \begin{vmatrix} -2 & -2 & 4 \\ -2 & 4 & -2 \\ 4 & -2 & -2 \end{vmatrix} = 16 + 16 + 16 - 64 + 8 + 8 = 0$$

Hence the vectors are coplanar.

Example 1.33 **Prove that the four points $6\hat{i} - 7\hat{j}, 16\hat{i} - 19\hat{j} - 4\hat{k}, 3\hat{j} - 6\hat{k}$ and $2\hat{i} + 5\hat{j} + 10\hat{k}$ form a tetrahedron in space.**

Sol.

Given points are $A(6\hat{i} - 7\hat{j}), B(16\hat{i} - 19\hat{j} - 4\hat{k}), C(3\hat{j} - 6\hat{k}), D(2\hat{i} + 5\hat{j} + 10\hat{k})$

Hence vectors $\overrightarrow{AB} = 10\hat{i} - 12\hat{j} - 4\hat{k}$, $\overrightarrow{AC} = -6\hat{i} + 10\hat{j} - 6\hat{k}$ and $\overrightarrow{AD} = -4\hat{i} + 12\hat{j} + 10\hat{k}$

Now determinant of coefficients of $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ is

$$\begin{vmatrix} 10 & -12 & -4 \\ -6 & 10 & -6 \\ -4 & 12 & 10 \end{vmatrix} = 10(100 + 72) + 12(-60 - 24) - 4(-72 + 40) \neq 0$$

Hence, the given points are non-coplanar and therefore form a tetrahedron in space.

Example 1.34 **If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors, show that points $l_1\vec{a}+m_1\vec{b}$, $l_2\vec{a}+m_2\vec{b}$ and $l_3\vec{a}+m_3\vec{b}$ are collinear if** $\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ 1 & 1 & 1 \end{vmatrix} = 0.$

Sol.

We know that three points having P.V.s $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear if there exists a relation of the form $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, where $x + y + z = 0$.

Now $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ gives

$x(l_1\vec{a} + m_1\vec{b}) + y(l_2\vec{a} + m_2\vec{b}) + z(l_3\vec{a} + m_3\vec{b}) = \vec{0}$

or $(xl_1 + yl_2 + zl_3)\vec{a} + (xm_1 + ym_2 + zm_3)\vec{b} = \vec{0}$

Since $\vec{a}$ and $\vec{b}$ are two non-collinear vectors, it follows that

$xl_1 + yl_2 + zl_3 = 0$ (i)

$xm_1 + ym_2 + zm_3 = 0$ (ii)

Because otherwise one is expressible as a scalar multiple of the other which would mean that $\vec{a}$ and $\vec{b}$ are collinear.

Also $x + y + z = 0$. (iii)

Eliminating x, y and z from (i), (ii) and (iii), we get

$$\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Alternative method:

$A(l_1\vec{a} + m_1\vec{b})$, $B(l_2\vec{a} + m_2\vec{b})$ and $C(l_3\vec{a} + m_3\vec{b})$ are collinear.

$\Rightarrow$ Vectors $= (l_2 - l_3)\vec{a} + (m_2 - m_3)\vec{b}$ and $\overrightarrow{AB} = (l_1 - l_2)\vec{a} + (m_1 - m_2)\vec{b}$ are collinear.

$$\Rightarrow \frac{l_1 - l_2}{l_2 - l_3} = \frac{m_1 - m_2}{m_2 - m_3}$$

$$\Rightarrow \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Example 1.35 **Vectors $\vec{a}$ and $\vec{b}$ are non-collinear. Find for what value of x vectors $\vec{c} = (x-2)\vec{a}+\vec{b}$ and $\vec{d} = (2x+1)\vec{a}-\vec{b}$ are collinear?**

Sol.

Both the vectors $\vec{c}$ and $\vec{d}$ are non-zero as the coefficients of $\vec{b}$ in both are non-zero.

Two vectors $\vec{c}$ and $\vec{d}$ are collinear if one of them is a linear multiple of the other. Therefore,

$\vec{d} = \lambda \vec{c}$

or $(2x+1)\vec{a} - \vec{b} = \lambda \{(x-2)\vec{a} + \vec{b}\}$ (i)

or $\{(2x+1) - \lambda(x-2)\}\vec{a} - (1+\lambda)\vec{b} = 0$

The above expression is of the form $p\vec{a} + q\vec{b} = 0$, where $\vec{a}$ and $\vec{b}$ are non-collinear, and hence we have $p = 0$ and $q = 0$. Therefore,

$2x + 1 - \lambda(x-2) = 0$ (ii)

and $1 + \lambda = 0$ (iii)

From (iii), $\lambda = -1$ and putting this value in (i), we get $x = \frac{1}{3}$

Alternative method:

$\vec{c} = (x-2)\vec{a} + \vec{b}$ and $\vec{d} = (2x+1)\vec{a} - \vec{b}$ are collinear.

If $\frac{x-2}{2x+1} = \frac{1}{-1} \Rightarrow x = \frac{1}{3}$

Example 1.36 **The median AD of the triangle ABC is bisected at E and BE meets AC at F. Find $AF : FC$.**

Sol.

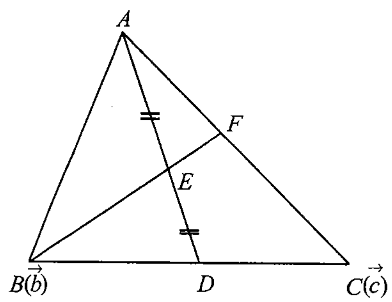


Fig. 1.38

Taking A at the origin

Let P.V. of B and C be $\vec{b}$ and $\vec{c}$, respectively.

P.V. of D is $\frac{\vec{b}+\vec{c}}{2}$ and P.V. of E is $\frac{\vec{b}+\vec{c}}{4}$

Let $AF{:}FC = p{:}1$.

Then position vector of F is $\frac{p\vec{c}}{p+1}$ (i)

Let $BF{:}EF = q : 1$.

The position vector of F is $\frac{q\frac{(\vec{b}+\vec{c})}{4} - \vec{b}}{q-1}$ (ii)

Comparing P.V. of F in (i) and (ii), we have

$$\frac{p\vec{c}}{p+1} = \frac{q\dfrac{(\vec{b}+\vec{c})}{4} - \vec{b}}{q-1}$$

Since vectors $\vec{b}$ and $\vec{c}$ are independent, we have

$$\frac{p}{p+1} = \frac{q}{4(q-1)} \text{ and } \frac{q-4}{4(q-1)} = 0$$

$\Rightarrow$ $p = 1/4$ and $q = 4$

$\Rightarrow$ $AF : FC = 1:2$

Example 1.37 **Prove that the necessary and sufficient condition for any four points in three-dimensional space to be coplanar is that there exists a linear relation connecting their position vectors such that the algebraic sum of the coefficients (not all zero) in it is zero.**

Sol.

Let us suppose that the points A, B, C and D whose position vectors are $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, respectively, are coplanar. In that case the lines AB and CD will intersect at some point P (it being assumed that AB and CD are not parallel, and if they are, then we will choose any other pair of non-parallel lines formed by the given points). If P divides AB in the ratio $q : p$ and CD in the ratio $n : m$, then the position vector of P written from AB and CD is

$$\frac{p\vec{a}+q\vec{b}}{p+q} = \frac{m\vec{c}+n\vec{d}}{m+n}$$

or $\dfrac{p}{p+q}\vec{a} + \dfrac{q}{p+q}\vec{b} - \dfrac{m}{m+n}\vec{c} - \dfrac{n}{m+n}\vec{d} = \vec{0}$

or $L\vec{a} + M\vec{b} + N\vec{c} + P\vec{d} = \vec{0}$

where $L + M + N + P = \dfrac{p}{p+q} + \dfrac{q}{p+q} - \dfrac{m}{m+n} - \dfrac{n}{m+n} = 1 - 1 = 0$

Hence the condition is necessary.

Converse: Let $l\vec{a} + m\vec{b} + n\vec{c} + p\vec{d} = \vec{0}$

where $l + m + n + p = 0$ (i)

We will show that the points A, B, C and D are coplanar.

Now of the three scalars $l + m$, $l + n$ and $l + p$, one at least is not zero, because if all of them are zero, then $l + m = 0, l + n = 0, l + p = 0$. Therefore,

$m = n = p = -l$

Hence $l + m + n + p = 0$ $\Rightarrow$ $l - 3l = 0$ $\Rightarrow$ $l = 0$

Hence $m = n = p = -l = 0$

Thus $l = 0, m = 0, n = 0, p = 0$, which is against the hypothesis.

Let us suppose that $l + m$ is not zero.

$l+m=-(n+p)\neq 0,$ [From (i)] (ii)

Also from the given relation, we have

$$l\vec{a}+m\vec{b}=-(n\vec{c}+p\vec{d})$$

or $\dfrac{l\vec{a}+m\vec{b}}{l+m}=\dfrac{n\vec{c}+p\vec{d}}{n+p}$ [From (ii)] (iii)

L.H.S. represents a point which divides AB in the ratio $m : l$ and R.H.S. represents a point which divides CD in the ratio $p : n$. These points being the same, it follows that a point on AB is the same as a point on CD, showing that the lines AB and CD intersect. Hence the four points A, B, C and D are coplanar.

Example 1.38 **a. If $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar vectors, prove that vectors $3\vec{a}-7\vec{b}-4\vec{c}$, $3\vec{a}-2\vec{b}+\vec{c}$ and $\vec{a}+\vec{b}+2\vec{c}$ are coplanar.**

b. If the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}-3\hat{k}$ and $3\hat{i}+a\hat{j}+5\hat{k}$ are coplanar, then prove that $a = 4$.

Sol.

a. If the given vectors are coplanar, then we should be able to express one of them as a linear combination of the other two.

Let us assume that $3\vec{a}-7\vec{b}-4\vec{c}=x(3\vec{a}-2\vec{b}+\vec{c})+y(\vec{a}+\vec{b}+2\vec{c})$,

where x and y are scalars. Since $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar, equating the coefficients of $\vec{a}$, $\vec{b}$ and $\vec{c}$, we get

$3x + y = 3, -2x + y = -7, x + 2y = -4$

Solving the first two, we find that $x = 2$ and $y = -3$. These values of x and y satisfy the third equation as well.

Hence the given vectors are coplanar.

b. Given vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}-3\hat{k}$ and $3\hat{i}+a\hat{j}+5\hat{k}$ are coplanar. Then $\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & a & 5 \end{vmatrix}=0$

$\Rightarrow$ $3-7a+25=0$

$\Rightarrow$ $a=4$

Example 1.39 **If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar vectors, prove that the four points $2\vec{a}+3\vec{b}-\vec{c}$, $\vec{a}-2\vec{b}+3\vec{c}$, $3\vec{a}+4\vec{b}-2\vec{c}$ and $\vec{a}-6\vec{b}+6\vec{c}$ are coplanar.**

Sol.

Let the given points be A, B, C and D. If they are coplanar, then the three coterminous vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ should be coplanar.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -\vec{a} - 5\vec{b} + 4\vec{c}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \vec{a} + \vec{b} - \vec{c}$

and $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = -\vec{a} - 9\vec{b} + 7\vec{c}$

Since the vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar, we must have $\begin{vmatrix} -1 & -5 & 4 \\ 1 & 1 & -1 \\ -1 & -9 & 7 \end{vmatrix} = 0$, which is true.

Hence proved.

Example 1.40 **Points $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$ and $D(\vec{d})$ are related as $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = 0$ and $x + y + z + w = 0$, where x, y, z and w are scalars (sum of any two of x, y, z and w is not zero).**

Prove that if A, B, C and D are concyclic, then $|xy||\vec{a} - \vec{b}|^2 = |wz||\vec{c} - \vec{d}|^2$

Sol.

From the given conditions, it is clear that points $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$ and $D(\vec{d})$ are coplanar.

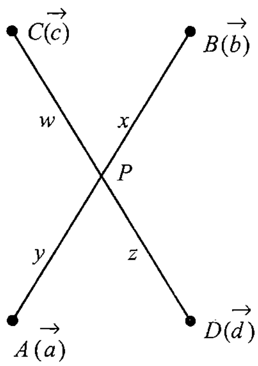


Fig. 1.39

Now, A, B, C and D are concyclic. Therefore,

$AP \times BP = CP \times DP$

$$\left|\frac{y}{x+y}\right| |\vec{a} - \vec{b}| \left|\frac{x}{x+y}\right| |\vec{a} - \vec{b}| = \left|\frac{w}{w+z}\right| |\vec{c} - \vec{d}| \left|\frac{z}{w+z}\right| |\vec{c} - \vec{d}|$$

$|xy||\vec{a} - \vec{b}|^2 = |wz||\vec{c} - \vec{d}|^2$

Concept Application Exercise 1.2

1. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are four vectors in three-dimensional space with the same initial point and such that $3\vec{a} - 2\vec{b} + \vec{c} - 2\vec{d} = \vec{0}$, show that terminals A, B, C and D of these vectors are coplanar. Find the point at which AC and BD meet. Find the ratio in which P divides AC and BD.
2. Show that the vectors $2\vec{a} - \vec{b} + 3\vec{c}$, $\vec{a} + \vec{b} - 2\vec{c}$ and $\vec{a} + \vec{b} - 3\vec{c}$ are non-coplanar vectors (where $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors).

3. Examine the following vectors for linear independence:

i. $\vec{i}+\vec{j}+\vec{k}, 2\vec{i}+3\vec{j}-\vec{k}, -\vec{i}-2\vec{j}+2\vec{k}$

ii. $3\vec{i}+\vec{j}-\vec{k}, 2\vec{i}-\vec{j}+7\vec{k}, 7\vec{i}-\vec{j}+13\vec{k}$

4. If $\vec{a}$ and $\vec{b}$ are non-collinear vectors and $\vec{A}=(p+4q)\vec{a}+(2p+q+1)\vec{b}$ and $\vec{B}=(-2p+q+2)\vec{a}+(2p-3q-1)\vec{b}$, and if $3\vec{A}=2\vec{B}$, then determine p and q.

5. If $\vec{a}, \vec{b}$ and $\vec{c}$ are any three non-coplanar vectors, then prove that points $l_1\vec{a}+m_1\vec{b}+n_1\vec{c}, l_2\vec{a}+m_2\vec{b}+n_2\vec{c}, l_3\vec{a}+m_3\vec{b}+n_3\vec{c}, l_4\vec{a}+m_4\vec{b}+n_4\vec{c}$ are coplanar if

$$\begin{vmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

6. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-zero, non-coplanar vectors, then find the linear relation between the following four vectors: $\vec{a}-2\vec{b}+3\vec{c}, 2\vec{a}-3\vec{b}+4\vec{c}, 3\vec{a}-4\vec{b}+5\vec{c}, 7\vec{a}-11\vec{b}+15\vec{c}$.

Exercises

Subjective Type

Solutions on page 1.57

1. The position vectors of the vertices A, B and C of a triangle are $\hat{i}+\hat{j}, \hat{j}+\hat{k}$ and $\hat{i}+\hat{k}$, respectively. Find a unit vector $\hat{r}$ lying in the plane of ABC and perpendicular to IA, where I is the incentre of the triangle.
2. A ship is sailing towards the north at a speed of 1.25 m/s. The current is taking it towards the east at the rate of 1 m/s and a sailor is climbing a vertical pole on the ship at the rate of 0.5 m/s. Find the velocity of the sailor in space.
3. Given four points P_1, P_2, P_3 and P_4 on the coordinate plane with origin O which satisfy the condition $\overrightarrow{OP}_{n-1}+\overrightarrow{OP}_{n+1}=\frac{3}{2}\overrightarrow{OP}_n$.

   i. If P_1 and P_2 lie on the curve $xy = 1$, then prove that P_3 does not lie on the curve.

   ii. If P_1, P_2 and P_3 lie on the circle $x^2 + y^2 = 1$, then prove that P_4 also lies on this circle.
4. $ABCD$ is a tetrahedron and O is any point. If the lines joining O to the vertices meet the opposite faces at P, Q, R and S, prove that $\frac{OP}{AP}+\frac{OQ}{BQ}+\frac{OR}{CR}+\frac{OS}{DS}=1$.
5. A pyramid with vertex at point P has a regular hexagonal base $ABCDEF$. Position vectors of points A and B are $\hat{i}$ and $\hat{i}+2\hat{j}$, respectively. Centre of the base has the position vector $\hat{i}+\hat{j}+\sqrt{3}\hat{k}$. Altitude drawn from P on the base meets the diagonal AD at point G. Find all possible position vectors of G. It is given that the volume of the pyramid is $6\sqrt{3}$ cubic units and $AP = 5$ units.

6. A straight line L cuts the lines AB, AC and AD of a parallelogram $ABCD$ at points B_1, C_1 and D_1, respectively. If $\overrightarrow{AB_1} = \lambda_1 \overrightarrow{AB}$, $\overrightarrow{AD_1} = \lambda_2 \overrightarrow{AD}$ and $\overrightarrow{AC_1} = \lambda_3 \overrightarrow{AC}$, then prove that $\frac{1}{\lambda_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$.

7. The position vectors of the points P and Q are $5\hat{i}+7\hat{j}-2\hat{k}$ and $-3\hat{i}+3\hat{j}+6\hat{k}$, respectively. Vector $\vec{A} = 3\hat{i}-\hat{j}+\hat{k}$ passes through point P and vector $\vec{B} = -3\hat{i}+2\hat{j}+4\hat{k}$ passes through point Q. A third vector $2\hat{i}+7\hat{j}-5\hat{k}$ intersects vectors A and B. Find the position vectors of points of intersection.

8. Show that $x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ and $x_3\hat{i} + y_3\hat{j} + z_3\hat{k}$ are non-coplanar if $|x_1| > |y_1| + |z_1|$, $|y_2| > |x_2| + |z_2|$ and $|z_3| > |x_3| + |y_3|$.

9. If $\vec{A}$ and $\vec{B}$ be two vectors and k be any scalar quantity greater than zero, then prove that $|\vec{A}+\vec{B}|^2 \le (1+k)|\vec{A}|^2 + \left(1+\frac{1}{k}\right)|\vec{B}|^2$.

10. Consider the vectors $\hat{i}+\cos(\beta-\alpha)\hat{j}+\cos(\gamma-\alpha)\hat{k}$, $\cos(\alpha-\beta)\hat{i}+\hat{j}+\cos(\gamma-\beta)\hat{k}$ and $\cos(\alpha-\gamma)\hat{i}+\cos(\beta-\gamma)\hat{j}+a\hat{k}$, where α, β and γ are different angles. If these vectors are coplanar, show that a is independent of α, β and γ.

11. In a triangle PQR, S and T are points on QR and PR, respectively, such that $QS = 3SR$ and $PT = 4TR$. Let M be the point of intersection of PS and QT. Determine the ratio $QM : MT$ using the vector method.

12. A boat moves in still water with a velocity which is k times less than the river flow velocity. Find the angle to the stream direction at which the boat should be rowed to minimize drifting.

13. If D, E and F are three points on the sides BC, CA and AB, respectively, of a triangle ABC such that the lines AD, BE and CF are concurrent, then show that

$$\frac{BD}{CD}\cdot\frac{CE}{AE}\cdot\frac{AF}{BF} = -1$$

14. In a quadrilateral $PQRS$, $\overrightarrow{PQ} = \vec{a}$, $\overrightarrow{QR} = \vec{b}$, $\overrightarrow{SP} = \vec{a}-\vec{b}$, M is the midpoint of $\overrightarrow{QR}$ and X is a point on SM such that $SX = \frac{4}{5}SM$. Prove that P, X and R are collinear.

Objective Type

Solutions on page 1.65

Each question has four choices *a, b, c* and *d*, out of which *only one* answer is correct. Find the correct answer.

1. Four non-zero vectors will always be
   - **a.** linearly dependent
   - **b.** linearly independent
   - **c.** either a or b
   - **d.** none of these

2. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three units vectors such that $3\vec{a}+4\vec{b}+5\vec{c}=0$. Then which of the following statements is true?
   - **a.** $\vec{a}$ is parallel to $\vec{b}$
   - **b.** $\vec{a}$ is perpendicular to $\vec{b}$
   - **c.** $\vec{a}$ is neither parallel nor perpendicular to $\vec{b}$
   - **d.** none of these

3. Let ABC be a triangle, the position vectors of whose vertices are respectively $\hat{i}+2\hat{j}+4\hat{k}$, $-2\hat{i}+2\hat{j}+\hat{k}$ and $2\hat{i}+4\hat{j}-3\hat{k}$. Then ΔABC is
   a. isosceles b. equilateral c. right angled d. none of these

4. If $|\vec{a}+\vec{b}|<|\vec{a}-\vec{b}|$, then the angle between $\vec{a}$ and $\vec{b}$ can lie in the interval
   a. $(-\pi/2, \pi/2)$ b. $(0, \pi)$ c. $(\pi/2, 3\pi/2)$ d. $(0, 2\pi)$

5. A point O is the centre of a circle circumscribed about a triangle ABC. Then $\overrightarrow{OA}\sin 2A + \overrightarrow{OB}\sin 2B + \overrightarrow{OC}\sin 2C$ is equal to
   a. $(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC})\sin 2A$
   b. $3\overrightarrow{OG}$, where G is the centroid of triangle ABC
   c. $\vec{0}$
   d. none of these

6. If G is the centroid of a triangle ABC, then $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}$ is equal to
   a. $\vec{0}$ b. $3\overrightarrow{GA}$ c. $3\overrightarrow{GB}$ d. $3\overrightarrow{GC}$

7. If $\vec{a}$ is a non-zero vector of modulus a and m is a non-zero scalar, then $m\vec{a}$ is a unit vector if
   a. $m=\pm 1$ b. $a=|m|$ c. $a=1/|m|$ d. $a=1/m$

8. $ABCD$ a parallelogram, and A_1 and B_1 are the midpoints of sides BC and CD, respectively. If $\overrightarrow{AA_1}+\overrightarrow{AB_1}=\lambda\overrightarrow{AC}$, then λ is equal to
   a. $\frac{1}{2}$ b. 1 c. $\frac{3}{2}$ d. 2

9. The position vectors of the points P and Q with respect to the origin O are $\vec{a}=\hat{i}+3\hat{j}-2\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}-2\hat{k}$, respectively. If M is a point on PQ, such that OM is the bisector of POQ, then $\overrightarrow{OM}$ is
   a. $2(\hat{i}-\hat{j}+\hat{k})$ b. $2\hat{i}+\hat{j}-2\hat{k}$ c. $2(-\hat{i}+\hat{j}-\hat{k})$ d. $2(\hat{i}+\hat{j}+\hat{k})$

10. $ABCD$ is a quadrilateral. E is the point of intersection of the line joining the midpoints of the opposite sides. If O is any point and $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}=x\overrightarrow{OE}$, then x is equal to
   a. 3 b. 9 c. 7 d. 4

11. If vectors $\overrightarrow{AB}=-3\hat{i}+4\hat{k}$ and $\overrightarrow{AC}=5\hat{i}-2\hat{j}+4\hat{k}$ are the sides of a ΔABC, then the length of the median through A is
   a. $\sqrt{14}$ b. $\sqrt{18}$ c. $\sqrt{29}$ d. 5

12. A, B, C and D have position vectors $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ respectively, such that $\vec{a}-\vec{b}=2(\vec{d}-\vec{c})$. Then
   a. AB and CD bisect each other b. BD and AC bisect each other
   c. AB and CD trisect each other d. BD and AC trisect each other

13. If $\vec{a}$ and $\vec{b}$ are two unit vectors and θ is the angle between them, then the unit vector along the angular bisector of $\vec{a}$ and $\vec{b}$ will be given by
   a. $\frac{\vec{a}-\vec{b}}{2\cos(\theta/2)}$ b. $\frac{\vec{a}+\vec{b}}{2\cos(\theta/2)}$ c. $\frac{\vec{a}-\vec{b}}{\cos(\theta/2)}$ d. none of these

14. Let us define the length of a vector $a\hat{i}+b\hat{j}+c\hat{k}$ as $|a|+|b|+|c|$. This definition coincides with the usual definition of length of a vector $a\hat{i}+b\hat{j}+c\hat{k}$ if and only if

a. $a=b=c=0$ **b.** any two of a, b and c are zero

c. any one of a, b and c is zero **d.** $a+b+c=0$

15. Given three vectors $\vec{a}=6\hat{i}-3\hat{j}$, $\vec{b}=2\hat{i}-6\hat{j}$ and $\vec{c}=-2\hat{i}+21\hat{j}$ such that $\vec{\alpha}=\vec{a}+\vec{b}+\vec{c}$. Then the resolution of the vector $\vec{\alpha}$ into components with respect to $\vec{a}$ and $\vec{b}$ is given by

a. $3\vec{a}-2\vec{b}$ **b.** $3\vec{b}-2\vec{a}$ **c.** $2\vec{a}-3\vec{b}$ **d.** $\vec{a}-2\vec{b}$

16. If $\vec{\alpha}+\vec{\beta}+\vec{\gamma}=a\vec{\delta}$ and $\vec{\beta}+\vec{\gamma}+\vec{\delta}=b\vec{\alpha}$, $\vec{\alpha}$ and $\vec{\delta}$ are non-collinear, then $\vec{\alpha}+\vec{\beta}+\vec{\gamma}+\vec{\delta}$ equals

a. $a\vec{\alpha}$ **b.** $b\vec{\delta}$ **c.** 0 **d.** $(a+b)\vec{\gamma}$

17. In triangle ABC, $\angle A=30°$, H is the orthocentre and D is the midpoint of BC. Segment HD is produced to T such that $HD=DT$. The length AT is equal to

a. $2BC$ **b.** $3BC$ **c.** $\frac{4}{3}BC$ **d.** none of these

18. Let $\vec{r}_1, \vec{r}_2, \vec{r}_3, \cdots, \vec{r}_n$ be the position vectors of points $P_1, P_2, P_3, \cdots, P_n$ relative to the origin O. If the vector equation $a_1\vec{r}_1+a_2\vec{r}_2+\cdots+a_n\vec{r}_n=0$ holds, then a similar equation will also hold w.r.t. to any other origin provided

a. $a_1+a_2+\cdots+a_n=n$ **b.** $a_1+a_2+\cdots+a_n=1$

c. $a_1+a_2+\cdots+a_n=0$ **d.** $a_1=a_2=a_3=\cdots=a_n=0$

19. Given three non-zero, non-coplanar vectors $\vec{a}, \vec{b}$ and $\vec{c}$. $\vec{r}_1=p\vec{a}+q\vec{b}+\vec{c}$ and $\vec{r}_2=\vec{a}+p\vec{b}+q\vec{c}$. If the vectors $\vec{r}_1+2\vec{r}_2$ and $2\vec{r}_1+\vec{r}_2$ are collinear, then (p, q) is

a. (0, 0) **b.** (1, –1) **c.** (–1, 1) d. (1, 1)

20. If the vectors $\vec{a}$ and $\vec{b}$ are linearly independent satisfying $(\sqrt{3}\tan\theta+1)\vec{a}+(\sqrt{3}\sec\theta-2)\vec{b}=0$, then the most general values of θ are

a. $n\pi-\frac{\pi}{6}$, $n\in Z$ **b.** $2n\pi\pm\frac{11\pi}{6}$, $n\in Z$ **c.** $n\pi\pm\frac{\pi}{6}$, $n\in Z$ **d.** $2n\pi+\frac{11\pi}{6}$, $n\in Z$

21. In a trapezium, vector $\overrightarrow{BC}=\alpha\overrightarrow{AD}$. We will then find that $\vec{p}=\overrightarrow{AC}+\overrightarrow{BD}$ is collinear with $\overrightarrow{AD}$. If $\vec{p}=\mu\overrightarrow{AD}$, then which of the following is true?

a. $\mu=\alpha+2$ **b.** $\mu+\alpha=1$ **c.** $\alpha=\mu+1$ **d.** $\mu=\alpha+1$

22. Vectors $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$; $\vec{b}=2\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}+4\hat{k}$ are so placed that the end point of one vector is the starting point of the next vector. Then the vectors are

a. not coplanar **b.** coplanar but cannot form a triangle

c. coplanar and form a triangle **d.** coplanar and can form a right-angled triangle

23. Vectors $\vec{a} = -4\hat{i} + 3\hat{k}$; $\vec{b} = 14\hat{i} + 2\hat{j} - 5\hat{k}$ are laid off from one point. Vector $\vec{d}$, which is being laid off from the same point dividing the angle between vectors $\vec{a}$ and $\vec{b}$ in equal halves and having the magnitude $\sqrt{6}$, is

a. $\hat{i} + \hat{j} + 2\hat{k}$ **b.** $\hat{i} - \hat{j} + 2\hat{k}$ **c.** $\hat{i} + \hat{j} - 2\hat{k}$ **d.** $2\hat{i} - \hat{j} - 2\hat{k}$

24. If $\hat{i} - 3\hat{j} + 5\hat{k}$ bisects the angle between $\hat{a}$ and $-\hat{i} + 2\hat{j} + 2\hat{k}$, where $\hat{a}$ is a unit vector, then

a. $\hat{a} = \frac{1}{105}(41\hat{i} + 88\hat{j} - 40\hat{k})$

b. $\hat{a} = \frac{1}{105}(41\hat{i} + 88\hat{j} + 40\hat{k})$

c. $\hat{a} = \frac{1}{105}(-41\hat{i} + 88\hat{j} - 40\hat{k})$

d. $\hat{a} = \frac{1}{105}(41\hat{i} - 88\hat{j} - 40\hat{k})$

25. If $4\hat{i} + 7\hat{j} + 8\hat{k}$, $2\hat{i} + 3\hat{j} + 4\hat{k}$ and $2\hat{i} + 5\hat{j} + 7\hat{k}$ are the position vectors of the vertices *A*, *B* and *C*, respectively, of triangle *ABC*, the position vector of the pôint where the bisector of angle *A* meets *BC*, is

a. $\frac{2}{3}(-6\hat{i} - 8\hat{j} - 6\hat{k})$ **b.** $\frac{2}{3}(6\hat{i} + 8\hat{j} + 6\hat{k})$ **c.** $\frac{1}{3}(6\hat{i} + 13\hat{j} + 18\hat{k})$ **d.** $\frac{1}{3}(5\hat{j} + 12\hat{k})$

26. If $\vec{b}$ is a vector whose initial point divides the join of $5\hat{i}$ and $5\hat{j}$ in the ratio $k : 1$ and whose terminal point is the origin and $|\vec{b}| \le \sqrt{37}$, then *k* lies in the interval

a. $[-6, -1/6]$

b. $(-\infty, -6] \cup [-1/6, \infty)$

c. $[0, 6]$

d. none of these

27. Find the value of λ so that the points *P*, *Q*, *R* and *S* on the sides *OA*, *OB*, *OC* and *AB*, respectively, of a regular tetrahedron *OABC* are coplanar. It is given that $\frac{OP}{OA} = \frac{1}{3}$, $\frac{OQ}{OB} = \frac{1}{2}$, $\frac{OR}{OC} = \frac{1}{3}$ and $\frac{OS}{AB} = \lambda$.

a. $\lambda = \frac{1}{2}$ **b.** $\lambda = -1$ **c.** $\lambda = 0$ **d.** for no value of λ

28. '*I*' is the incentre of triangle *ABC* whose corresponding sides are *a*, *b*, *c*, respectively. $a\overrightarrow{IA} + b\overrightarrow{IB} + c\overrightarrow{IC}$ is always equal to

a. $\vec{0}$

b. $(a+b+c)\overrightarrow{BC}$

c. $(\vec{a} + \vec{b} + \vec{c})\overrightarrow{AC}$

d. $(a+b+c)\overrightarrow{AB}$

29. Let $x^2 + 3y^2 = 3$ be the equation of an ellipse in the *x*-*y* plane. *A* and *B* are two points whose position vectors are $-\sqrt{3}\,\hat{i}$ and $-\sqrt{3}\,\hat{i} + 2\hat{k}$. Then the position vector of a point *P* on the ellipse such that $\angle APB = \pi/4$ is

a. $\pm\hat{j}$ **b.** $\pm(\hat{i} + \hat{j})$ **c.** $\pm\hat{i}$ **d.** none of these

30. Locus of the point *P*, for which $\overrightarrow{OP}$ represents a vector with direction cosine $\cos\alpha = \frac{1}{2}$ ('*O*' is the origin) is

a. a circle parallel to the y-z plane with centre on the x-axis
b. a cone concentric with the positive x-axis having vertex at the origin and the slant height equal to the magnitude of the vector
c. a ray emanating from the origin and making an angle of 60° with the x-axis
d. a disc parallel to the y-z plane with centre on the x-axis and radius equal to $|\overrightarrow{OP}| \sin 60°$

31. If $\vec{x}$ and $\vec{y}$ are two non-collinear vectors and ABC is a triangle with side lengths a, b and c satisfying $(20a-15b)\vec{x} + (15b-12c)\vec{y} + (12c-20a)(\vec{x}\times\vec{y}) = \vec{0}$, then triangle ABC is

a. an acute-angled triangle
b. an obtuse-angled triangle
c. a right-angled triangle
d. an isosceles triangle

32. A uni-modular tangent vector on the curve $x = t^2 + 2$, $y = 4t - 5$, $z = 2t^2 - 6t$ at $t = 2$ is

a. $\frac{1}{3}(2\hat{i} + 2\hat{j} + \hat{k})$
b. $\frac{1}{3}(\hat{i} - \hat{j} - \hat{k})$
c. $\frac{1}{6}(2\hat{i} + \hat{j} + \hat{k})$
d. $\frac{2}{3}(\hat{i} + \hat{j} + \hat{k})$

33. If $\vec{x}$ and $\vec{y}$ are two non-collinear vectors and a, b and c represent the sides of a ΔABC satisfying $(a-b)\vec{x} + (b-c)\vec{y} + (c-a)(\vec{x}\times\vec{y}) = 0$, then ΔABC is (where $\vec{x}\times\vec{y}$ is perpendicular to the plane of $\vec{x}$ and $\vec{y}$)

a. an acute-angled triangle
b. an obtuse-angled triangle
c. a right-angled triangle
d. a scalene triangle

34. $\vec{A}$ is a vector with direction cosines $\cos\alpha$, $\cos\beta$ and $\cos\gamma$. Assuming the y-z plane as a mirror, the direction cosines of the reflected image of $\vec{A}$ in the y-z plane are

a. $\cos\alpha, \cos\beta, \cos\gamma$
b. $\cos\alpha, -\cos\beta, \cos\gamma$
c. $-\cos\alpha, \cos\beta, \cos\gamma$
d. $-\cos\alpha, -\cos\beta, -\cos\gamma$

Multiple Correct Answers Type

Solutions on page 1.74

Each question has four choices *a*, *b*, *c*, and d, out of which *one or more* are correct.

1. The vectors $x\hat{i} + (x+1)\hat{j} + (x+2)\hat{k}$, $(x+3)\hat{i} + (x+4)\hat{j} + (x+5)\hat{k}$ and $(x+6)\hat{i} + (x+7)\hat{j} + (x+8)\hat{k}$ are coplanar if x is equal to

a. 1
b. −3
c. 4
d. 0

2. The sides of a parallelogram are $2\hat{i}+4\hat{j}-5\hat{k}$ and $\hat{i}+2\hat{j}+3\hat{k}$. The unit vector parallel to one of the diagonals is

a. $\frac{1}{7}(3\hat{i}+6\hat{j}-2\hat{k})$
b. $\frac{1}{7}(3\hat{i}-6\hat{j}-2\hat{k})$
c. $\frac{1}{\sqrt{69}}(\hat{i}+2\hat{j}+8\hat{k})$
d. $\frac{1}{\sqrt{69}}(-\hat{i}-2\hat{j}+8\hat{k})$

3. A vector $\vec{a}$ has the components $2p$ and 1 w.r.t. a rectangular Cartesian system. This system is rotated through a certain angle about the origin in the counterclockwise sense. If, with respect to a new system, $\vec{a}$ has components $(p + 1)$ and 1, then p is equal to

a. -1 b. $-1/3$ c. 1   d. 2

4. If points $\hat{i} + \hat{j}, \hat{i} - \hat{j}$ and $p\hat{i} + q\hat{j} + r\hat{k}$ are collinear, then

a. $p = 1$ b. $r = 0$ c. $q \in R$ d. $q \neq 1$

5. If $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar vectors and λ is a real number, then the vectors $\vec{a} + 2\vec{b} + 3\vec{c}, \lambda\vec{b} + \mu\vec{c}$ and $(2\lambda - 1)\vec{c}$ are coplanar when

a. $\mu \in R$ b. $\lambda = \frac{1}{2}$ c. $\lambda = 0$ d. no value of λ

6. If the resultant of three forces $\vec{F_1} = p\hat{i} + 3\hat{j} - \hat{k}, \vec{F_2} = 6\hat{i} - \hat{k}$ and $\vec{F_3} = -5\hat{i} + \hat{j} + 2\hat{k}$ acting on a particle has a magnitude equal to 5 units, then the value of p is

a. -6 b. -4 c. 2   d. 4

7. If the vectors $\hat{i} - \hat{j}, \hat{j} + \hat{k}$ and $\vec{a}$ form a triangle, then $\vec{a}$ may be

a. $-\hat{i} - \hat{k}$ b. $\hat{i} - 2\hat{j} - \hat{k}$ c. $2\hat{i} + \hat{j} + \hat{k}$ d. $\hat{i} + \hat{k}$

8. The vector $\hat{i} + x\hat{j} + 3\hat{k}$ is rotated through an angle θ and doubled in magnitude. It now becomes $4\hat{i} + (4x - 2)\hat{j} + 2\hat{k}$. The values of x are

a. 1   b. $-2/3$ c. 2   d. $4/3$

9. $\vec{a}, \vec{b}$ and $\vec{c}$ are three coplanar unit vectors such that $\vec{a} + \vec{b} + \vec{c} = 0$. If three vectors $\vec{p}, \vec{q}$ and $\vec{r}$ are parallel to $\vec{a}, \vec{b}$ and $\vec{c}$, respectively, and have integral but different magnitudes, then among the following options, $|\vec{p} + \vec{q} + \vec{r}|$ can take a value equal to

a. 1   b. 0   c. $\sqrt{3}$ d. 2

10. If non-zero vectors $\vec{a}$ and $\vec{b}$ are equally inclined to coplanar vector $\vec{c}$, then $\vec{c}$ can be

a. $\frac{|\vec{a}|}{|\vec{a}| + 2|\vec{b}|}\vec{a} + \frac{|\vec{b}|}{|\vec{a}| + |\vec{b}|}\vec{b}$ b. $\frac{|\vec{b}|}{|\vec{a}| + |\vec{b}|}\vec{a} + \frac{|\vec{a}|}{|\vec{a}| + |\vec{b}|}\vec{b}$

c. $\frac{|\vec{a}|}{|\vec{a}| + 2|\vec{b}|}\vec{a} + \frac{|\vec{b}|}{|\vec{a}| + 2|\vec{b}|}\vec{b}$ d. $\frac{|\vec{b}|}{2|\vec{a}| + |\vec{b}|}\vec{a} + \frac{|\vec{a}|}{2|\vec{a}| + |\vec{b}|}\vec{b}$

11. If $A(-4, 0, 3)$ and $B(14, 2, -5)$, then which one of the following points lie on the bisector of the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ (O is the origin of reference)?

a. (2, 2, 4)   b. (2, 11, 5)   c. (–3, –3, –6)   d. (1, 1, 2)

12. In a four-dimensional space where unit vectors along the axes are $\hat{i}, \hat{j}, \hat{k}$ and $\hat{l}$, and $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ are four non-zero vectors such that no vector can be expressed as linear combination of others and $(\lambda - 1)(\vec{a}_1 - \vec{a}_2) + \mu(\vec{a}_2 + \vec{a}_3) + \gamma(\vec{a}_3 + \vec{a}_4 - 2\vec{a}_2) + \vec{a}_3 + \delta\vec{a}_4 = \vec{0}$, then

a. $\lambda = 1$ **b.** $\mu = -2/3$ **c.** $\gamma = 2/3$ **d.** $\delta = 1/3$

13. Let ABC be a triangle, the position vectors of whose vertices are $7\hat{j} + 10\hat{k}$, $-\hat{i} + 6\hat{j} + 6\hat{k}$ and $-4\hat{i} + 9\hat{j} + 6\hat{k}$. Then ΔABC is

a. isosceles **b.** equilateral **c.** right angled **d.** none of these

Reasoning Type

Solutions on page 1.77

Each question has four choices *a, b, c,* and *d*, out of which *only one* is correct. Each question contains Statement 1 and Statement 2.

a. Both the statements are true, and Statement 2 is the correct explanation for Statement 1.
b. Both the statements are true, but Statement 2 is not the correct explanation for Statement 1.
c. Statement 1 is true and Statement 2 is false.
d. Statment 1 is false and Statement 2 is true.

1. A vector has components p and 1 with respect to a rectangular Cartesian system. The axes are rotated through an angle α about the origin in the anticlockwise sense.
Statement 1: If the vector has component $p + 2$ and 1 with respect to the new system, then $p = -1$
Statement 2: Magnitude of the original vector and the new vector remains the same.

2. **Statement 1:** If three points P, Q and R have position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, respectively, and $2\vec{a} + 3\vec{b} - 5\vec{c} = 0$, then the points P, Q and R must be collinear.
Statement 2: If for three points A, B and C; $\overrightarrow{AB} = \lambda\,\overrightarrow{AC}$, then points A, B and C must be collinear.

3. **Statement 1:** If $\vec{u}$ and $\vec{v}$ are unit vectors inclined at an angle α and $\vec{x}$ is a unit vector bisecting the angle between them, then $\vec{x} = (\vec{u} + \vec{v})/(2\sin(\alpha/2))$.
Statement 2: If ΔABC is an isosceles triangle with $AB = AC = 1$, then the vector representing the bisector of angle A is given by $\overrightarrow{AD} = (\overrightarrow{AB} + \overrightarrow{AC})/2$.

4. **Statement 1:** If $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the direction cosines of any line segment, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.
Statement 2: If $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the direction cosines of a line segment, $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$.

5. **Statement 1:** The direction cosines of one of the angular bisectors of two intersecting lines having direction cosines as l_1, m_1, n_1 and l_2, m_2, n_2 are proportional to $l_1 + l_2$, $m_1 + m_2$, $n_1 + n_2$.
Statement 2: The angle between the two intersecting lines having direction cosines as l_1, m_1, n_1 and l_2, m_2, n_2 is given by $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$.

6. **Statement 1:** In ΔABC, $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=0$

Statement 2: If $\overrightarrow{OA}=\vec{a}$, $\overrightarrow{OB}=\vec{b}$, then $\overrightarrow{AB}=\vec{a}+\vec{b}$

7. **Statement 1:** $\vec{a}=3\vec{i}+p\vec{j}+3\vec{k}$ and $\vec{b}=2\vec{i}+3\vec{j}+q\vec{k}$ are parallel vectors if $p=9/2$ and $q=2$.

Statement 2: If $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ and $\vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$ are parallel, $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$.

8. **Statement 1:** If $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$, then $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

Statement 2 : If the diagonals of a parallelogram are equal in magnitude, then the parallelogram is a rectangle.

9. **Statement 1:** Let $A(\vec{a})$, $B(\vec{b})$ and $C(\vec{c})$ be three points such that $\vec{a}=2\hat{i}+\hat{k}$, $\vec{b}=3\hat{i}-\hat{j}+3\hat{k}$ and $\vec{c}=-\hat{i}+7\hat{j}-5\hat{k}$. Then $OABC$ is a tetrahedron.

Statement 2: Let $A(\vec{a})$, $B(\vec{b})$ and $C(\vec{c})$ be three points such that vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar. Then $OABC$ is a tetrahedron, where O is the origin.

10. **Statement 1:** Let $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be the position vectors of four points A, B, C and D and $3\vec{a}-2\vec{b}+5\vec{c}-6\vec{d}=\vec{0}$. Then points A, B, C and D are coplanar.

Statement 2: Three non-zero, linearly dependent coinitial vectors ($\overrightarrow{PQ}, \overrightarrow{PR}$ and $\overrightarrow{PS}$) are coplanar. Then $\overrightarrow{PQ}=\lambda\overrightarrow{PR}+\mu\overrightarrow{PS}$, where λ and μ are scalars.

11. **Statement 1:** If $|\vec{a}|=3$, $|\vec{b}|=4$ and $|\vec{a}+\vec{b}|=5$, then $|\vec{a}-\vec{b}|=5$.

Statement 2: The length of the diagonals of a rectangle is the same.

Linked Comprehension Type

Solutions on page 1.79

Based on each paragraph, some multiple choice questions have to be answered. Each question has four choices *a, b, c,* and *d,* out of which *only one* is correct.

For Problems 1–3

ABCD is a parallelogram. *L* is a point on *BC* which divides *BC* in the ratio 1 : 2. *AL* intersects *BD* at *P*. *M* is a point on *DC* which divides *DC* in the ratio 1 : 2 and *AM* intersects *BD* in *Q*.

1. Point *P* divides *AL* in the ratio

   a. 1 : 2 b. 1 : 3 c. 3 : 1 d. 2 : 1

2. Point *Q* divides *DB* in the ratio

   a. 1 : 2 b. 1 : 3 c. 3 : 1 d. 2 : 1

3. *PQ* : *DB* is equal to

   a. 2/3 b. 1/3 c. 1/2 d. 3/4

For Problems 4–5

Let *OABCD* be a pentagon in which the sides *OA* and *CB* are parallel and the sides *OD* and *AB* are parallel. Also *OA* : *CB* = 2 : 1 and *OD* : *AB* = 1 : 3.

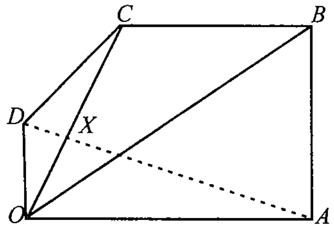


Fig. 1.40

4. The ratio $\dfrac{OX}{XC}$ is

a. 3/4 **b.** 1/3 **c.** 2/5 **d.** 1/2

5. The ratio $\dfrac{AX}{XD}$ is

a. 5/2 **b.** 6 **c.** 7/3 **d.** 4

For Problems 6–7

Consider the regular hexagon *ABCDEF* with centre at *O* (origin).

6. $\overrightarrow{AD} + \overrightarrow{EB} + \overrightarrow{FC}$ is equal to

a. $2\overrightarrow{AB}$ **b.** $3\overrightarrow{AB}$ **c.** $4\overrightarrow{AB}$ **d.** none of these

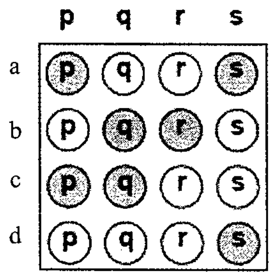
7. Five forces $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}, \overrightarrow{AE}, \overrightarrow{AF}$ act at the vertex *A* of a regular hexagon *ABCDEF*. Then their resultant is

a. $6\overrightarrow{AO}$ **b.** $2\overrightarrow{AO}$ **c.** $4\overrightarrow{AO}$ **d.** $6\overrightarrow{AO}$

Matrix-Match Type

Solutions on page 1.82

Each question contains statements given in two columns which have to be matched. Statements (a, b, c, d) in Column I have to be matched with statements (p, q, r, s) in Column II. If the correct matches are a → p, s; b → q, r; c → p, q and d → s, then the correctly bubbled 4 × 4 matrix should be as follows:



1. Refer to the following diagram:

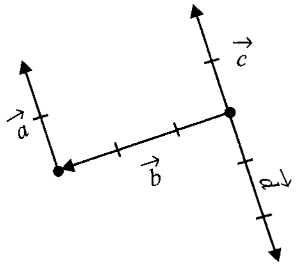


Fig. 1.41

Column I	Column II
a. Collinear vectors	p. $\vec{a}$
b. Coinitial vectors	q. $\vec{b}$
c. Equal vectors	r. $\vec{c}$
d. Unlike vectors (same initial point)	s. $\vec{d}$

2. $\vec{a}$ and $\vec{b}$ form the consecutive sides of a regular hexagon $ABCDEF$.

Column I	Column II
a. If $\overrightarrow{CD} = x\vec{a} + y\vec{b}$, then	p. $x = -2$
b. If $\overrightarrow{CE} = x\vec{a} + y\vec{b}$, then	q. $x = -1$
c. If $\overrightarrow{AE} = x\vec{a} + y\vec{b}$, then	r. $y = 1$
d. $\overrightarrow{AD} = -x\vec{b}$, then	s. $y = 2$

Integer Answer Type

Solutions on page 1.83

1. Let ABC be a triangle whose centroid is G, orhtocentre is H and circumcentre is the origin 'O'. If D is any point in the plane of the triangle such that no three of O, A, C and D are collinear satisfying the relation $\vec{AD} + \vec{BD} + \vec{CH} + 3\vec{HG} = \lambda\,\vec{HD}$, then what is the value of the scalar 'λ'?
2. If the resultant of three forces $\vec{F_1} = p\hat{i} + 3\hat{j} - \hat{k}$, $\vec{F_2} = -5\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{F_3} = 6\hat{i} - \hat{k}$ acting on a particle has a magnitude equal to 5 units, then what is difference in the values of p?
3. Let $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors such that $\vec{a} + \vec{b} - \vec{c} = 0$. If the area of triangle formed by vectors $\vec{a}$ and $\vec{b}$ is A, then what is the value of $4A^2$?

4. Find the least positive integral value of x for which the angle between vectors $\vec{a} = x\hat{i} - 3\hat{j} - \hat{k}$ and $\vec{b} = 2x\hat{i} + x\hat{j} - \hat{k}$ is acute.
5. Vectors along the adjacent sides of parallelogram are $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} + \hat{k}$. Find the length of the longer diagonal of the parallelogram.
6. If vectors $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \lambda\hat{i} + \hat{j} + 2\hat{k}$ are coplanar, then find the value of $(\lambda - 4)$.

Archives *Solutions on page 1.84*

Subjective Type

1. Find all values of λ such that $x, y, z \neq (0, 0, 0)$ and $(\hat{i}+\hat{j}+3\hat{k})x+(3\hat{i}-3\hat{j}+\hat{k})y+(-4\hat{i}+5\hat{j})z = \lambda(x\hat{i}+y\hat{j}+z\hat{k})$, where, $\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors along the coordinate axes. **(IIT-JEE, 1998)**

2. A vector has components A_1, A_2 and A_3 in a right-handed rectangular Cartesian coordinate system $OXYZ$. The coordinate system is rotated about the x-axis through an angle $\pi/2$. Find the components of A in the new coordinate system in terms of A_1, A_2 and A_3. **(IIT-JEE, 1983)**

3. The position vectors of the point A, B, C and D are $3\hat{i}-2\hat{j}-\hat{k}$, $2\hat{i}+3\hat{j}-4\hat{k}$, $-\hat{i}+\hat{j}+2\hat{k}$ and $4\hat{i}+5\hat{j}+\lambda\hat{k}$, respectively. If the points A, B, C and D lie on a plane, find the value of λ. **(IIT-JEE, 1986)**

4. Let $OACB$ be a parallelogram with O at the origin and OC a diagonal. Let D be the midpoint of OA. Using vector methods prove that BD and CO intersect in the same ratio. Determine this ratio. **(IIT-JEE, 1988)**

5. In a triangle ABC, D and E are points on BC and AC, respectively, such that $BD = 2DC$ and $AE = 3EC$. Let P be the point of intersection of AD and BE. Find BP/PE using the vector method. **(IIT-JEE,1993)**

6. Prove, by vector method or otherwise, that the point of intersection of the diagonals of a trapezium lies on the line passing through the midpoint of the parallel sides (you may assume that the trapezium is not a parallelogram.) **(IIT-JEE, 1998)**

7. Show, by vector method, that the angular bisectors of a triangle are concurrent and find an expression for the position vector of the point of concurrency in terms of the position vectors of the vertices. **(IIT-JEE, 2001)**

8. Let $\vec{A}(t) = f_1(t)\hat{i} + f_2(t)\hat{j}$ and $\vec{B}(t) = g_1(t)\hat{i} + g_2(t)\hat{j}$, $t \in [0,1]$, where f_1, f_2, g_1, g_2 are continuous functions. If $\vec{A}(t)$ and $\vec{B}(t)$ are non-zero vectors for all and $\vec{A}(0) = 2\hat{i}+3\hat{j}$, $\vec{A}(1) = 6\hat{i}+2\hat{j}$, $\vec{B}(0) = 3\hat{i}+2\hat{j}$ and $\vec{B}(1) = 2\hat{i}+6\hat{j}$, then show that $\vec{A}(t)$ and $\vec{B}(t)$ are parallel for some t. **(IIT-JEE, 2001)**

9. In a triangle OAB, E is the midpoint of BO and D is a point on AB such that $AD : DB = 2 : 1$. If OD and AE intersect at P, determine the ratio $OP : PD$ using the vector method. **(IIT-JEE, 1989)**

Objective Type

Fill in the blanks

1. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and the vectors $\vec{A} = (1, a, a^2)$, $\vec{B} = (1, b, b^2)$, $\vec{C} = (1, c, c^2)$ are non-coplanar, then the product abc = ___________. **(IIT-JEE,1985)**

2. If the vectors $a\hat{i}+\hat{j}+\hat{k}$, $\hat{i}+b\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+c\hat{k}$ $(a, b, c \neq 1)$ are coplanar, then the value of $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}$ = ___________. **(IIT-JEE,1987)**

True or false

1. The points with position vectors $\vec{a}+\vec{b}$, $\vec{a}-\vec{b}$ and $\vec{a}+k\vec{b}$ are collinear for all real values of k. **(IIT-JEE, 1984)**

Multiple choice questions with one correct answer

1. The points with position vectors $60\hat{i}+3\hat{j}$, $40\hat{i}-8\hat{j}$, $a\hat{i}-52\hat{j}$ are collinear if
   a. $a = -40$ b. $a = 40$
   c. $a = 20$ d. none of these **(IIT-JEE, 1983)**

2. Let a, b and c be distinct non-negative numbers. If vectors $a\hat{i}+a\hat{j}+c\hat{k}$, $\hat{i}+\hat{k}$ and $c\hat{i}+c\hat{j}+b\hat{k}$ are coplanar, then c is
   a. the arithmetic mean of a and b b. the geometric mean of a and b
   c. the harmonic mean of a and b d. equal to zero **(IIT-JEE, 1993)**

3. Let $\vec{a}=\vec{i}-\vec{k}$, $\vec{b}=x\vec{i}+\vec{j}+(1-x)\vec{k}$ and $\vec{c}=y\vec{i}+x\vec{j}+(1+x-y)\vec{k}$. Then $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar for
   a. some values of x b. some values of y
   c. no values of x and y d. for all values of x and y **(IIT-JEE, 2000)**

4. Let α, β and γ be distinct and real numbers. The points with position vectors $\alpha i+\beta\hat{j}+\gamma\hat{k}$, $\beta\hat{i}+\gamma\hat{j}+\alpha\hat{k}$ and $\gamma\hat{i}+\alpha\hat{j}+\beta\hat{k}$
   a. are collinear b. form an equilateral triangle
   c. form a scalene triangle d. form a right-angled triangle
   (IIT-JEE, 1994)

5. The number of distinct real values of λ, for which the vectors $-\lambda^2\hat{i}+\hat{j}+\hat{k}$, $\hat{i}-\lambda^2\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}-\lambda^2\hat{k}$ are coplanar is
   a. zero b. one c. two d. three
   (IIT-JEE, 2007)

6. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=4\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{c}=\hat{i}+\alpha\hat{j}+\beta\hat{k}$ are linearly dependent vectors and $|\vec{c}| = \sqrt{3}$, then
   a. $a = 1, b = -1$ b. $a = 1, b = \pm 1$ c. $\alpha = -1, \beta = \pm 1$ d. $\alpha = \pm 1, \beta = 1$
   (IIT-JEE, 1998)

ANSWERS AND SOLUTIONS

Subjective Type

1. Since $|\overrightarrow{AB}| = |\overrightarrow{BC}| = |\overrightarrow{CA}|$, the incentre is same as the circumcentre, and hence IA is perpendicular to BC. Therefore, $\vec{r}$ is parallel to BC.

$\vec{r} = \lambda(\hat{i} - \hat{j})$

Hence, unit vector $\vec{r} = \pm\frac{1}{\sqrt{2}}(\hat{i} - \hat{j})$

2. We take the unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ parallel to the east, north and vertically upwards in the direction of pole, respectively. Then the velocity vectors of the current, ship and the sailor are, respectively, $\hat{i}, 1.25\,\hat{j}$ and $0.5\,\hat{k}$. Velocity $\vec{v}$ of the sailor in space is the resultant of these vectors.

Hence $\vec{v} = \hat{i} + 1.25\,\hat{j} + 0.5\,\hat{k}$

Then $|\vec{v}| = \sqrt{1 + (1.25)^2 + (0\cdot5)^2}$

$= \sqrt{1 + 1.5625 + .25}$

$= \sqrt{2.8125} = 1.677$ m/s

3. (i) Put $n = 2$ in $\overrightarrow{OP}_{n-1} + \overrightarrow{OP}_{n+1} = \frac{3}{2}\overrightarrow{OP}_n$

$\overrightarrow{OP}_3 = \frac{3}{2}\overrightarrow{OP}_2 - \overrightarrow{OP}_1$ (i)

$$\left.\begin{aligned}\overrightarrow{OP}_1 &= a\,\hat{i} + \frac{1}{a}\hat{j}\\ \overrightarrow{OP}_2 &= b\,\hat{i} + \frac{1}{b}\hat{j}\end{aligned}\right\} ab \neq 0$$

$\therefore\ \overrightarrow{OP}_3 = \frac{3}{2}\left(b\hat{i} + \frac{1}{b}\hat{j}\right) - \left(a\hat{i} + \frac{1}{a}\hat{j}\right) = \left(\frac{3b}{2} - a\right)\hat{i} + \left(\frac{3}{2b} - \frac{1}{a}\right)\hat{j}$

If P_3 lies on $xy = 1$

$\left(\frac{3b}{2} - a\right)\left(\frac{3}{2b} - \frac{1}{a}\right) = 1$

$\Rightarrow (3b - 2a)(3a - 2b) = 4ab$

$\Rightarrow 9ab - 6b^2 - 6a^2 + 4ab = 4ab$

$\Rightarrow 2a^2 - 3ab + 2b^2 = 0$

which is not possible as Discriminant < 0 ($a = 0$ and $b = 0$ not possible)

(ii) $\overrightarrow{OP} = \cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}$ and $\overrightarrow{OP}_2 = \cos\beta\,\hat{i} + \sin\beta\,\hat{j}$

$\therefore\ \overrightarrow{OP}_3 = \frac{3}{2}(\cos\beta\,\hat{i} + \sin\beta\,\hat{j}) - (\cos\alpha\,\hat{i} + \sin\alpha\,\hat{j})$

$= \left(\frac{3}{2}\cos\beta - \cos\alpha\right)\hat{i} + \left(\frac{3}{2}\sin\beta - \sin\alpha\right)\hat{j}$

Since P_3 lies on $x^2 + y^2 = 1$

$\Rightarrow \left(\frac{3}{2}\cos\beta - \cos\alpha\right)^2 + \left(\frac{3}{2}\sin\beta - \sin\alpha\right)^2 = 1$

$\Rightarrow \frac{9}{4} + 1 - 3\,(\cos\beta\cos\alpha + \sin\beta\sin\alpha) = 1$

$\Rightarrow \frac{9}{4} - 3\cos(\beta - \alpha) = 0 \Rightarrow \cos(\beta - \alpha) = \frac{3}{4}$ (ii)

Put $n = 3$ in the given relation.

$\overrightarrow{OP_2} + \overrightarrow{OP_4} = \frac{3}{2}\overrightarrow{OP_3}$; $\overrightarrow{OP_4} = \frac{3}{2}\overrightarrow{OP_3} - \overrightarrow{OP_2}$

$\Rightarrow \overrightarrow{OP_4} = \frac{3}{2}\left(\frac{3}{2}\overrightarrow{OP_2} - \overrightarrow{OP_1}\right) - \overrightarrow{OP_2} = \frac{5}{4}\overrightarrow{OP_2} - \frac{3}{2}\overrightarrow{OP_1}$

$\Rightarrow \overrightarrow{OP_4} = \frac{5}{4}(\cos\beta\,\hat{i} + \sin\beta\,\hat{j}) - \frac{3}{2}(\cos\alpha\,\hat{i} + \sin\alpha\,\hat{j})$, which lies on $x^2 + y^2 = 1$

4.

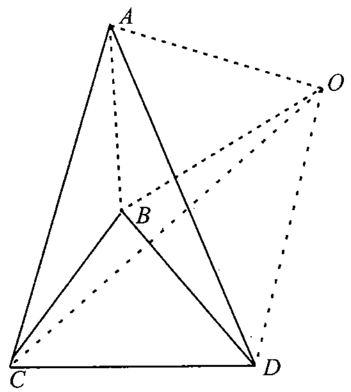


Fig. 1.42

Here $ABCD$ is a tetrahedron. Let O be the origin and the P.V. of A, B, C and D be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, respectively. We know that four linearly dependent vectors can be expressed as

$x\vec{a} + y\vec{b} + z\vec{c} + t\vec{d} = 0$ (where x, y, z and t are scalars)

or $y\vec{b} + z\vec{c} + t\vec{d} = -x\vec{a}$

$\Rightarrow \frac{y\vec{b} + z\vec{c} + t\vec{d}}{y + z + t} = -\frac{x\vec{a}}{y + z + t}$

where L.H.S. is P.V. of a point in the plane BCD and R.H.S. is a point on $\overrightarrow{AO}$

Therefore, there must be a point common to both the plane and the straight line. That is

$$\overrightarrow{OP} = \frac{-x\vec{a}}{y+z+t}$$

But, $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = -\frac{x\vec{a}}{y+z+t} - \vec{a} = -\left(\frac{x+y+z+t}{y+z+t}\right)\vec{a}$

$$\overrightarrow{OP} = \frac{x}{y+z+t}\left(\frac{y+z+t}{x+y+z+t}\right)\overrightarrow{AP}$$

$$\overrightarrow{OP} = \left(\frac{x}{x+y+z+t}\right)\overrightarrow{AP}$$

$$\Rightarrow \frac{OP}{AP} = \frac{x}{x+y+z+t}$$

Similarly, $\frac{OQ}{BQ} = \frac{y}{x+y+z+t}$

$$\frac{OR}{CR} = \frac{z}{x+y+z+t} \text{ and } \frac{OS}{DS} = \frac{t}{x+y+z+t}$$

$$\Rightarrow \frac{OP}{AP} + \frac{OQ}{BQ} + \frac{OR}{CR} + \frac{OS}{DS} = 1$$

5. Let the centre of the base be (0). Therefore,

$|\overrightarrow{AB}| = 2$

$\Delta OAB = \frac{1}{4} \times 4 \times \sqrt{3} = \sqrt{3}$

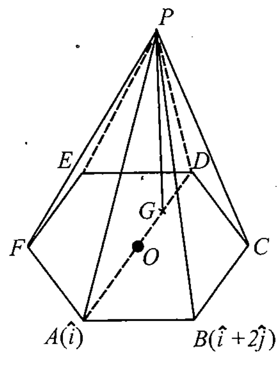


Fig. 1.43

Base area = $6\sqrt{3}$ sq. units

Let height of the pyramid be h. Therefore,

$$\frac{1}{3}\times 6\sqrt{3}h = 6\sqrt{3} \Rightarrow h = 3 \text{ units}$$

It is given that $|\overrightarrow{AP}| = 5$. Therefore,

$AG = \sqrt{25-9} = 4$ units

$\Rightarrow |\overrightarrow{AG}| = 4$ units

Now $|\overrightarrow{AG}|$ and $|\overrightarrow{AO}|$ are collinear. Therefore,

$$\overrightarrow{AG} = \lambda \overrightarrow{AO} \Rightarrow |\overrightarrow{AG}| = |\lambda||\overrightarrow{AO}| \Rightarrow 2|\lambda| = 4 \Rightarrow |\lambda| = 2$$

$$\Rightarrow \overrightarrow{AG} = \pm 2(\hat{i}+\hat{j}+\sqrt{3}\hat{k}) \Rightarrow \overrightarrow{OG} = \pm 2(\hat{i}+\hat{j}+\sqrt{3}\hat{k})+\hat{i}$$

$$\overrightarrow{OG} = -(\hat{i}+2\hat{j}+2\sqrt{3}\hat{k}),\ 3\hat{i}+2\hat{j}+2\sqrt{3}\hat{k}$$

6. Let $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{AD} = \vec{b}$; then $\overrightarrow{AC} = \vec{a}+\vec{b}$.

Given $\overrightarrow{AB_1} = \lambda_1 \vec{a}$, $\overrightarrow{AD_1} = \lambda_2 \vec{b}$, $\overrightarrow{AC_1} = \lambda_3(\vec{a}+\vec{b})$

$$\overrightarrow{B_1D_1} = \overrightarrow{AD_1} - \overrightarrow{AB_1} = \lambda_2 \vec{b} - \lambda_1 \vec{a}$$

Since vectors $\overrightarrow{D_1C_1}$ and $\overrightarrow{B_1D_1}$ are collinear, we have

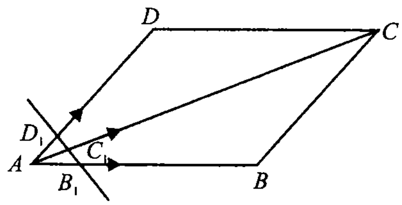


Fig. 1.44

$$\overrightarrow{D_1C_1} = k\overrightarrow{B_1D_1} \text{ for some } k \in R$$

$$\Rightarrow \overrightarrow{AC_1} - \overrightarrow{AD_1} = k\overrightarrow{B_1D_1}$$

$$\Rightarrow \lambda_3(\vec{a}+\vec{b}) - \lambda_2 \vec{b} = k(\lambda_2 \vec{b} - \lambda_1 \vec{a})$$

$$\Rightarrow \lambda_3 \vec{a} + (\lambda_3 - \lambda_2)\vec{b} = k\lambda_2 \vec{b} - k\lambda_1 \vec{a}$$

Hence, $\lambda_3 = -k\lambda_1$ and $\lambda_3 - \lambda_2 = k\lambda_2$

$$\Rightarrow k = -\frac{\lambda_3}{\lambda_1} = \frac{\lambda_3 - \lambda_2}{\lambda_2}$$

$$\Rightarrow \lambda_1\lambda_2 = \lambda_1\lambda_3 + \lambda_2\lambda_3$$

$$\Rightarrow \frac{1}{\lambda_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

7. Let vector $2\hat{i}+7\hat{j}-5\hat{k}$ intersect vectors $\vec{A}$ and $\vec{B}$ at points L and M, respectively, which have to be determined. Take them to be (x_1, y_1, z_1) and (x_2, y_2, z_2), respectively.

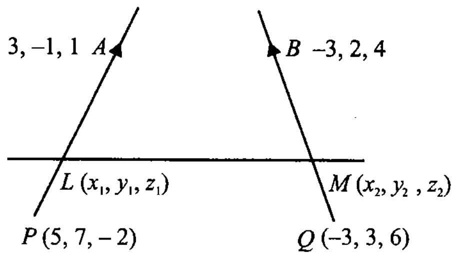


Fig. 1.45

PL is collinear with vector $\vec{A}$. Therefore,

$\therefore\ \overrightarrow{PL} = \lambda\vec{A}$

Comparing the coefficient of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get $\dfrac{x_1-5}{3} = \dfrac{y_1-7}{-1} = \dfrac{z_1+2}{1} = \lambda$ (say)

L is $3\lambda+5, -\lambda+7, \lambda-2$

Similarly, $\overrightarrow{QM} = \mu\vec{B}$. Therefore,

$$\frac{x_3+3}{-3} = \frac{y_2-3}{2} = \frac{z_2-6}{4} = \mu \text{ (say)}$$

$\therefore\ M$ is $-3\mu-3, 2\mu+3, 4\mu+6$

Again LM is collinear with vector $2\hat{i}+7\hat{j}-5\hat{k}$. Therefore,

$$\frac{x_2-x_1}{2} = \frac{y_2-y_1}{7} = \frac{z_2-z_1}{-5} = v \text{ (say)}$$

$$\frac{-3\mu-3\lambda-8}{2} = \frac{2\mu+\lambda-4}{7} = \frac{4\mu-\lambda+8}{-5} = v$$

$$3\mu+3\lambda+2v = -8$$
$$2\mu+\lambda-7v = 4$$
$$4\mu-\lambda+5v = -8$$

Solving, we get

$\lambda = \mu = v = -1$

Therefore, point L is $(2, 8, -3)$ or $2\hat{i}+8\hat{j}-3\hat{k}$

and M is $(0, 1, 2)$ or $\hat{j} + 2\hat{k}$

8. If the given vectors are coplanar, then $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$

or the set of equations

$x_1x + y_1y + z_1z = 0,$

$x_2x + y_2y + z_2z = 0$ and

$x_3x + y_3y + z_3z = 0$ has a non-trivial solution.

Let the given set has a non-trivial solution x, y, z without the loss of generality; we can assume that $x \geq y \geq z$.

For the given equation $x_1x + y_1y + z_1z = 0$, we have $x_1x = -y_1y - z_1z$. Therefore,

$|x_1x| = |y_1y + z_1z| \leq |y_1y| + |z_1z| \Rightarrow |x_1x| \leq |y_1x| + |z_1x| \Rightarrow |x_1| \leq |y_1| + |z_1|$,

which is a contradiction to the given inequality, i.e., $|x_1| > |y_1| + |z_1|$.

Similarly, the other inequalities rule out the possibility of a non-trivial solution.

Hence the given equation has only a trivial solution. Hence the given vectors are non-coplanar.

9. We know: $(1+k)\,|\vec{A}|^2 + \left(1+\frac{1}{k}\right)|\vec{B}|^2 = |\vec{A}|^2 + k|\vec{A}|^2 + |\vec{B}|^2 + \frac{1}{k}|\vec{B}|^2$ (i)

Also,

$$k|\vec{A}|^2 + \frac{1}{k}|\vec{B}|^2 \geq 2\left(k + |\vec{A}|^2 . \frac{1}{k}|\vec{B}|^2\right)^{1/2} = 2|\vec{A}|\cdot|\vec{B}| \quad \text{(ii)}$$

(Since arithmetic mean ≥ geometric mean)

$\therefore\ (1+k)|\vec{A}|^2 + \left(1+\frac{1}{k}\right)|\vec{B}|^2 \geq |\vec{A}|^2 + |\vec{B}|^2 + 2|\vec{A}|\cdot|\vec{B}| = (|\vec{A}| + |\vec{B}|)^2$ (Using (i) and (ii))

And also $|\vec{A}| + |\vec{B}| \geq |\vec{A} + \vec{B}|$

Hence, $(1+k)\,|\vec{A}|^2 + \left(1+\frac{1}{k}\right)^2 |\vec{B}|^2 \geq |\vec{A} + \vec{B}|^2$

10. Since the vectors are coplanar

$$\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & a \end{vmatrix} = 0$$

$$\begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & a-1 \end{vmatrix} \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 1 \end{vmatrix} = 0$$

$\Rightarrow a = 1$

11.

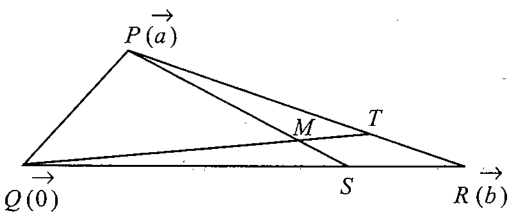


Fig. 1.46

Let $QM : MT = \lambda : 1$ and $PM : MS = \mu : 1$

and $\overrightarrow{QP} = \vec{a}$, $\overrightarrow{QR} = \vec{b}$

$$\Rightarrow \overrightarrow{QT} = \frac{4\vec{b} + \vec{a}}{5}$$

$$\text{and } \overrightarrow{QM} = \frac{\lambda}{\lambda+1}\left(\frac{4\vec{b} + \vec{a}}{5}\right) \quad \text{(i)}$$

$$\overrightarrow{QS} = \frac{3}{4}\vec{b}, \overrightarrow{QM} = \frac{\mu\left(\frac{3}{4}\vec{b}\right) + \vec{a}}{\mu+1} \quad \text{(ii)}$$

From (i) and (ii), $\frac{1}{\mu+1} = \frac{\lambda}{5(\lambda+1)}$ and $\frac{4\lambda}{5(\lambda+1)} = \frac{3\mu}{4(\mu+1)}$

$\Rightarrow \lambda = 15/4$ and $\mu = 16/3$

$\therefore QM : MT = 15 : 4$

12. Let the flow velocity of river be u and the velocity of boat in still water be v.

Thus, $v = \frac{u}{K}$

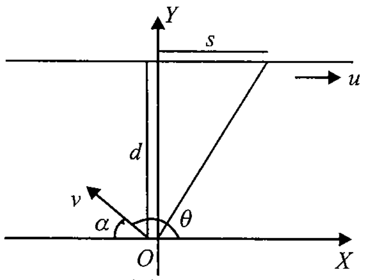


Fig. 1.47

Also, let the boat move at an angle θ with the stream direction.

Now the velocity of boat in the river is the vector resultant of the velocity of boat and flow velocity of river, which can be written as

$$\vec{v}_B = (u - v\cos\alpha)\hat{i} + (u\sin\alpha)\hat{j} = (u + v\cos\theta)\hat{i} + (u\sin\theta)\hat{j}$$

Hence, the time taken to cross the river = $\frac{d}{u\sin\theta}$ (d = width of the river)

Thus, the drift $s = (u + v\cos\theta)\cdot d$

$$\Rightarrow s = d\,(\text{cosec}\,\theta + \frac{v}{u}\cot\theta)$$

$$\Rightarrow \frac{ds}{d\theta} = d\,(\text{cosec}\,\theta\cot\theta - \frac{v}{\mu}\,\text{cosec}^2\,\theta) = 0$$

$$\Rightarrow \frac{v}{u}\text{cosec}^2\theta = \text{cosec}\,\theta\cot\theta$$

$$\Rightarrow \cos\theta = \frac{1}{k} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{k}\right)$$

13. $x\vec{a}+y\vec{b}+z\vec{c}+t\vec{h}=0$ such that

$x+y+z+t=0$ (i)

$x\vec{a}+y\vec{b}=-(z\vec{c}+t\vec{h})$

and $x+y=-(z+t)$

$\therefore \dfrac{x\vec{a}+y\vec{b}}{x+y}=\dfrac{z\vec{c}+t\vec{h}}{z+t}$ (ii)

Position vector of $F=\dfrac{x\vec{a}+y\vec{b}}{x+y}$

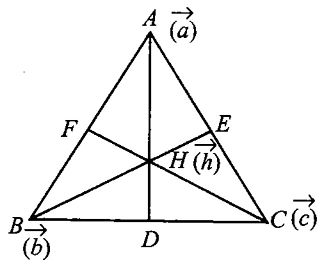


Fig. 1.48

Hence, F divides AB in the ratio y/x.

$\dfrac{AF}{FB}=\dfrac{y}{x}$

Similarly, $\dfrac{BD}{CD}=\dfrac{z}{y}$ and $\dfrac{CE}{AE}=\dfrac{x}{z}$

$\Rightarrow \dfrac{AF}{FB}\cdot\dfrac{BD}{CD}\cdot\dfrac{CE}{AE}=-1$

14. $\overrightarrow{OM}=\dfrac{\vec{b}}{2} \Rightarrow \overrightarrow{PM}=\vec{a}+\dfrac{\vec{b}}{2}$

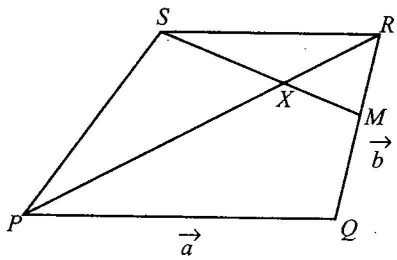


Fig. 1.49

$\overrightarrow{SM}=\overrightarrow{PM}-\overrightarrow{PS}=2\vec{a}-\dfrac{1}{2}\vec{b}$

$\overrightarrow{SX}=\dfrac{4}{5}\overrightarrow{SM}=\dfrac{8}{5}\vec{a}-\dfrac{2}{5}\vec{b}$

$\overrightarrow{PX} = \overrightarrow{PS} + \overrightarrow{SX}$

$= -\vec{a} + \vec{b} + \frac{8}{5}\vec{a} - \frac{2}{5}\vec{b} = \frac{3}{5}(\vec{a} + \vec{b})$

Also $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} = \vec{a} + \vec{b} = \frac{5}{3}\overrightarrow{PX}$

Hence P, X and R are collinear.

Objective Type

1. **a.** Four or more than four non-zero vectors are always linearly dependent.
2. **d.** $3\vec{a} + 4\vec{b} + 5\vec{c} = 0$

   $\Rightarrow \vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

   No other conclusion can be derived from it.
3. **c.** $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = 4\hat{i} + 2\hat{j} - 4\hat{k}$

   $\overrightarrow{AB} = -3\hat{i} - 3\hat{k}, \overrightarrow{AC} = \hat{i} + 2\hat{j} - 7\hat{k}$

   $BC^2 = 36, AB^2 = 18, AC^2 = 54$

   Clearly, $AC^2 = BC^2 + AB^2$

   $\therefore \angle B = 90°$
4. **c.** $|\vec{a} + \vec{b}| < |\vec{a} - \vec{b}|$

   $\Rightarrow \frac{\pi}{2} < \theta < \frac{3\pi}{2}$
5. **c.** The position vector of the point O with respect to itself is

   $$\frac{\overrightarrow{OA}\sin 2A + \overrightarrow{OB}\sin 2B + \overrightarrow{OC}\sin 2C}{\sin 2A + \sin 2B + \sin 2C}$$

   $$\Rightarrow \frac{\overrightarrow{OA}\sin 2A + \overrightarrow{OB}\sin 2B + \overrightarrow{OC}\sin 2C}{\sin 2A + \sin 2B + \sin 2C} = \vec{0}$$

   $$\Rightarrow \overrightarrow{OA}\sin 2A + \overrightarrow{OB}\sin 2B + \overrightarrow{OC}\sin 2C = \vec{0}$$
6. **a.** We have $\overrightarrow{GB} + \overrightarrow{GC} = (1+1)\overrightarrow{GD} = 2\overrightarrow{GD}$, where D is the midpoint of BC.

   $\because \overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \overrightarrow{GA} + 2\overrightarrow{GD} = \overrightarrow{GA} - \overrightarrow{GA} = 0$

   ($\because$ G divides AC in the ratio 2 : 1, $\therefore 2\overrightarrow{GD} = -\overrightarrow{GA}$)
7. **c.** $m\vec{a}$ is a unit vector if and only if

   $|m\vec{a}| = 1 \Rightarrow |m|\,|\vec{a}| = 1 \Rightarrow |m|\,a = 1 \Rightarrow a = \frac{1}{|m|}$

8. c.

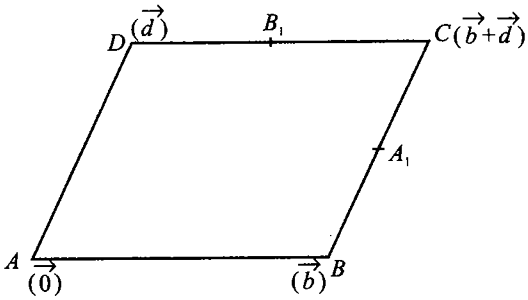


Fig. 1.50

Let P.V. of A, B and D be $\vec{0}, \vec{b}$ and $\vec{d}$, respectively.

Then P.V. of C, $\vec{c} = \vec{b}+\vec{d}$

Also P.V. of $A_1 = \vec{b}+\frac{\vec{d}}{2}$

and P.V. of $B_1 = \vec{d}+\frac{\vec{b}}{2}$

$\Rightarrow \overrightarrow{AA_1}+\overrightarrow{AB_1} = \frac{3}{2}(\vec{b}+\vec{d}) = \frac{3}{2}\overrightarrow{AC}$

9. b. Since $|\overrightarrow{OP}|=|\overrightarrow{OQ}| \neq \sqrt{14}$, ΔOPQ is isosceles.

Hence the internal bisector OM is perpendicular to PQ and M is the midpoint of P and Q.

$\therefore \overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OP}+\overrightarrow{OQ}) = 2\hat{i}+\hat{j}-2\hat{k}$

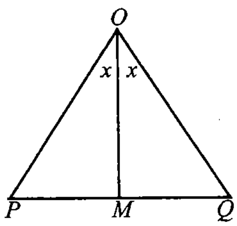


Fig. 1.51

10. d.

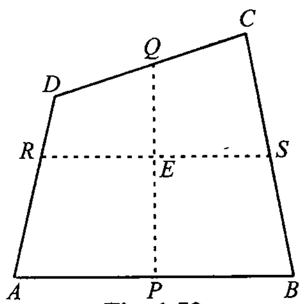


Fig. 1.52

Let $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = \vec{c}$ and $\overrightarrow{OD} = \vec{d}$. Therefore,

$$\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD} = \vec{a}+\vec{b}+\vec{c}+\vec{d}$$

P, the midpoint of AB, is $\dfrac{\vec{a}+\vec{b}}{2}$

Q, the midpoint of $\overrightarrow{CD}$, is $\dfrac{\vec{c}+\vec{d}}{2}$

Therefore, the midpoint of $\overrightarrow{PQ}$ is $\dfrac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$.

Similarly the midpoint of RS is $\dfrac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$, i.e., $\overrightarrow{OE} = \dfrac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4} \Rightarrow x=4$.

11. b.

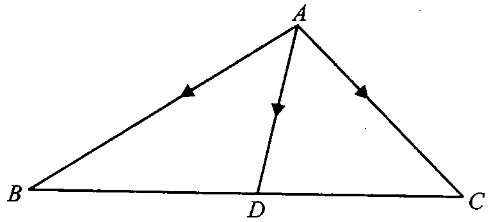


Fig. 1.53

$$\overrightarrow{AB}+\overrightarrow{AC} = 2\overrightarrow{AD}$$

$$\therefore\ \overrightarrow{AD} = \frac{1}{2}\{(-3\hat{i}+4\hat{k})+(5\hat{i}-2\hat{j}+4\hat{k})\}$$

$$= \hat{i}-\hat{j}+4\hat{k}$$

Length of $AD = \sqrt{1+1+16} = \sqrt{18}$

12. d. $\vec{a} - \vec{b} = 2\,(\vec{d}-\vec{c})$

$$\therefore\ \frac{\vec{a}+2\vec{c}}{2+1} = \frac{\vec{b}+2\vec{d}}{2+1}$$

$\Rightarrow$ AC and BD trisect each other as L.H.S. is the position vector of a point trisecting A an C, and R.H.S. that of B and D.

13. b. Vector in the direction of angular bisector of $\vec{a}$ and $\vec{b}$ is $\dfrac{\vec{a}+\vec{b}}{2}$

Unit vector in this direction is $\dfrac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|}$

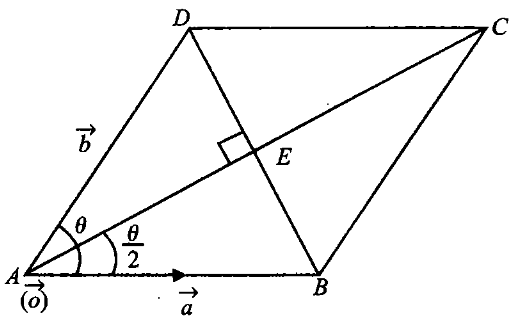


Fig. 1.54

From the figure, position vector of E is $\dfrac{\vec{a}+\vec{b}}{2}$

Now in triangle AEB, $AE = AB\cos\dfrac{\theta}{2}$

$$\Rightarrow \left|\frac{\vec{a}+\vec{b}}{2}\right| = \cos\frac{\theta}{2}$$

Hence unit vector along the bisector is $\dfrac{\vec{a}+\vec{b}}{2\cos\dfrac{\theta}{2}}$

14. **b.** $|a|+|b|+|c| = \sqrt{a^2+b^2+c^2} \Leftrightarrow 2|ab|+2|bc|+2|ca| = 0$

$\Leftrightarrow ab = bc = ca = 0 \Leftrightarrow$ any two of a, b and c are zero

15. **c.** $\vec{\alpha} = \vec{a}+\vec{b}+\vec{c} = 6\hat{i}+12\hat{j}$

Let $\vec{\alpha} = x\vec{a}+y\vec{b} \Rightarrow 6x+2y = 6$

and $-3x-6y = 12$

$\therefore x = 2, y = -3$

$\therefore \vec{\alpha} = 2\vec{a}-3\vec{b}$

16. **c.** Given $\vec{\alpha}+\vec{\beta}+\vec{\gamma} = a\vec{\delta}$ (i)

$\vec{\beta}+\vec{\gamma}+\vec{\delta} = b\vec{\alpha}$ (ii)

From (i), $\vec{\alpha}+\vec{\beta}+\vec{\gamma}+\vec{\delta} = (a+1)\vec{\delta}$ (iii)

From (ii), $\vec{\alpha}+\vec{\beta}+\vec{\gamma}+\vec{\delta} = (b+1)\vec{\alpha}$ (iv)

From (iii) and (iv),

$(a+1)\vec{\delta} = (b+1)\vec{\alpha}$ (v)

Since $\vec{\alpha}$ is not parallel to $\vec{\delta}$,

From (v), $a + 1 = 0$ and $b + 1 = 0$

From (iii), $\vec{\alpha} + \vec{\beta} + \vec{\gamma} + \vec{\delta} = 0$

17. **a.** Let the origin of reference be the circumcentre of the triangle.

Let $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = \vec{c}$ and $\overrightarrow{OT} = \vec{t}$

Then $|\vec{a}| = |\vec{b}| = |\vec{c}| = R$ (circumradius)

Again $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OA} + 2\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AH} = \overrightarrow{OH}$

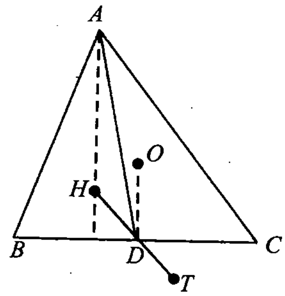


Fig. 1.55

Therefore, the P.V. of H is $\vec{a} + \vec{b} + \vec{c}$. Since D is the midpoint of HT, we have

$$\frac{\vec{a} + \vec{b} + \vec{c} + \vec{t}}{2} = \frac{\vec{b} + \vec{c}}{2} \Rightarrow \vec{t} = -\vec{a}$$

$\therefore \overrightarrow{AT} = -2\vec{a} \Rightarrow \overrightarrow{AT} = |-2\vec{a}| = 2|\vec{a}| = 2R$. But $BC = 2R \sin A = R$, therefore $AT = 2BC$

18. **c.** Given $a_1\vec{r_1} + a_2\vec{r_2} + \cdots + a_n\vec{r_n} = 0$

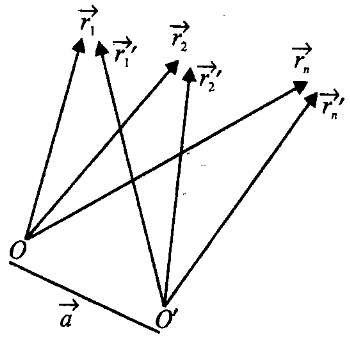


Fig. 1.56

Now $\vec{a}+\vec{r_1'}=\vec{r_1}$ and so on

Hence $a_1\ (\vec{a}+\vec{r_1'})+a_2\ (\vec{a}+\vec{r_2'})+\cdots+a_n\ (\vec{a}+\vec{r_n'})=0$

$a_1\vec{r_1'}+a_2\vec{r_2'}+\cdots+a_n\vec{r_n'}+\vec{a}(a_1+a_2+\cdots+a_n)=0$

Hence $a_1\vec{r_1'}+a_2\vec{r_2'}+\cdots+a_n\vec{r_n'}=0$ if $a_1+a_2+\cdots+a_n=0.$

19. d. $\vec{r_1}+2\vec{r_2}=(p\vec{a}+q\vec{b}+\vec{c})+2(\vec{a}+p\vec{b}+q\vec{c})=(p+2)\vec{a}+(q+2p)\vec{b}+(1+2q)\vec{c}$

$2\vec{r_1}+\vec{r_2}=(2p+1)\ \vec{a}+(2q+p)\ \vec{b}+(2+q)\ \vec{c}$

$$\frac{p+2}{2p+1}=\frac{q+2p}{2q+p}=\frac{1+2q}{2+q}=\frac{p+q+2p+2q+3}{p+q+2p+2q+3}=1$$

$\Rightarrow p=1$ and $q=1$

20. d. $\sqrt{3}\tan\theta+1=0$ and $\sqrt{3}\ \sec\theta-2=0$

$\Rightarrow \theta=\dfrac{11\pi}{6}$

$\Rightarrow \theta=2n\pi+\dfrac{11\pi}{6},\ n\in Z$

21. d. $\vec{c}-\vec{b}=\alpha\vec{d}$ and $\vec{P}=\overrightarrow{AC}+\overrightarrow{BD}=\mu\ \overrightarrow{AD}$

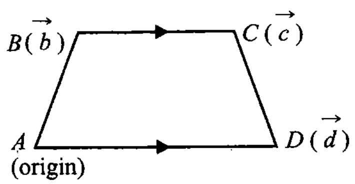


Fig. 1.57

Hence $\vec{p}=\vec{c}+\vec{d}-\vec{b}=\mu\ \vec{d}$ (using $\vec{c}-\vec{b}=\alpha\vec{d}$)

or $\alpha+1=\mu$

22. b. Note that $\vec{a}+\vec{b}=\vec{c}$

23. a. $\hat{a}=\dfrac{-4i+3\hat{k}}{5};\hat{b}=\dfrac{14\hat{i}+2\hat{j}-5\hat{k}}{15}$

A vector $\vec{V}$ bisecting the angle between $\vec{a}$ and $\vec{b}$ is $\vec{V}=\hat{a}+\hat{b}$

$$=\frac{-12\hat{i}+9\hat{k}+14\hat{i}+2\hat{j}-5\hat{k}}{15}$$

$$=\frac{2\hat{i}+2\hat{j}+4\hat{k}}{15}$$

Required vector $\vec{d} = \sqrt{6}\,\hat{V} = \hat{i} + \hat{j} + 2\hat{k}$

24. d. We must have $\lambda(\hat{i} - 3\hat{j} + 5\hat{k}) = \hat{a} + \dfrac{2\hat{k} + 2\hat{j} - \hat{i}}{3}$. Therefore,

$3\hat{a} = 3\lambda(\hat{i} - 3\hat{j} + 5\hat{k}) - (2\hat{k} + 2\hat{j} - \hat{i}) = \hat{i}(3\lambda + 1) - \hat{j}(2 + 9\lambda) + \hat{k}(15\lambda - 2)$

$\Rightarrow 3|\hat{a}| = \sqrt{(3\lambda+1)^2 + (2+9\lambda)^2 + (15\lambda-2)^2}$

$\Rightarrow 9 = (3\lambda + 1)^2 + (2 + 9\lambda)^2 + (15\lambda - 2)^2 \Rightarrow 315\lambda^2 - 18\lambda = 0 \Rightarrow \lambda = 0, \dfrac{2}{35}$

If $\lambda = 0$, $\vec{a} = \hat{i} - 2\hat{j} - 2\hat{k}$ (not acceptable)

For $\lambda = \dfrac{2}{35}$, $\vec{a} = \dfrac{41}{105}\hat{i} - \dfrac{88}{105}\hat{j} - \dfrac{40}{105}\hat{k}$

25. c. Suppose the bisector of angle A meets BC at D. Then AD divides BC in the ratio $AB : AC$.

So, P.V. of $D = \dfrac{|\overrightarrow{AB}|(2\hat{i} + 5\hat{j} + 7\hat{k}) + |\overrightarrow{AC}|\ (2\hat{i} + 3\hat{j} + 4\hat{k})}{|\overrightarrow{AB}| + |\overrightarrow{AC}|}$

But $\overrightarrow{AB} = -2\hat{i} - 4\hat{j} - 4\hat{k}$ and $\overrightarrow{AC} = -2\hat{i} - 2\hat{j} - \hat{k}$

$\Rightarrow |\overrightarrow{AB}| = 6$ and $|\overrightarrow{AC}| = 3$

$\therefore$ P.V. of $D = \dfrac{6(2\hat{i} + 5\hat{j} + 7\hat{k}) + 3(2\hat{i} + 3\hat{j} + 4\hat{k})}{6+3}$

$= \dfrac{1}{3}(6\hat{i} + 13\hat{j} + 18\hat{k})$

26. b. The point that divides $5\hat{i}$ and $5\hat{j}$ in the ratio of $k : 1$ is $\dfrac{(5\hat{j})k + (5\hat{i})1}{k+1}$

$\therefore\ \vec{b} = \dfrac{5\hat{i} + 5k\hat{j}}{k+1}$

Also, $|\vec{b}| \le \sqrt{37}$

Fig. 1.58

$\Rightarrow \dfrac{1}{k+1}\sqrt{25 + 25k^2} \le \sqrt{37}$

$\Rightarrow 5\sqrt{1+k^2} \le \sqrt{37}(k+1)$

Squaring both sides

$25(1+k^2) \le 37(k^2+2k+1)$

or $6k^2+37k+6 \ge 0 \Rightarrow (6k+1)(k+6) \ge 0$

$k \in (-\infty, -6] \cup \left[-\frac{1}{6}, \infty\right)$

27. b. Let $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$, then $\overrightarrow{AB} = \vec{b} - \vec{a}$ and $\overrightarrow{OP} = \frac{1}{3}\vec{a}, \overrightarrow{OQ} = \frac{1}{2}\vec{b}, \overrightarrow{OR} = \frac{1}{3}\vec{c}$.

Since P, Q, R and S are coplanar, then

$\overrightarrow{PS} = \alpha\,\overrightarrow{PQ} + \beta\,\overrightarrow{PR}$ ($\overrightarrow{PS}$ can be written as a linear combination of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$)

$= \alpha(\overrightarrow{OQ} - \overrightarrow{OP}) + \beta(\overrightarrow{OR} - \overrightarrow{OP})$

i.e., $\overrightarrow{OS} - \overrightarrow{OP} = -(\alpha+\beta)\frac{\vec{a}}{3} + \frac{\alpha}{2}\vec{b} + \frac{\beta}{3}\vec{c}$

$\Rightarrow \overrightarrow{OS} = (1-\alpha-\beta)\frac{\vec{a}}{3} + \frac{\alpha}{2}\vec{b} + \frac{\beta}{3}\vec{c}$ (i)

Given $\overrightarrow{OS} = \lambda\,\overrightarrow{AB} = \lambda(\vec{b} - \vec{a})$ (ii)

From (i) and (ii), $\beta = 0, \frac{1-\alpha}{3} = -\lambda$ and $\frac{\alpha}{2} = \lambda$

$\Rightarrow 2\lambda = 1 + 3\lambda$

$\Rightarrow \lambda = -1$

28. a. Let the incentre be at the origin and be $A(\vec{p}), B(\vec{q})$ and $C(\vec{r})$. Then

$\overrightarrow{IA} = \vec{p}, \overrightarrow{IB} = \vec{q}$ and $\overrightarrow{IC} = \vec{r}$.

Incentre I is $\frac{a\vec{p}+b\vec{q}+c\vec{r}}{a+b+c}$, where $p = BC, q = AC$ and $r = AB$

Incentre is at the origin. Therefore,

$\frac{a\vec{p}+b\vec{q}+c\vec{r}}{a+b+c} = \vec{0}$, or $a\vec{p}+b\vec{q}+c\vec{r} = \vec{0}$

$\Rightarrow a\,\overrightarrow{IA} + b\,\overrightarrow{IB} + c\,\overrightarrow{IC} = \vec{0}$

29. a.

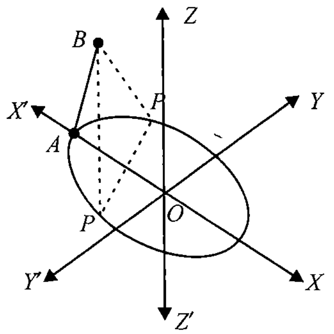


Fig. 1.59

Point P lies on $x^2 + 3y^2 = 3$ (i)

Now from the diagram, according to the given conditions, $AP = AB$

or $(x + \sqrt{3})^2 + (y - 0)^2 = 4$ or $(x + \sqrt{3})^2 + y^2 = 4$ (ii)

Solving (i) and (ii), we get $x = 0$ and $y = \pm 1$

Hence point P has position vector $\pm \hat{j}$

30. **b.**

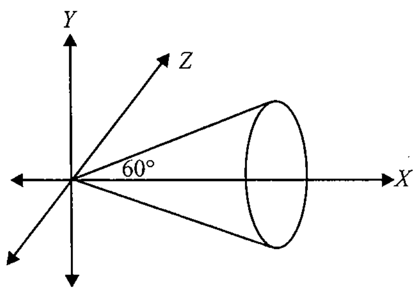


Fig. 1.60

From the diagram, it is obvious that locus is a cone concentric with the positive x-axis having vertex at the origin and the slant height equal to the magnitude of the vector.

31. **c.** Since $\vec{x}, \vec{y}$ and $\vec{x} \times \vec{y}$ are linearly independent,

$20a - 15b = 15b - 12c = 12c - 20a = 0$

$\Rightarrow \frac{a}{3} = \frac{b}{4} = \frac{c}{5}$

$\Rightarrow c^2 = a^2 + b^2$

Hence, ΔABC is right angled.

32. **a.** The position vector of any point at t is

$\vec{r} = (2+t^2)\hat{i} + (4t-5)\hat{j} + (2t^2-6)\hat{k}$

$\Rightarrow \dfrac{d\vec{r}}{dt} = 2t\,\hat{i} + 4\hat{j} + (4t-6)\hat{k}$

$\Rightarrow \left.\dfrac{d\vec{r}}{dt}\right|_{t=2} = 4\hat{i} + 4\hat{j} + 2\hat{k}$ and $\left.\left|\dfrac{d\vec{r}}{dt}\right|\right|_{t=2} = \sqrt{16+16+4} = 4$

Hence, the required unit tangent vector at $t = 2$ is $\frac{1}{3}(2\hat{i} + 2\hat{j} + \hat{k})$.

33. **a.** As $\vec{x}$, $\vec{y}$ and $\vec{x} \times \vec{y}$ are non-collinear vectors, vectors are linearly independent.

$\Rightarrow a - b = 0 = b - c = c - a$

$\Rightarrow a = b = c$

Therefore, the triangle is equilateral.

34. **c.**

Multiple Correct Answers Type

1. **a.,b.,c.,d.**

$x\hat{i} + (x+1)\hat{j} + (x+2)\hat{k}$, $(x+3)\hat{i} + (x+4)\hat{j} + (x+5)\hat{k}$ and $(x+6)\hat{i} + (x+7)\hat{j} + (x+8)\hat{k}$ are coplanar

We have determinant of their coefficients as $\begin{vmatrix} x & x+1 & x+2 \\ x+3 & x+4 & x+5 \\ x+6 & x+7 & x+8 \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have

$$\begin{vmatrix} x & 1 & 2 \\ x+3 & 1 & 2 \\ x+6 & 1 & 2 \end{vmatrix} = 0$$

Hence $x \in R$

2. **a., d.** Let $\vec{a} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.

Then the diagonals of the parallelogram are $\vec{p} = \vec{a} + \vec{b}$ and $\vec{q} = \vec{b} - \vec{a}$,

i.e., $\vec{p} = 3\hat{i} + 6\hat{j} - 2\hat{k}$, $\vec{q} = -\hat{i} - 2\hat{j} + 8\hat{k}$

So, unit vectors along the diagonals are $\frac{1}{7}(3\hat{i} + 6\hat{j} - 2\hat{k})$ and $\frac{1}{\sqrt{69}}(-\hat{i} - 2\hat{j} + 8\hat{k})$.

3. **b., c.** We have, $\vec{a} = 2p\,\hat{i} + \hat{j}$

On rotation, let $\vec{b}$ be the vector with components $(p + 1)$ and 1 so that $\vec{b} = (p+1)\hat{i} + \hat{j}$.

Now, $|\vec{a}| = |\vec{b}| \Rightarrow a^2 = b^2$

$\Rightarrow 4p^2 + 1 = (p + 1)^2 + 1$

$\Rightarrow 4p^2 = (p + 1)^2$

$\Rightarrow 2p = \pm (p + 1)$

$\Rightarrow 3p = -1$ or $p = 1$

$\therefore p = -1/3$ or $p = 1$

4. a.,b.,d.

Points $A(\hat{i} + \hat{j}), B(\hat{i} - \hat{j})$ and $C(p\hat{i} + q\hat{j} + r\hat{k})$ are collinear

Now $\vec{AB} = -2\hat{j}$ and $\vec{BC} = (p - 1)\hat{i} + (q - 1)\hat{j} + r\hat{k}$

Vectors $\vec{AB}$ and $\vec{BC}$ must be collinear

$\Rightarrow \quad p = 1, r = 0$ and $q \neq 1$

5. a., b., c.

For coplanar vectors, $\begin{vmatrix} 1 & 2 & 3 \\ 0 & \lambda & \mu \\ 0 & 0 & 2\lambda - 1 \end{vmatrix} = 0$

$\Rightarrow (2\lambda - 1)\lambda = 0 \Rightarrow \lambda = 0, \dfrac{1}{2}$

6. b., c.

Let $\vec{R}$ be the resultant.

Then $\vec{R} = \vec{F_1} + \vec{F_2} + \vec{F_3} = (p+1)\hat{i} + 4\hat{j}$

Given, $|\vec{R}| = 5$. Therefore,

$(p + 1)^2 + 16 = 25$

$\Rightarrow p + 1 = \pm 3$

$\therefore \ p = 2, -4$

7. a., b., d.

$$\vec{a} = \left[\pm\left(\hat{i} - \hat{j}\right) \pm \left(\hat{j} + \hat{k}\right)\right]$$

$$= \pm\left(\hat{i} + \hat{k}\right), \pm\left(\hat{i} - 2\hat{j} - \hat{k}\right)$$

8. b., c. Let $\vec{\alpha} = \hat{i} + x\hat{j} + 3\hat{k}, \vec{\beta} = 4\hat{i} + (4x - 2)\hat{j} + 2\hat{k}$

Given, $2|\vec{\alpha}| = |\vec{\beta}|$

$\Rightarrow 2\sqrt{10 + x^2} = \sqrt{20 + 4(2x - 1)^2}$

$\Rightarrow 10 + x^2 = 5 + (4x^2 - 4x + 1)$

$\Rightarrow 3x^2 - 4x - 4 = 0$

$\Rightarrow x = 2, -\dfrac{2}{3}$

9. c., d.

Let $\vec a, \vec b$ and $\vec c$ lie in the x-y plane.

Let $\vec a = \hat i, \vec b = -\frac{1}{2}\hat i + \frac{\sqrt3}{2}\hat i$ and $\vec c = -\frac{1}{2}\hat i - \frac{\sqrt3}{2}\hat j$. Therefore,

$$|\vec p+\vec q+\vec r| = |\lambda\vec a+\mu\vec b+\nu\vec c|$$

$$= \left|\lambda\hat i+\mu\left(-\frac{1}{2}\hat i+\frac{\sqrt3}{2}\hat j\right)+\nu\left(-\frac{1}{2}\hat i-\frac{\sqrt3}{2}\hat j\right)\right|$$

$$= \left|\left(\lambda-\frac{\mu}{2}-\frac{\nu}{2}\right)\hat i+\frac{\sqrt3}{2}(\mu-\nu)\hat j\right|$$

$$= \sqrt{\left(\lambda-\frac{\mu}{2}-\frac{\nu}{2}\right)^2+\frac{3}{4}(\mu-\nu)^2}$$

$$= \sqrt{\lambda^2+\mu^2+\nu^2-\lambda\mu-\lambda\nu-\mu\nu}$$

$$= \frac{1}{\sqrt2}\sqrt{(\lambda-\mu)^2+(\mu-\nu)^2+(\nu-\lambda)^2}$$

$$\ge \frac{1}{\sqrt2}\sqrt{1+1+4} = \sqrt3$$

$\Rightarrow |\vec p+\vec q+\vec r|$ can take a value equal to $\sqrt3$ and 2.

10. b., d. Since $\vec a$ and $\vec b$ are equally inclined to $\vec c$, $\vec c$ must be of the form $t\left(\frac{\vec a}{|\vec a|}+\frac{\vec b}{|\vec b|}\right)$.

Now $\frac{|\vec b|}{|\vec a|+|\vec b|}\vec a+\frac{|\vec a|}{|\vec a|+|\vec b|}\vec b = \frac{|\vec a||\vec b|}{|\vec a|+|\vec b|}\left(\frac{\vec a}{|\vec a|}+\frac{\vec b}{|\vec b|}\right)$

Also, $\frac{|\vec b|}{2|\vec a|+|\vec b|}\vec a+\frac{|\vec a|}{2|\vec a|+|\vec b|}\vec b = \frac{|\vec a||\vec b|}{2|\vec a|+|\vec b|}\left(\frac{\vec a}{|\vec a|}+\frac{\vec b}{|\vec b|}\right)$

Other two vectors cannot be written in the form $t\left(\frac{\vec a}{|\vec a|}+\frac{\vec b}{|\vec b|}\right)$.

11. a., c., d.

$\overrightarrow{OA} = -4\hat i+3\hat k$; $\overrightarrow{OB} = 14\hat i+2\hat j-5\hat k$

$\hat a = \frac{-4\hat i+3\hat k}{5}$; $\hat b = \frac{14\hat i+2\hat j-5\hat k}{15}$

$\vec r = \frac{\lambda}{15}[-12\hat i+9\hat j+14\hat i+2\hat j-5\hat k]$

$\vec{r} = \frac{\lambda}{15}[2\hat{i} + 2\hat{j} + 4\hat{k}]$

$\vec{r} = \frac{2\lambda}{15}[\hat{i} + \hat{j} + 2\hat{k}]$

12. a., b., d.

$(\lambda - 1)\ (\vec{a}_1 - \vec{a}_2) + \mu\ (\vec{a}_2 + \vec{a}_3) + \gamma\ (\vec{a}_3 + \vec{a}_4 - 2\vec{a}_2) + \vec{a}_3 + \delta\vec{a}_4 = \vec{0}$

i.e., $(\lambda - 1)\ \vec{a}_1 + (1 - \lambda + \mu - 2\gamma)\vec{a}_2 + (\mu + \gamma + 1)\ \vec{a}_3 + (\gamma + \delta)\ \vec{a}_4 = \vec{0}$

Since $\vec{a}_1, \vec{a}_2, \vec{a}_3$ and $\vec{a}_4$ are linearly independent

$\lambda - 1 = 0, 1 - \lambda + \mu - 2\gamma = 0, \mu + \gamma + 1 = 0$ and $\gamma + \delta = 0$

i.e., $\lambda = 1, \mu = 2\gamma, \mu + \gamma + 1 = 0, \gamma + \delta = 0$

i.e., $\lambda = 1, \mu = -\frac{2}{3}, \gamma = -\frac{1}{3}, \delta = \frac{1}{3}$

13. a., c. We have, $\overrightarrow{AB} = -\hat{i} - \hat{j} - 4\hat{k}$, $\overrightarrow{BC} = -3\hat{i} + 3\hat{j}$ and $\overrightarrow{CA} = 4\hat{i} - 2\hat{j} + 4\hat{k}$. Therefore,

$|\overrightarrow{AB}| = |\overrightarrow{BC}| = 3\sqrt{2}$ and $|\overrightarrow{CA}| = 6$

Clearly, $|\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2 = |\overrightarrow{AC}|^2$

Hence, the triangle is right-angled isosceles triangle.

Reasoning Type

1. a.

$\sqrt{(p+2)^2 + 1} = \sqrt{p^2 + 1}$

$\Rightarrow p^2 + 4 + 4p + 1 = p^2 + 1$

$\Rightarrow 4p = -4$

$\Rightarrow p = -1$

Hence a is the correct option.

2. a. $2\vec{a} + 3\vec{b} - 5\vec{c} = 0$

$\Rightarrow 3(\vec{b} - \vec{a}) = 5(\vec{c} - \vec{a}) \Rightarrow \overrightarrow{AB} = \frac{5}{3}\overrightarrow{AC}$

$\Rightarrow \overrightarrow{AB}$ and $\overrightarrow{AC}$ must be parallel since there is a common point A. The points A, B and C must be collinear.

3. d. We know that the unit vector along bisector of unit vectors $\vec{u}$ and $\vec{v}$ is $\frac{\vec{u} + \vec{v}}{2\cos\frac{\theta}{2}}$, where θ is the angle between vectors $\vec{u}$ and $\vec{v}$.

Hence Statement 1 is false, however Statement 2 is true.

4. **b.** Obviously, Statement 1 is true.

$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 2\cos^2 \alpha - 1 + 2\cos^2 \beta - 1 + 2\cos^2 \gamma - 1$

$= 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - 3 = 2 - 3 = -1$

Hence, Statement 2 is true but does not explain Statement 1 as it is result derived using the result in the statement.

5. **b.**

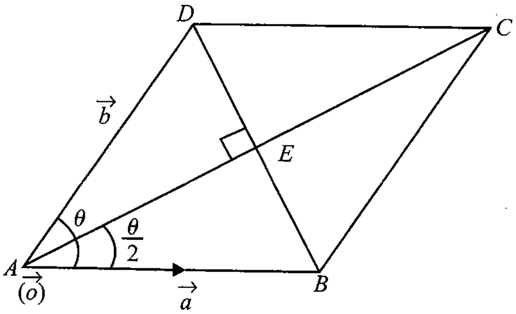


Fig. 1.61

We know that vector in the direction of angular bisector of unit vectors $\vec{a}$ and $\vec{b}$ is $\dfrac{\vec{a}+\vec{b}}{2\cos\frac{\theta}{2}}$

where $\vec{a} = \vec{AB} = l_1\hat{i} + m_1\hat{j} + n_1\hat{k}$ and $\vec{b} = \vec{AD} = l_2\hat{i} + m_2\hat{j} + n_3\hat{k}$

Thus unit vector along the bisector is $\dfrac{l_1 + l_2}{\cos\frac{\theta}{2}}\hat{i} + \dfrac{m_1 + m_2}{\cos\frac{\theta}{2}}\hat{j} + \dfrac{n_1 + n_2}{\cos\frac{\theta}{2}}\hat{k}$

Hence Statement 1 is true.

Also, in triangle ABD, by cosine rule

$$\cos\theta = \frac{AB^2 + AD^2 - BD^2}{2AB \cdot AD}$$

$$\Rightarrow \quad \cos\theta = \frac{1 + 1 - |(l_1 - l_2)\hat{i} + (m_1 - m_2)\hat{j} + (n_1 - n_2)\hat{k}|^2}{2}$$

$$\Rightarrow \quad \cos\theta = \frac{2 - [(l_1 - l_2)^2 + (m_1 - m_2)^2 + (n_1 - n_2)^2]}{2}$$

$$= \frac{2 - [2 - 2(l_1l_2 + m_1m_2 + n_1n_2)]}{2}$$

$$= l_1l_2 + m_1m_2 + n_1n_2$$

Hence, Statement 2 is true but does not explain Statement 1.

6. **c.** In ΔABC, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} = -\overrightarrow{CA} \Rightarrow \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{O}$

$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ is the triangle law of addition.

Hence Statement 1 is true and Statement 2 is false.

7. a.

$$\frac{3}{2}=\frac{p}{3}=\frac{3}{q} \Rightarrow p=\frac{9}{2} \text{ and } q=2$$

Thus, both the statements are true and Statement 2 is the correct explanation for Statement 1.

8. **a.** $\vec{a}+\vec{b} = \vec{a}-\vec{b}$ are the diagonals of a parallelogram whose sides are $\vec{a}$ and $\vec{b}$.

$|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$

$\Rightarrow$ Diagonals of the parallelogram have the same length.

$\Rightarrow$ The parallelogram is a rectangle $\Rightarrow \vec{a} \perp \vec{b}$

9. **a.** Given vectors are non-coplanar. Hence the answer is (a)

10. **a.** $3\vec{a}-2\vec{b}+5\vec{c}-6\vec{d}=(2\vec{a}-2\vec{b})+(-5\vec{a}+5\vec{c})+(6\vec{a}-6\vec{d})$

$$=-2\overrightarrow{AB}+5\overrightarrow{AC}-6\overrightarrow{AD}=\vec{0}$$

Therefore, $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are linearly dependent. Hence by Statement 2, Statement 1is true.

11. **a.** We have adjacent sides of triangle $|\vec{a}| = 3$, $|\vec{b}| = 4$.

The length of the diagonal is $|\vec{a}+\vec{b}| = 5$.

Since it satisfies the Pythagoras theorem, $\vec{a} \perp \vec{b}$.

Hence the parallelogram is a rectangle.

Hence length of the other diagonal is $\left|\vec{a}-\vec{b}\right|=5$

Linked Comprehension Type

For Problems 1–3

1. c., 2. b., 3. c.

Sol.

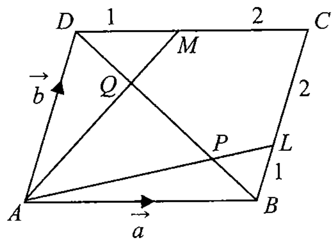


Fig. 1.62

$$\overrightarrow{BL}=\frac{1}{3}\vec{b}$$

$$\therefore\ \overrightarrow{AL}=\vec{a}+\frac{1}{3}\vec{b}$$

Let $\overrightarrow{AP} = \lambda \overrightarrow{AL}$ and P divides DB in the ratio $\mu : 1 - \mu$

Then $\overrightarrow{AP} = \lambda \vec{a} + \frac{\lambda}{3}\vec{b}$ (i)

Also $\overrightarrow{AP} = \mu \vec{a} + (1-\mu)\vec{b}$ (ii)

From (i) and (ii), $\lambda \vec{a} + \frac{\lambda}{3}\vec{b} = \mu \vec{a} + (1-\mu)\vec{b}$

$\therefore \lambda = \mu$ and $\frac{\lambda}{3} = 1 - \mu$

$\therefore \lambda = \frac{3}{4}$

Hence, P divides AL in the ratio 3 : 1 and P divides DB in the ratio 3 : 1.
Similarly Q divides DB in the ratio 1 : 3.

Thus $DQ = \frac{1}{4}DB$ and $PB = \frac{1}{4}DB$

$\therefore PQ = \frac{1}{2}DB$, i.e., $PQ : DB = 1 : 2$

For Problems 4–5

4. c., 5. b.

Sol.

Let the position vectors of A, B, C and D be $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$, respectively.
Then, $OA : CB = 2 : 1$

$\Rightarrow \overrightarrow{OA} = 2\overrightarrow{CB}$

$\Rightarrow \vec{a} = 2(\vec{b} - \vec{c})$ (i)

and $OD : AB = 1 : 3$

$3\overrightarrow{OD} = \overrightarrow{AB}$

$\Rightarrow 3\vec{d} = (\vec{b} - \vec{a}) = \vec{b} - 2(\vec{b} - \vec{c})$ (using (i))

$\Rightarrow 3\vec{d} = -\vec{b} + 2\vec{c}$ (ii)

Let $OX : XC = \lambda : 1$ and $AX : XD = \mu : 1$

Now, X divides OC in the ratio $\lambda : 1$. Therefore,

P.V. of $X = \frac{\lambda \vec{c}}{\lambda + 1}$ (iii)

X also divides AD in the ratio $\mu : 1$

P.V. of $X = \frac{\mu \vec{d} + \vec{a}}{\mu + 1}$ (iv)

From (iii) and (iv), we get

$$\frac{\lambda \vec{c}}{\lambda+1}=\frac{\mu \vec{d}+\vec{a}}{\mu+1}$$

$$\Rightarrow \left(\frac{\lambda}{\lambda+1}\right)\vec{c}=\left(\frac{\mu}{\mu+1}\right)\vec{d}+\left(\frac{1}{\mu+1}\right)\vec{a}$$

$$\Rightarrow \left(\frac{\lambda}{\lambda+1}\right)\vec{c}=\left(\frac{\mu}{\mu+1}\right)\left(\frac{-\vec{b}+2\vec{c}}{3}\right)+\left(\frac{1}{\mu+1}\right)2\left(\vec{b}-\vec{c}\right) \quad \text{(using (i) and (ii))}$$

$$\Rightarrow \left(\frac{\lambda}{\lambda+1}\right)\vec{c}=\left(\frac{6-\mu}{3(\mu+1)}\right)\vec{b}+\left(\frac{2\mu}{3(\mu+1)}-\frac{2}{\mu+1}\right)\vec{c}$$

$$\Rightarrow \left(\frac{\lambda}{\lambda+1}\right)\vec{c}=\left(\frac{6-\mu}{3(\mu+1)}\right)\vec{b}+\left(\frac{2\mu-6}{3(\mu+1)}\right)\vec{c}$$

$$\Rightarrow \left(\frac{6-\mu}{3(\mu+1)}\right)\vec{b}+\left(\frac{2\mu-6}{3(\mu+1)}-\frac{\lambda}{\lambda+1}\right)\vec{c}=\vec{0}$$

$$\Rightarrow \frac{6-\mu}{3(\mu+1)}=0 \text{ and } \frac{2\mu-6}{3(\mu+1)}-\frac{\lambda}{\lambda+1}=0 \text{ (as } \vec{b} \text{ and } \vec{c} \text{ are non-collinear)}$$

$$\Rightarrow \mu=6,\ \lambda=\frac{2}{5}$$

Hence $OX : XC = 2 : 5$

For Problems 6–7

6. c., 7. d.

Sol.

Consider the regular hexagon $ABCDEF$ with centre at O (origin).

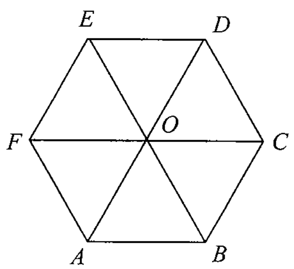


Fig. 1.63

$$\overrightarrow{AD}+\overrightarrow{EB}+\overrightarrow{FC}=2\overrightarrow{AO}+2\overrightarrow{OB}+2\overrightarrow{OC}$$
$$=2(\overrightarrow{AO}+\overrightarrow{OB})+2\overrightarrow{OC}$$

$= 2\overrightarrow{AB} + 2\overrightarrow{AB}$ $(\because \overrightarrow{OC} = \overrightarrow{AB})$

$= 4\overrightarrow{AB}$

$\vec{R} = \overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF}$

$= \overrightarrow{ED} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{CD}$ $(\because \overrightarrow{AB} = \overrightarrow{ED}$ and $\overrightarrow{AF} = \overrightarrow{CD})$

$= (\overrightarrow{AC} + \overrightarrow{CD}) + (\overrightarrow{AE} + \overrightarrow{ED}) + \overrightarrow{AD}$

$= \overrightarrow{AD} + \overrightarrow{AD} + \overrightarrow{AD} = 3\overrightarrow{AD} = 6\overrightarrow{AO}$

Matrix-Match Type

1. **a → p, r, s; b → q, r, s; c → p, r; d → r, s**
2. **a → q, r; b → p, r; c → q, s; d → p**

Sol.

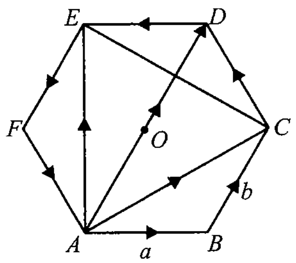


Fig. 1.64

$\overrightarrow{AB} = \vec{a}$, $\overrightarrow{BC} = \vec{b}$

$\therefore \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \vec{a} + \vec{a}$ (i)

$\overrightarrow{AD} = 2\overrightarrow{BC} = 2\vec{b}$ (ii)

(because AD is parallel to BC and twice its length).

$\overrightarrow{CD} = \overrightarrow{AD} - \overrightarrow{AC} = 2\vec{b} - (\vec{a} + \vec{b})$

$= \vec{b} - \vec{a}$

$\overrightarrow{FA} = -\overrightarrow{CD} = \vec{a} - \vec{b}$ (iii)

$\overrightarrow{DE} = -\overrightarrow{AB} = -\vec{a}$ (iv)

$\overrightarrow{EF} = -\overrightarrow{BC} = -\vec{b}$ (v)

$\overrightarrow{AE} = \overrightarrow{AD} + \overrightarrow{DE} = 2\vec{b} - \vec{a}$ (vi)

$\overrightarrow{CE} = \overrightarrow{CD} + \overrightarrow{DE} = \vec{b} - \vec{a} - \vec{a} = \vec{b} - 2\vec{a}$ (vii)

Integer Answer Type

1. **(2)** L.H.S. $= \vec{d} - \vec{a} + \vec{d} - \vec{b} + \vec{h} - \vec{c} + 3(\vec{g} - \vec{h})$

$$= 2\vec{d} - (\vec{a} + \vec{b} + \vec{c}) + 3\frac{(\vec{a} + \vec{b} + \vec{c})}{3} - 2\vec{h}$$

$$= 2\vec{d} - 2\vec{h} = 2(\vec{d} - \vec{h}) = 2\overrightarrow{HD}$$

$\Rightarrow \quad \lambda = 2$

2. **(6)** Let $\vec{R}$ be the resultant

Then $\vec{R} = \vec{F_1} + \vec{F_2} + \vec{F_3} = (p+1)\hat{i} + 4\hat{j}$

Given, $|\vec{R}| = 5$, therefore $R^2 = 25$

$\because \quad (p + 1)^2 + 16 = 25 \Rightarrow p + 1 = \pm 3$

$\therefore \quad p = 2, -4$

3. **(3)** Given, $\vec{a} + \vec{b} = \vec{c}$

Now vector $\vec{c}$ is along the diagonal of the parallelogram which has adjacent side vectors $\vec{a}$ and $\vec{b}$.

Since $\vec{c}$ is also a unit vector, triangle formed by vectors $\vec{a}$ and $\vec{b}$ is an equilateral triangle.

Then, area of triangle is $\frac{\sqrt{3}}{4}$.

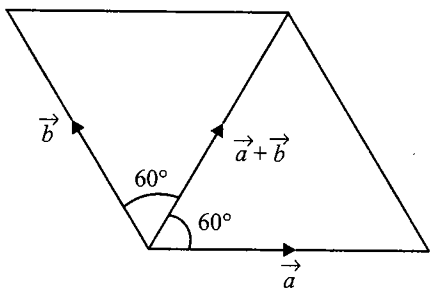


Fig. 1.65

4. **(2)** Let $\vec{a} = x\hat{i} - 3\hat{j} - \hat{k}$ and $\vec{b} = 2x\hat{i} + x\hat{j} - \hat{k}$ be the adjacent sides of the parallelogram.

Now angle between $\vec{a}$ and $\vec{b}$ is acute,

$\Rightarrow \quad |\vec{a} + \vec{b}| > |\vec{a} - \vec{b}|$

$\Rightarrow \quad \left|3x\hat{i} + (x-3)\hat{j} - 2\hat{k}\right|^2 > \left|-x\hat{i} - (x+3)\hat{j}\right|^2$

$\Rightarrow \quad 9x^2 + (x-3)^2 + 4 > x^2 + (x+3)^2$

$\Rightarrow \quad 8x^2 - 12x + 4 > 0$

$\Rightarrow \quad 2x^2 - 3x + 1 > 0$

$\Rightarrow \quad (2x-1)(x-1) > 0$

$\Rightarrow \quad x < 1/2 \quad \text{or} \quad x > 1$

Hence the least positive integral value is 2

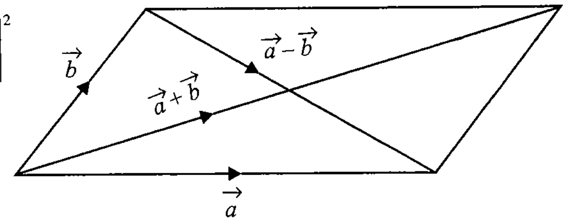


Fig. 1.66

5. **(7)** Vectors along the sides are $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} + \hat{k}$

Clearly the vector along the longer diagonal is $\vec{a} + \vec{b} = 3\hat{i} + 6\hat{j} + 2\hat{k}$

Hence length of the longer diagonal is $|\vec{a} + \vec{b}| = |3\hat{i} + 6\hat{j} + 2\hat{k}| = 7$

6. **(9)** Vector $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}, \vec{b} = 2\hat{i} - \hat{j} + \hat{k}, \vec{c} = \lambda\hat{i} + \hat{j} + 2\hat{k}$ are coplanar

$$\Rightarrow \quad \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ \lambda & 1 & 2 \end{vmatrix} = 0$$

$\Rightarrow \quad \lambda - 3 + 2(-5) = 0$

$\Rightarrow \quad \lambda = 13$

Archives

Subjective Type

1. $(\hat{i} + \hat{j} + 3\hat{k})x + (3\hat{i} - 3\hat{j} + \hat{k})y + (-4\hat{i} + 5\hat{j})z = \lambda(x\hat{i} + y\hat{j} + z\hat{k})$

Comparing coefficient of $\hat{i}$, $x + 3y - 4z = \lambda x$

$\Rightarrow (1 - \lambda)x + 3y - 4z = 0$ (i)

Comparing coefficient of $\hat{j}$, $x - 3y + 5z = \lambda y$

$\Rightarrow x - (3 + \lambda)y + 5z = 0$ (ii)

Comparing coefficient of $\hat{k}$, $3x + y + 0z = \lambda z$

$3x + y - \lambda z = 0$ (iii)

All the above three equations are satisfied for x, y and z not all zero if

$$\begin{vmatrix} 1-\lambda & 3 & -4 \\ 1 & -(3+\lambda) & 5 \\ 3 & 1 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)[3\lambda + \lambda^2 - 5] - 3[-\lambda - 15] - 4[1 + 9 + 3\lambda] = 0$

$\Rightarrow \lambda^3 + 2\lambda^2 + \lambda = 0$

$\Rightarrow \lambda(\lambda + 1)^2 = 0$

$\Rightarrow \lambda = 0, -1$

2. Since vector $\vec{A}$ has components A_1, A_2 and A_3, in the coordinate system $OXYZ$,

$\vec{A} = \hat{i}A_1 + \hat{j}A_2 + \hat{k}A_3$

When given system is rotated through $\pi/2$, the new x-axis is along the old y-axis and the new y-axis is along the old negative x-axis; z remains same as before.

Hence the components of A in the new system are A_2, $-A_1$ and A_3.

Therefore, $\vec{A}$ becomes $A_2\hat{i} - A_1\hat{j} + A_3\hat{k}$.

3. Given that P.V.'s of points A, B, C and D are $3\hat{i} - 2\hat{j} - \hat{k}$, $2\hat{i} + 3\hat{j} - 4\hat{k}$, $-\hat{i} + \hat{j} + 2\hat{k}$ and $4\hat{i} + 5\hat{j} + \lambda\hat{k}$, respectively.

Given that A, B, C and D lie in a plane if $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar. Therefore,

$$\begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & 1+\lambda \end{vmatrix} = 0$$

$\Rightarrow -1(3 + 3\lambda - 21) - 5(-4 - 4\lambda - 3) - 3(-28 - 3) = 0$

$\Rightarrow -3\lambda + 18 + 20\lambda + 35 + 93 = 0$

$\Rightarrow 17\lambda = -146$

$\Rightarrow \lambda = -\dfrac{146}{17}$

4. $OACB$ is a parallelogram with O as origin. Let with respect to O, position vectors of A and B be $\vec{a}$ and $\vec{b}$, respectively. Then P.V. of C is $\vec{a} + \vec{b}$.

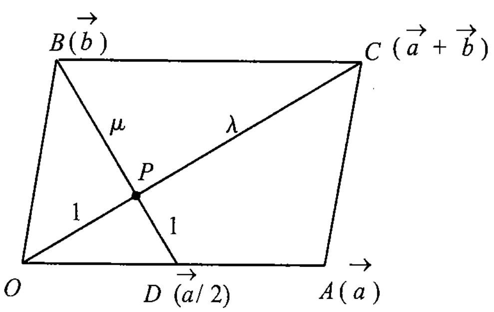


Fig. 1.67

Also D is the midpoint of OA; therefore, the position vector of D is $\vec{a}/2$.
CO and BD intersect each other at P.
Let P divide CO in the ratio $\lambda : 1$ and BD in the ratio $\mu : 1$. Then by section theorem, position vector of point P dividing CO in ratio $\lambda : 1$ is

$$\frac{\lambda \times 0 + 1 \times (\vec{a}+\vec{b})}{\lambda+1} = \frac{\vec{a}+\vec{b}}{\lambda+1} \quad \text{(i)}$$

and position vector of point P dividing BD in the ratio $\mu : 1$ is

$$\frac{\mu\left(\frac{\vec{a}}{2}\right)+1(\vec{b})}{\mu+1} = \frac{\mu\vec{a}+2\vec{b}}{2(\mu+1)} \quad \text{(ii)}$$

As (i) and (ii) represent the position vector of the same point, hence

$$\frac{\vec{a}+\vec{b}}{\lambda+1} = \frac{\mu\vec{a}+2\vec{b}}{2(\mu+1)}$$

Equating the coefficients of $\vec{a}$ and $\vec{b}$, we get

$$\frac{1}{\lambda+1} = \frac{\mu}{2(\mu+1)} \quad \text{(iii)}$$

$$\frac{1}{\lambda+1} = \frac{1}{\mu+1} \quad \text{(iv)}$$

From (iv) we get $\lambda = \mu \Rightarrow P$ divides CO and BD in the same ratio.
Putting $\lambda = \mu$ in Eq. (iii), we get $\mu = 2$
Thus the required ratio is 2 : 1.

5. Let the vertices of the triangle be $A(\vec{0}), B(\vec{b})$ and $C(\vec{c})$.

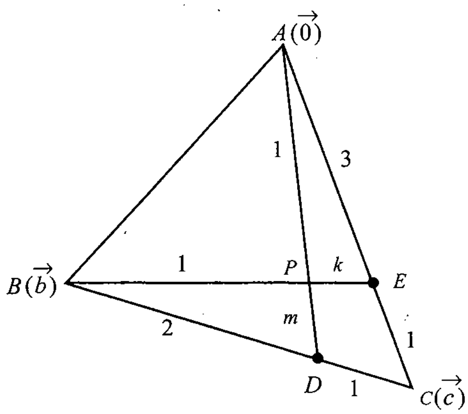


Fig. 1.68

Given that D divides BC in the ratio 2 : 1.

Therefore, position vector of D is $\dfrac{\vec{b}+2\vec{c}}{3}$.

E divides AC in the ratio 3 : 1.

Therefore, position vector of E is $\dfrac{\vec{0}+3\vec{c}}{4}=\dfrac{3\vec{c}}{4}$.

Let point of intersection P of AD and BE divide BE in the ratio $1 : k$ and AD in the ratio $1 : m$. Then position vectors of P in these two cases are $\dfrac{k\vec{b}+1(3\vec{c}/4)}{k+1}$ and $\dfrac{m\vec{0}+m\ ((\vec{b}+2\vec{c})/3)}{m+1}$, respectively.

Equating the position vectors of P in these two cases, we get

$$\frac{k\vec{b}}{k+1}+\frac{3\vec{c}}{4(k+1)}=\frac{m\vec{b}}{3(m+1)}+\frac{2m\vec{c}}{3(m+1)}$$

$$\Rightarrow \frac{k}{k+1}=\frac{m}{3(m+1)} \text{ and } \frac{3}{4(k+1)}=\frac{2m}{3(m+1)}$$

Dividing, we have $\dfrac{4k}{3}=\dfrac{1}{2}\Rightarrow k=\dfrac{3}{8}$

Required ratio is 8 : 3.

6. Let the P.V.s of the points A, B, C and D be $\vec{O}$, $B(\vec{b})$, $D(\vec{d})$ and $C(\vec{d}+t\vec{b})$

For any point $\vec{r}$ on $\overrightarrow{AC}$ and $\overrightarrow{BD}$, $\vec{r}=\lambda\ (\vec{d}+t\vec{b})$ and $\vec{r}=(1-\mu)\ \vec{b}+\mu\vec{d}$, respectively.

For the point of intersection, say T, compare the coefficients.

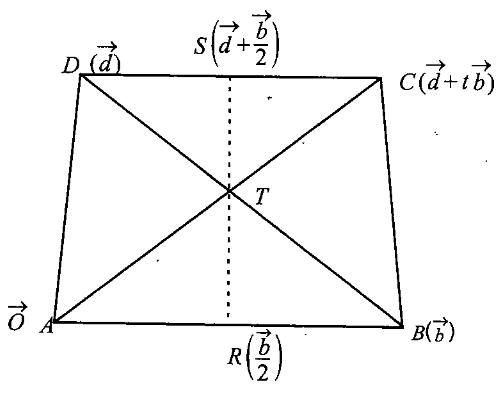


Fig. 1.69

$\lambda=\mu,\ t\lambda=1-\mu=1-\lambda$ or $(t+1)\lambda=1$

$\therefore \lambda = \dfrac{1}{t+1} = \mu$

Therefore, $\vec{r}$ (position vector of T) is $\dfrac{\vec{d}+t\vec{b}}{t+1}$. (i)

Let R and S be the midpoints of the parallel sides AB and DC; then R is $\dfrac{b}{2}$ and S is $d + t\dfrac{b}{2}$.
Let T divide SR in the ratio m:1.

Position vector of T is $\dfrac{m\dfrac{\vec{b}}{2}+\vec{d}+t\dfrac{\vec{b}}{2}}{m+1}$, which is equivalent to $\dfrac{\vec{d}+t\vec{b}}{t+1}$.

Comparing coefficients of $\vec{b}$ and $\vec{d}$, $\dfrac{1}{m+1} = \dfrac{1}{t+1}$ and $\dfrac{m+t}{2(m+1)} = \dfrac{t}{t+1}$.

From the first relation, $m = t$, which satisfies the second relation. Hence proved.

7. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be the position vectors of A, B and C, respectively.

Let AD, BE and CF be the bisectors of $\angle A$, $\angle B$ and $\angle C$, respectively.

a, b and c are the lengths of sides BC, CA and AB, respectively.

Now AD divides BC in the ratio $BD : DC = AB : AC = c : b$.

Hence, the position vector of D is $\vec{d} = \dfrac{b\vec{b}+c\vec{c}}{b+c}$.

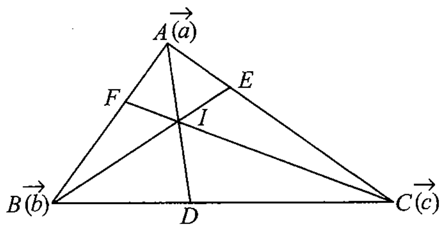


Fig. 1.70

Let I be the point of intersection of BE and AD.

Then in ΔABC, BI is bisector of $\angle B$. Therefore,
$DI : IA = BD : BA$

But $\dfrac{BD}{DC} = \dfrac{c}{b} \Rightarrow \dfrac{BD}{BD+DC} = \dfrac{c}{c+b}$

$\Rightarrow \dfrac{BD}{BC} = \dfrac{c}{c+b}$

$\Rightarrow BD = \dfrac{ac}{b+c}$

$\therefore\ DI : IA = \dfrac{ac}{b+c} : c = a : (b + c)$

$\therefore$ P.V. of $I = \dfrac{a\vec{a}+\vec{d}(b+c)}{a+b+c}$

$$= \frac{a\vec{a}+\left(\dfrac{b\vec{b}+c\vec{c}}{b+c}\right)(b+c)}{a+b+c} = \frac{a\vec{a}+b\vec{b}+c\vec{c}}{a+b+c}$$

As P.V. of I is symmetrical in $\vec{a}$, $\vec{b}$, $\vec{c}$ and a, b, c, it must lie on CF as well.

8. $\vec{A}(t)$ is parallel to $\vec{B}(t)$ for some $t \in [0, 1]$ if and only if $\dfrac{f_1(t)}{g_1(t)} = \dfrac{f_2(t)}{g_2(t)}$ for some $t \in [0, 1]$

or $f_1(t)\cdot g_2(t) = f_2(t)g_1(t)$ for some $t \in [0, 1]$

Let $h(t) = f_1(t)\cdot g_2(t) - f_2(t)\cdot g_1(t)$

$h(0) = f_1(0)\cdot g_2(0) - f_2(0)\cdot g_1(0)$

$= 2\times 2 - 3\times 3 = -5 < 0$

$h(1) = f_1(1)\cdot g_2(1) - f_2(1)\cdot g_1(1)$

$= 6\times 6 - 2\times 2 = 32 > 0$

Since h is a continuous function, and $h(0)\cdot h(1) < 0$, there are some $t \in [0, 1]$ for which $h(t) = 0$, i.e., $\vec{A}(t)$ and $\vec{B}(t)$ are parallel vectors for this t.

9. With O as origin let $\vec{a}$ and $\vec{b}$ be the position vectors of A and B, respectively.

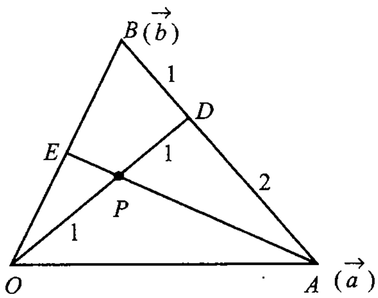


Fig. 1.71

Then the position vector of E, the midpoint of OB, is $\vec{b}/2$.

Again since $AD : DB = 2 : 1$, the position vector of D is

$$\frac{1\cdot\vec{a}+2\vec{b}}{1+2} = \frac{\vec{a}+2\vec{b}}{3}$$

Let $\dfrac{OP}{OD} = \dfrac{1}{\lambda}$

$\Rightarrow$ P.V. of P is $\dfrac{\vec{a}+2\vec{b}}{3(\lambda+1)}$

Let $\dfrac{AP}{PE} = \dfrac{1}{\mu}$

$\Rightarrow$ P.V. of P is $\dfrac{\mu\vec{a}+\dfrac{\vec{b}}{2}}{\mu+1}$

Comparing P.V. of P, we have

$$\frac{1}{3(\lambda+1)} = \frac{\mu}{\mu+1} \text{ and } \frac{2}{3(\lambda+1)} = \frac{1}{2(\mu+1)}$$

Dividing $\mu = \dfrac{1}{4} \Rightarrow \lambda = \dfrac{2}{3}$

$$\Rightarrow \frac{OP}{PA} = \frac{3}{2}$$

Objective Type

Fill in the blanks

1. Given that $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$

$$\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

Operating $C_2 \leftrightarrow C_3$ and then $C_1 \leftrightarrow C_2$ in first determinant

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$$(1 + abc)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

either $1 + abc = 0$ or $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$

Also given that vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ are non-coplanar.

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0$$

So we must have $1 + abc = 0$

$abc = -1$

2. Given that the vectors $\vec{u} = a\hat{i} + \hat{j} + \hat{k}$, $\vec{v} = \hat{i} + b\hat{j} + \hat{k}$ and $\vec{w} = \hat{i} + \hat{j} + c\hat{k}$, where $a, b, c \neq 1$ are coplanar. Therefore,

$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

Operating $C_1 \rightarrow C_1 - C_2$, $C_2 \rightarrow C_2 - C_3$

$$\begin{vmatrix} a-1 & 0 & 1 \\ 1-b & b-1 & 1 \\ 0 & 1-c & c \end{vmatrix} = 0$$

Expanding

$c(a-1)(b-1)+(1-b)(1-c)-(1-c)(a-1)=0$

$$\frac{c}{1-c}+\frac{1}{1-a}+\frac{1}{1-b}=0$$

$$\frac{c}{1-c}+1+\frac{1}{1-a}+\frac{1}{1-b}=1$$

$$\frac{1}{1-c}+\frac{1}{1-a}+\frac{1}{1-b}=1$$

True or false

1. Let position vectors of points A, B and C be $\vec{a}+\vec{b}, \vec{a}-\vec{b}$ and $\vec{a}+k\vec{b}$, respectively.

   Then $\overrightarrow{AB}=(\vec{a}-\vec{b})-(\vec{a}+\vec{b})=-2\vec{b}$

   Similarly, $\overrightarrow{BC}=(\vec{a}+k\vec{b})-(\vec{a}-\vec{b})=(k+1)\vec{b}$

   Clearly $\overrightarrow{AB} \parallel \overrightarrow{BC} \quad \forall k \in \mathrm{R}$

   $\Rightarrow A, B$ and C are collinear $\quad \forall k \in R$

   Therefore, the statement is true.

Multiple choice questions with one correct answer

1. **a.** Three points $A(\vec{a}), B(\vec{b}), C(\vec{c})$ are collinear if $\overline{AB} \parallel \overline{AC}$

   $\overrightarrow{AB}=-20\hat{i}-11\hat{j};\ \overrightarrow{AC}=(a-60)\hat{i}-55\hat{j}$

   $$\Rightarrow \overrightarrow{AB} \parallel \overrightarrow{AC} \Rightarrow \frac{a-60}{-20}=\frac{-55}{-11} \Rightarrow a=-40$$

2. **b.** a, b and c are distinct negative numbers and vectors $a\hat{i}+a\hat{j}+c\hat{k},\ \hat{i}+\hat{k}$ and $c\hat{i}+c\hat{j}+b\hat{k}$ are coplanar

   $$\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix}=0$$

   $\Rightarrow ac+c^2-ab-ac=0$

   $\Rightarrow c^2=ab$

   $\Rightarrow a, c, b$ are in G.P.

   So c is the G.M. of a and b.

3. **c.** $\vec{a}=\hat{i}-\hat{k}$

   $\vec{b}=x\hat{i}+\hat{j}+(1-x)\hat{k}$

   $\vec{c}=y\vec{i}+x\vec{j}+(1+x-y)\vec{k}$

$$= \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1+x-y \end{vmatrix}$$

$= (1 + x - y - x + x^2) - 1(x^2 - y)$
$= 1$

4. b. Let the given position vectors be of points A, B and C, respectively. Then

$$|\overrightarrow{AB}| = \sqrt{(\beta-\alpha)^2 + (\gamma-\beta)^2 + (\alpha-\gamma)^2}$$

$$|\overrightarrow{BC}| = \sqrt{(\gamma-\beta)^2 + (\alpha-\gamma)^2 + (\alpha-\beta)^2}$$

$$|\overrightarrow{CA}| = \sqrt{(\alpha-\gamma)^2 + (\beta-\alpha)^2 + (\gamma-\beta)^2}$$

$\because |\overrightarrow{AB}| = |\overrightarrow{BC}| = |\overrightarrow{CA}|$

Hence, ΔABC is an equilateral triangle.

5. c. We know that three vectors are coplanar if their scalar triple product is zero.

$$\Rightarrow \begin{vmatrix} -\lambda^2 & 1 & 1 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{vmatrix} = 0 \quad R_1 \to R_1 + R_2 + R_3$$

$$\Rightarrow \begin{vmatrix} 2-\lambda^2 & 2-\lambda^2 & 2-\lambda^2 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda^2)\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda^2)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -(1+\lambda^2) & 0 \\ 0 & 0 & -(1+\lambda^2) \end{vmatrix} = 0 \qquad (R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$$

$\Rightarrow (2 - \lambda^2)(1 + \lambda^2)^2 = 0 \Rightarrow \lambda = \pm\sqrt{2}$
Hence two real solutions.

6. d. Given that $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 4\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{c} = \hat{i} + \alpha\hat{j} + \beta\hat{k}$ are linearly dependent,

$$\begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 1 & \alpha & \beta \end{vmatrix} = 0$$

$\Rightarrow 1 - \beta = 0$

$\Rightarrow \beta = 1$

Also given that $|\vec{c}| = \sqrt{3} \Rightarrow 1 + \alpha^2 + \beta^2 = 3$

Substituting the value of β, we get $\alpha^2 = 1$

$\Rightarrow \alpha = \pm 1$

CHAPTER

2

Different Products of Vectors and Their Geometrical Applications

- Dot (Scalar) Product
- Applications of Dot (Scalar) Product
- Vector (or Cross) Product of Two Vectors
- Scalar Triple Product
- Vector Triple Product
- Reciprocal System of Vectors

DOT (SCALAR) PRODUCT

The scalar product of vectors $\vec{a}$ and $\vec{b}$, written as $\vec{a}\cdot\vec{b}$, is defined to be the number $|\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

i.e., $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where $0 \le \theta \le \pi$.

Notes:

1. *$\vec{a}\cdot\vec{b}$ is positive if θ is acute.*
2. *$\vec{a}\cdot\vec{b}$ is negative if θ is obtuse.*
3. *$\vec{a}\cdot\vec{b}$ is zero if θ is a right angle.*

Physical Interpretation of Scalar Product

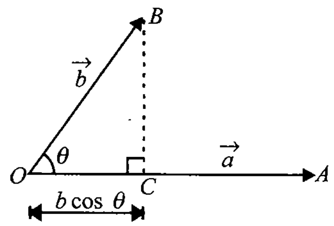


Fig. 2.1

Let $\overrightarrow{OA} = \vec{a}$ represent a force acting on a particle at O and let $\overrightarrow{OB} = \vec{b}$ represent the displacement of the particle from O to B as shown in the figure. Then the displacement in the direction of the force $= OC = b\cos\theta$.

Therefore the work done by a force is a scalar quantity equal to the product of the magnitude of the force and the resolved part of the displacement in the direction of force work done by force $\vec{a}$ in moving its point of application from O to $B = |\vec{a}||\vec{b}|\cos\theta = \vec{a}\cdot\vec{b}$.

Geometrical Interpretation of Scalar Product

Let $\vec{a}$ and $\vec{b}$ be two vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$, respectively.

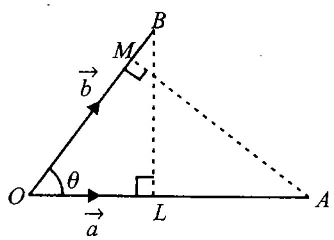


Fig. 2.2

Here OL and OM are known as projections of $\vec{b}$ on $\vec{a}$ and $\vec{a}$ on $\vec{b}$, respectively.

Now, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$= |\vec{a}|(OB\cos\theta)$

$= |\vec{a}|(OL)$

$=$ (magnitude of $\vec{a}$) (projection of $\vec{b}$ on $\vec{a}$) (i)

Again, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$= |\vec{b}|(|\vec{a}|\cos\theta)$

$= |\vec{b}|(OA\cos\theta)$

$= |\vec{b}|(OM)$

$=$ (magnitude of $\vec{b}$) (projection of $\vec{a}$ on $\vec{b}$) (ii)

Thus, geometrically interpreted, the scalar product of two vectors is the product of modulus of either vectors and the projection of the other in its direction.

Thus projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \vec{a}\cdot\dfrac{\vec{b}}{|\vec{b}|} = \vec{a}\cdot\hat{b}$

Projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \dfrac{\vec{a}}{|\vec{a}|}\cdot\vec{b} = \hat{a}\cdot\vec{b}$

Properties of Dot (Scalar) Product

i. $\vec{a}\cdot\vec{a} = |\vec{a}||\vec{a}|\cos 0° = |\vec{a}|^2 = a^2 \Rightarrow \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

ii. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (commutative)

iii. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ (distributive)

Proof:

Let $\vec{OA} = \vec{a}, \vec{OB} = \vec{b}, \vec{BC} = \vec{c}$ so that

$\vec{OC} = \vec{OB} + \vec{BC} = \vec{b} + \vec{c}$

From B draw $BM \perp OA$ and from C, drawn $CN \perp OA$

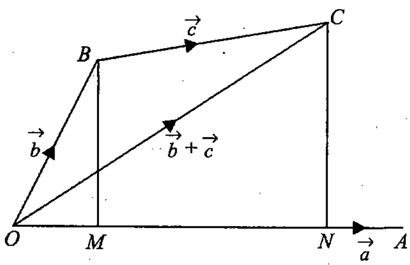


Fig. 2.3

L.H.S. $= \vec{a}\cdot(\vec{b}+\vec{c})$

$= \vec{OA}\cdot\vec{OC}$

$= (OA)(OC)\cos\theta$ (where $\theta = \angle CON$)

$= (OA)(ON)$ (as $ON = OC\cos\theta$)

$= (OA)(OM + MN)$

$= (OA)(OM) + (OA)(MN)$

$= \vec{OA} \cdot \vec{OB} + \vec{OA} \cdot \vec{BC}$

$= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ =R.H.S.

iv. $(l\vec{a}) \cdot (m\vec{b}) = lm(\vec{a} \cdot \vec{b})$, where l and m are scalars

v. If $\vec{a}$ and $\vec{b}$ are two non-zero vectors, then $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a}$ and $\vec{b}$ are perpendicular to each other

$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

vi. $(\vec{a} \pm \vec{b})^2 = (\vec{a} \pm \vec{b}) \cdot (\vec{a} \pm \vec{b})$

$= |\vec{a}|^2 + |\vec{b}|^2 \pm 2\vec{a} \cdot \vec{b}$

$= |\vec{a}|^2 + |\vec{b}|^2 \pm 2|\vec{a}||\vec{b}|\cos\theta$

vii. $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$

viii. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$(\because \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ and $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0)$

ix. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Taking dot product with $\hat{i}, \hat{j}$ and $\hat{k}$ alternatively, we have

$x = \vec{r} \cdot \hat{i},\ y = \vec{r} \cdot \hat{j}$ and $z = \vec{r} \cdot \hat{k}$

$\Rightarrow \vec{r} = (\vec{r} \cdot \hat{i})\hat{i} + (\vec{r} \cdot \hat{j})\hat{j} + (\vec{r} \cdot \hat{k})\hat{k}$

APPLICATIONS OF DOT (SCALAR) PRODUCT

Finding Angle between Two Vectors

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are non-zero vectors, then the angle between them is given by

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\ \sqrt{b_1^2 + b_2^2 + b_3^2}}$$

Also

$$\frac{(a_1b_1 + a_2b_2 + a_3b_3)^2}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)} = \cos^2\theta \le 1$$

$$\Rightarrow \quad (a_1b_1 + a_2b_2 + a_3b_3)^2 \le (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

Cosine Rule Using Dot Product

Using vector method, prove that in a triangle $a^2 = b^2 + c^2 - 2bc\cos A$ (Cosine law)

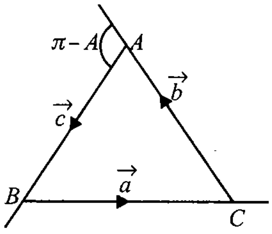


Fig. 2.4

In ΔABC,

Let $\overrightarrow{AB} = \vec{c}, \overrightarrow{BC} = \vec{a}, \overrightarrow{CA} = \vec{b}$,

Since $\vec{a} + \vec{b} + \vec{c} = 0$, we have $\vec{a} = -(\vec{b} + \vec{c})$

$\therefore |\vec{a}| = |-(\vec{b} + \vec{c})|$

$\Rightarrow |\vec{a}|^2 = |\vec{b} + \vec{c}|^2$

$\Rightarrow |\vec{a}|^2 = |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{b}\cdot\vec{c}$

$\Rightarrow |\vec{a}|^2 = |\vec{b}|^2 + |\vec{c}|^2 + 2|\vec{b}||\vec{c}|\cos(\pi - A)$

(Since angle between $\vec{b}$ and $\vec{c}$ = the angle between CA produced and AB)

$\Rightarrow a^2 = b^2 + c^2 - 2bc\cos A$

Finding Components of a Vector $\vec{b}$ Along and Perpendicular to Vector $\vec{a}$ or Resolving a Given Vector in the Direction of Given Two Perpendicular Vectors

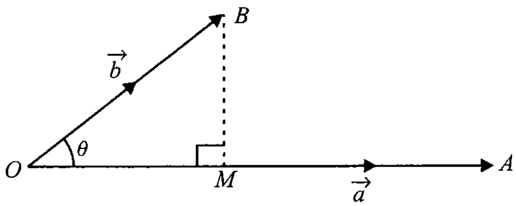


Fig. 2.5

Let $\vec{a}$ and $\vec{b}$ be two vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ and let θ be the angle between $\vec{a}$ and $\vec{b}$.

$\therefore \vec{b} = \overrightarrow{OM} + \overrightarrow{MB}$

Also $\overrightarrow{OM} = (OM)\hat{a}$

$= (OB\cos\theta)\hat{a}$

$= (|\vec{b}|\cos\theta)\hat{a}$

$$= \left(|\vec{b}| \frac{(\vec{a}\cdot\vec{b})}{|\vec{a}||\vec{b}|} \right) \hat{a}$$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} \right) \hat{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{a}|} \vec{a} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2} \right) \vec{a}$$

Also $\vec{b} = \overrightarrow{OM} + \overrightarrow{MB}$

$$\Rightarrow \overrightarrow{MB} = \vec{b} - \overrightarrow{OM} = \vec{b} - \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2} \right) \vec{a}$$

Thus, the components of $\vec{b}$ along and perpendicular to $\vec{a}$ are $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2} \right) \vec{a}$ and $\vec{b} - \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2} \right) \vec{a}$, respectively.

Example 2.1 **If $\vec{a}, \vec{b}$ and $\vec{c}$ are non-zero vectors such that $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$, then find the geometrical relation between the vectors.**

Sol. $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$

$\Rightarrow \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c} = \vec{0}$

$\Rightarrow \vec{a}\cdot(\vec{b} - \vec{c}) = \vec{0}$

$\Rightarrow$ Either $\vec{b} - \vec{c} = \vec{0}$ or $\vec{a} \perp (\vec{b} - \vec{c})$

$\Rightarrow \vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b} - \vec{c})$

Example 2.2 **If $\vec{r}\cdot\hat{i} = \vec{r}\cdot\hat{j} = \vec{r}\cdot\hat{k}$ and $|\vec{r}| = 3$, then find vector $\vec{r}$.**

Sol. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Since $\vec{r}\cdot\hat{i} = \vec{r}\cdot\hat{j} = \vec{r}\cdot\hat{k}$

$x = y = z$

Also $|\vec{r}| = \sqrt{x^2 + y^2 + z^2} = 3$

$\Rightarrow x = \pm\sqrt{3}$

Hence, the required vector $\vec{r} = \pm\sqrt{3}(\hat{i} + \hat{j} + \hat{k})$

Example 2.3 **If $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then find the value of $\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}$.**

Sol. Squaring $(\vec{a} + \vec{b} + \vec{c}) = \vec{0}$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{a}\cdot\vec{b} + 2\vec{b}\cdot\vec{c} + 2\vec{c}\cdot\vec{a} = 0$

$\Rightarrow 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = -3$

$\Rightarrow \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = -\frac{3}{2}$

Example 2.4 **If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular vectors of equal magnitudes, then find the angle between vectors $\vec{a}$ and $\vec{a}+\vec{b}+\vec{c}$.**

Sol. Since $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular, $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$

Angle between $\vec{a}$ and $\vec{a}+\vec{b}+\vec{c}$ is

$$\cos\theta=\frac{\vec{a}\cdot(\vec{a}+\vec{b}+\vec{c})}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|} \quad \text{(i)}$$

Now $|\vec{a}|=|\vec{b}|=|\vec{c}|=a$

$$|\vec{a}+\vec{b}+\vec{c}|^2=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\vec{a}\cdot\vec{b}+2\vec{b}\cdot\vec{c}+2\vec{c}\cdot\vec{a}$$

$$=a^2+a^2+a^2+0+0+0$$

$\Rightarrow |\vec{a}+\vec{b}+\vec{c}|^2=3a^2$

$\Rightarrow |\vec{a}+\vec{b}+\vec{c}|=\sqrt{3}a$

Putting this value in (i), we get $\theta=\cos^{-1}\frac{1}{\sqrt{3}}$

Example 2.5 **If $|\vec{a}|+|\vec{b}|=|\vec{c}|$ and $\vec{a}+\vec{b}=\vec{c}$, then find the angle between $\vec{a}$ and $\vec{b}$.**

Sol. $\vec{a}+\vec{b}=\vec{c}$

$$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}=|\vec{c}|^2 \quad \text{(i)}$$

and $|\vec{a}|+|\vec{b}|=|\vec{c}|$

$$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+2|\vec{a}||\vec{b}|=|\vec{c}|^2 \quad \text{(ii)}$$

$\therefore\ \vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|$ (from (i) and (ii))

$\Rightarrow \cos\theta=1\Rightarrow\theta=0°$

Example 2.6 **If three unit vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, then find the angle between $\vec{a}$ and $\vec{b}$.**

Sol. $\vec{a}+\vec{b}=-\vec{c}$

$\Rightarrow |\vec{a}+\vec{b}|^2=|\vec{c}|^2=1$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}=1$

$\Rightarrow \vec{a}\cdot\vec{b}=-\frac{1}{2}$

$\Rightarrow |\vec{a}||\vec{b}|\cos\theta=-\frac{1}{2}$

$\Rightarrow \cos\theta=-\frac{1}{2}$

$\Rightarrow \theta=\frac{2\pi}{3}$

Example 2.7 **If θ be the angle between the unit vectors $\vec{a}$ and $\vec{b}$, then prove that**

i. $\cos\dfrac{\theta}{2} = \dfrac{1}{2}|\vec{a}+\vec{b}|$

ii. $\sin\dfrac{\theta}{2} = \dfrac{1}{2}|\vec{a}-\vec{b}|$

Sol. i. $(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b}$

$= 1 + 1 + 2(1)(1)\cos\theta$

$= 2 + 2\cos\theta$

$\Rightarrow |\vec{a}+\vec{b}|^2 = 2\cdot 2\cos^2\dfrac{\theta}{2}$

$\Rightarrow \cos\dfrac{\theta}{2} = \dfrac{1}{2}|\vec{a}+\vec{b}|$

ii. $(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\cdot\vec{b}$

$= 1 + 1 - 2(1)(1)\cos\theta$

$= 2 - 2\cos\theta$

$\Rightarrow |\vec{a}-\vec{b}|^2 = 2\cdot 2\sin^2\dfrac{\theta}{2}$

$\Rightarrow \sin\dfrac{\theta}{2} = \dfrac{1}{2}|\vec{a}+\vec{b}|$

Example 2.8 **If the scalar projection of vector $x\hat{i}-\hat{j}+\hat{k}$ on vector $2\hat{i}-\hat{j}+5\hat{k}$ is $\dfrac{1}{\sqrt{30}}$, then find the value of x.**

Sol. Projection of $x\hat{i}-\hat{j}+\hat{k}$ on $2\hat{i}-\hat{j}+5\hat{k}$ $= \dfrac{(x\hat{i}-\hat{j}+\hat{k})\cdot(2\hat{i}-\hat{j}+5\hat{k})}{\sqrt{4+1+25}}$

$= \dfrac{2x+1+5}{\sqrt{30}}$

But, given $\dfrac{2x+6}{\sqrt{30}} = \dfrac{1}{\sqrt{30}}$ $\Rightarrow 2x+6 = 1 \Rightarrow x = \dfrac{-5}{2}$

Example 2.9 **If $\vec{a} = x\hat{i} + (x-1)\hat{j} + \hat{k}$ and $\vec{b} = (x+1)\hat{i} + \hat{j} + a\hat{k}$ make an acute angle $\forall\ x \in R$, then find the values of a.**

Sol. $\vec{a}\cdot\vec{b} = (x\hat{i}+(x-1)\hat{j}+\hat{k})\cdot((x+1)\hat{i}+\hat{j}+a\hat{k})$

$= x(x+1)+x-1+a$

$= x^2+2x+a-1$

We must have $\vec{a}\cdot\vec{b} > 0\ \forall\ x \in R$

$\Rightarrow x^2 + 2x + a - 1 > 0\ \forall\ x \in R$

$\Rightarrow 4 - 4(a-1) < 0$

$\Rightarrow a > 2$

Example 2.10 **If $\vec{a}\cdot\hat{i} = \vec{a}\cdot(\hat{i}+\hat{j}) = \vec{a}\cdot(\hat{i}+\hat{j}+\hat{k})$, then find the unit vector $\vec{a}$.**

Sol. Let $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$

Then, $\vec{a}\cdot\hat{i} = (x\hat{i}+y\hat{j}+z\hat{k})\cdot\hat{i} = x$ and $\vec{a}\cdot(\hat{i}+\hat{j}) = x+y$

and $\vec{a}\cdot(\hat{i}+\hat{j}+\hat{k}) = x+y+z$ (given that $x = x+y = x+y+z$)

Now $x = x+y \Rightarrow y = 0$ and $x+y = x+y+z \Rightarrow z = 0$

Hence $x = 1$ (Since $\vec{a}$ is a unit vector)

$\therefore\ \vec{a} = \hat{i}$

Example 2.11 **Prove by vector method that $\cos(A+B) = \cos A\cos B - \sin A\sin B$.**

Sol. Let $\hat{i}$ and $\hat{j}$ be unit vectors along OX and OY, respectively.

Let $\vec{OP}, \vec{OQ}$ be two unit vectors drawn in the plane XOY such that

$\angle XOP = A,\ \angle XOQ = B$

$\therefore\ POQ = A + B$

Now $\vec{OP} = \hat{i}\cos A + \hat{j}\sin A$

$\vec{OQ} = \hat{i}\cos B - \hat{j}\sin B$

$\therefore\ \vec{OP}\cdot\vec{OQ} = \cos A\cos B - \sin A.\sin B$

$\Rightarrow\ (1)(1)\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\Rightarrow\ \cos(A+B) = \cos A\cos B - \sin A\sin B$

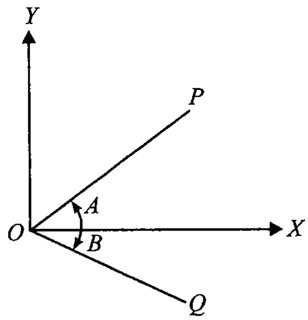


Fig. 2.6

Example 2.12 **In any triangle ABC, prove the projection formula $a = b\cos C + c\cos B$ using vector method.**

Sol. Let $\vec{BC} = \vec{a}, \vec{CA} = \vec{b}, \vec{AB} = \vec{c}$, so that

$BC = a,\ CA = b,\ AB = c$

Now $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\therefore\ \vec{a}\cdot(\vec{a}+\vec{b}+\vec{c}) = 0$

$\vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = 0$

$a^2 + ab\cos(180° - C) + ac\cos(180° - B) = 0$

$a^2 - ab\cos C - ac\cos B = 0$

$a - b\cos C - c\cos B = 0$

$a = b\cos C + c\cos B$

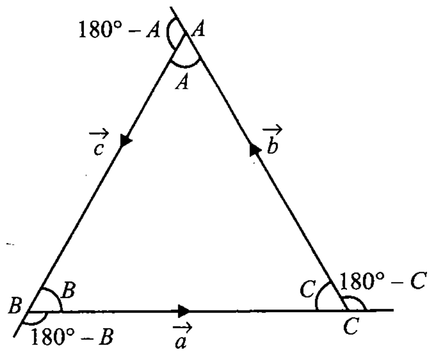


Fig. 2.7

Example 2.13 **Prove that an angle inscribed in a semi-circle is a right angle using vector method.**

Sol. Let O be the centre of the semi-circle and BA be the diameter. Let P be any point on the circumference of the semi-circle.

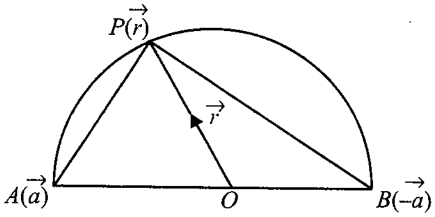


Fig. 2.8

Let $\vec{OA} = \vec{a}$, then $\vec{OB} = -\vec{a}$

Let $\vec{OP} = \vec{r}$

$\therefore$ $\vec{AP} = \vec{OP} - \vec{OA} = \vec{r} - \vec{a}$

$$\vec{BP} = \vec{OP} - \vec{OB} = \vec{r} - (-\vec{a}) = \vec{r} + \vec{a}$$

$$\vec{AP} \cdot \vec{BP} = (\vec{r} - \vec{a}) \cdot (\vec{r} + \vec{a})$$

$$= \vec{r}^2 - \vec{a}^2$$

$$= a^2 - a^2 \quad [\because r = a \text{ as } OP = OA]$$

$\therefore$ $\overrightarrow{AP}$ is perpendicular to $\overrightarrow{BP}$

$\Rightarrow \angle APB = 90°$

Example 2.14 **Using dot product of vectors, prove that a parallelogram, whose diagonals are equal, is a rectangle.**

Sol. Let $OACB$ be a parallelogram such that $OC = AB$

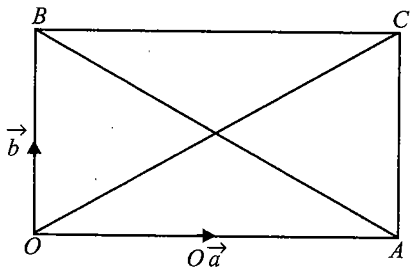


Fig. 2.9

Let $\vec{OA} = \vec{a}, \vec{OB} = \vec{b}$

Now $OC = AB$

$\Rightarrow \quad OC^2 = AB^2$

$$\Rightarrow \quad \left(\vec{OA} + \vec{AC}\right)^2 = \left(\vec{AO} + \vec{OB}\right)^2$$

$$\Rightarrow \quad \left(\vec{OA} + \vec{OB}\right)^2 = \left(-\vec{OA} + \vec{OB}\right)^2$$

$$\Rightarrow \quad (\vec{a} + \vec{b}) = (-\vec{a} + \vec{b})^2$$

$$\Rightarrow \quad \vec{a}^2 + \vec{b}^2 + 2\vec{a}\cdot\vec{b} = \vec{a}^2 + \vec{b}^2 - 2\vec{a}\cdot\vec{b}$$

$$\Rightarrow \quad 2\vec{a}\cdot\vec{b} = -2\vec{a}\cdot\vec{b}$$

$$\Rightarrow \quad 4\vec{a}\cdot\vec{b} = 0$$

$$\Rightarrow \quad \vec{a}\cdot\vec{b} = \vec{0}$$

$\Rightarrow \quad \vec{a}$ and $\vec{b}$ are perpendicular

$\Rightarrow \quad \angle AOB = 90°$

$\Rightarrow \quad OACB$ is a rectangle

Example 2.15 **If $a + 2b + 3c = 4$, then find the least value of $a^2 + b^2 + c^2$.**

Sol. Consider vectors $\vec{p} = a\hat{i} + b\hat{j} + c\hat{k}$ and $\vec{q} = \hat{i} + 2\hat{j} + 3\hat{k}$

$$\text{Now } \cos\theta = \frac{a + 2b + 3c}{\sqrt{a^2 + b^2 + c^2}\sqrt{1^2 + 2^2 + 3^2}}$$

$$\text{or} \qquad \cos^2\theta = \frac{(a + 2b + 3c)^2}{14(a^2 + b^2 + c^2)} \le 1$$

$$\Rightarrow \qquad a^2 + b^2 + c^2 \ge \frac{8}{7}$$

$\Rightarrow \qquad$ Hence least value of $a^2 + b^2 + c^2$ is $\frac{8}{7}$

Example 2.16 **Find a unit vector $\vec{a}$ which makes an angle of $\pi/4$ with the z-axis and it is such that $(\vec{a}+\hat{i}+\hat{j})$ is a unit vector.**

Sol. Let $\vec{a} = x\hat{i}+y\hat{j}+z\hat{k}$

Given $|\vec{a}|=1$, therefore

$x^2+y^2+z^2=1$ (i)

Angle between $\vec{a}$ and z-axis is $\pi/4$, therefore

$$\cos\left(\frac{\pi}{4}\right) = \frac{\vec{a}\cdot\hat{k}}{|\vec{a}||\hat{k}|}$$

$$\Rightarrow z = \frac{1}{\sqrt{2}}$$

Now $\vec{a}+\hat{i}+\hat{j} = (x+1)\hat{i}+(y+1)\hat{j}+z\hat{k}$

Given that $\vec{a}+\hat{i}+\hat{j}$ is a unit vector. Therefore,

$|\vec{a}+\hat{i}+\hat{j}| = \sqrt{[(x+1)^2+(y+1)^2+z^2]} = 1$

$\Rightarrow x^2+y^2+z^2+2x+2y+1=0$

$\Rightarrow 1+2x+2y+1=0$, using (i)

$\Rightarrow y = -(x+1)$

From (i), we have

$x^2+(x+1)^2+(1/2)=1$

$\Rightarrow 4x^2+4x+1=0$ or $(2x+1)^2=0$

$$x = -\frac{1}{2} \Rightarrow y = -\frac{1}{2}$$

Hence $\vec{a} = -\frac{1}{2}\hat{i}-\frac{1}{2}\hat{j}+\frac{1}{\sqrt{2}}\hat{k}$

Example 2.17 **Vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are of the same length and taken pair-wise they form equal angles. If $\vec{a}=\hat{i}+\hat{j}$ and $\vec{b}=\hat{j}+\hat{k}$, then find vector $\vec{c}$.**

Sol. Let $\vec{c} = x\hat{i}+y\hat{j}+z\hat{k}$. Then $|\vec{a}|=|\vec{b}|=|\vec{c}| \Rightarrow x^2+y^2+z^2=2$

It is given that the angles between the vectors taken in pairs are equal, say θ. Therefore,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{0+1+0}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$

$$\Rightarrow \frac{\vec{a}\cdot\vec{c}}{|\vec{a}||\vec{c}|} = \frac{1}{2} \text{ and } \frac{\vec{b}\cdot\vec{c}}{|\vec{b}||\vec{c}|} = \frac{1}{2}$$

$$\Rightarrow \frac{x+y}{\sqrt{2}\sqrt{2}} = \frac{1}{2} \text{ and } \frac{y+z}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$

$\Rightarrow x+y=1$ and $y+z=1$

$\Rightarrow y = 1-x$ and $z = 1-y = 1-(1-x) = x$

Also $x^2 + y^2 + z^2 = 2 \Rightarrow x^2 + (1-x)^2 + x^2 = 2$

$\Rightarrow (3x+1)(x-1) = 0 \Rightarrow x = 1, -1/3$

Now, $y = 1 - x \Rightarrow y = 0$ for $x = 1$ and $y = 4/3$ for $x = -1/3$

Hence, $\vec{c} = \hat{i} + 0\hat{j} + \hat{k}$ and $\vec{c} = -\frac{1}{3}\hat{i} + \frac{4}{3}\hat{j} - \frac{1}{3}\hat{k}$

Example 2.18 **If $\vec{a}, \vec{b}$ and $\vec{c}$ are three mutually perpendicular unit vectors and $\vec{d}$ is a unit vector which makes equal angles with $\vec{a}, \vec{b}$ and $\vec{c}$, then find the value of $|\vec{a}+\vec{b}+\vec{c}+\vec{d}|^2$.**

Sol. $|\vec{a}+\vec{b}+\vec{c}+\vec{d}|^2 = \Sigma|\vec{a}|^2 + 2\Sigma\vec{a}\cdot\vec{b} = 4 + 2\vec{d}\cdot(\vec{a}+\vec{b}+\vec{c})$ ($\because$ $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular)

Let $\vec{d} = \lambda\vec{a} + \mu\vec{b} + \nu\vec{c}$. Then $\vec{d}\cdot\vec{a} = \vec{d}\cdot\vec{b} = \vec{d}\cdot\vec{c} = \cos\theta$. Therefore,

$\lambda = \mu = \nu = \cos\theta$

Also $\lambda^2 + \mu^2 + \nu^2 = 1 \Rightarrow 3\cos^2\theta = 1 \Rightarrow \cos\theta = \pm\frac{1}{\sqrt{3}}$

$\therefore |\vec{a}+\vec{b}+\vec{c}+\vec{d}|^2 = 4 \pm \frac{2\cdot 3}{\sqrt{3}} = 4 \pm 2\sqrt{3}$

Example 2.19 **A particle acted by constant forces $4\hat{i} + \hat{j} - 3\hat{k}$ and $3\hat{i} + \hat{j} - \hat{k}$ is displaced from point $\hat{i} + 2\hat{j} + 3\hat{k}$ to point $5\hat{i} + 4\hat{j} + \hat{k}$. Find the total work done by the forces in units.**

Sol. Here $\vec{F} = \vec{F_1} + \vec{F_2} = (4\hat{i} + \hat{j} - 3\hat{k}) + (3\hat{i} + \hat{j} - \hat{k}) = 7\hat{i} + 2\hat{j} - 4\hat{k}$

and $\vec{d} = \vec{d_2} - \vec{d_1} = (5\hat{i} + 4\hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 4\hat{i} + 2\hat{j} - 2\hat{k}$

$\therefore$ Work done $= \vec{F}\cdot\vec{d}$

$= (7\hat{i} + 2\hat{j} - 4\hat{k})\cdot(4\hat{i} + 2\hat{j} - 2\hat{k})$

$= (7)(4) + (2)(2) + (-4)(-2)$

$= 28 + 4 + 8 = 40$ units

Example 2.20 **If $\vec{a} = 4\hat{i} + 6\hat{j}$ and $\vec{b} = 3\hat{j} + 4\hat{k}$, then find the component of $\vec{a}$ along $\vec{b}$.**

Sol. The component of vector $\vec{a}$ along $\vec{b}$ is $\frac{(\vec{a}.\vec{b})\vec{b}}{|\vec{b}|^2} = \frac{18}{25}(3\hat{i} + 4\hat{k})$

Example 2.21 **If $|\vec{a}| = |\vec{b}| = |\vec{a}+\vec{b}| = 1$, then find the value of $|\vec{a}-\vec{b}|$.**

Sol. We have

$|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2(|\vec{a}|^2 + |\vec{b}|^2)$

$\Rightarrow 1 + |\vec{a}-\vec{b}|^2 = 4 \Rightarrow |\vec{a}-\vec{b}| = \sqrt{3}$

Example 2.22 **If $\vec{a} = -\hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} + 0\hat{j} + \hat{k}$, then find vector $\vec{c}$ satisfying the following conditions: (i) that it is coplanar with $\vec{a}$ and $\vec{b}$, (ii) that it is $\perp$ to $\vec{b}$ and (iii) that $\vec{a}\cdot\vec{c} = 7$.**

Sol. Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$

Then from condition (i)

$$\begin{vmatrix} x & y & z \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = 0 \text{ or } x+3y-2z=0 \quad \text{(i)}$$

From condition (ii)

$2x+z=0$ (ii)

From condition (iii)

$-x+y+z=7$ (iii)

Solving (i), (ii) and (iii), we get the values of x, y and z and hence vector $\vec{c} = \frac{1}{2}(-3\hat{i}+5\hat{j}+6\hat{k})$

Example 2.23 **Let $\vec{a}, \vec{b}$ and $\vec{c}$ are vectors such that $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{c}| = 5$, and $(\vec{a}+\vec{b})$ is perpendicular to $\vec{c}$, $(\vec{b}+\vec{c})$ is perpendicular to $\vec{a}$ and $(\vec{c}+\vec{a})$ is perpendicular to $\vec{b}$. Then find the value of $|\vec{a}+\vec{b}+\vec{c}|$.**

Sol. Given, $(\vec{a}+\vec{b})\cdot\vec{c} = 0 \Rightarrow \vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c} = 0$

$(\vec{b}+\vec{c})\cdot\vec{a} = 0 \Rightarrow \vec{a}\cdot\vec{b}+\vec{c}\cdot\vec{a} = 0$

$(\vec{c}+\vec{a})\cdot\vec{b} = 0 \Rightarrow \vec{b}\cdot\vec{c}+\vec{a}\cdot\vec{b} = 0$

$\therefore 2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = 0$

Now, $|\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = 50$

$\Rightarrow |\vec{a}+\vec{b}+\vec{c}| = 5\sqrt{2}$

Example 2.24 **Prove that in a tetrahedron if two pairs of opposite edges are perpendicular, then the third pair is also perpendicular.**

Sol. Let $ABCD$ be the tetrahedron and A be at the origin.

Let $\overrightarrow{AB} = \vec{b}$, $\overrightarrow{AC} = \vec{c}$ and $\overrightarrow{AD} = \vec{d}$

Let the edge AB be perpendicular to the opposite edge CD.

$\Rightarrow \overrightarrow{AB}\cdot\overrightarrow{CD} = 0$

$\Rightarrow \vec{b}\cdot(\vec{d}-\vec{c}) = 0$

$\Rightarrow \vec{b}\cdot\vec{d} = \vec{b}\cdot\vec{c}$ (i)

Also let AC be perpendicular to the opposite edge BD. Therefore,

$\overrightarrow{AC}\cdot\overrightarrow{BD} = 0$

$\Rightarrow \vec{c}\cdot(\vec{d}-\vec{b}) = 0$

$\Rightarrow \vec{c}\cdot\vec{d} = \vec{b}\cdot\vec{c}$ (ii)

Now from (i) and (ii), we have

$\Rightarrow \vec{b}\cdot\vec{d} = \vec{c}\cdot\vec{d}$

$\Rightarrow (\vec{c}-\vec{b})\cdot\vec{d} = 0$

$\Rightarrow \overrightarrow{BC}\cdot\overrightarrow{AD} = 0$

$\Rightarrow AD$ is perpendicular to opposite edge BC.

Example 2.25 **In isosceles triangle ABC, $|\overrightarrow{AB}| = |\overrightarrow{BC}| = 8$, a point E divides AB internally in the ratio $1:3$, then find the angle between $\overrightarrow{CE}$ and $\overrightarrow{CA}$ (where $|\overrightarrow{CA}| = 12$).**

Sol.

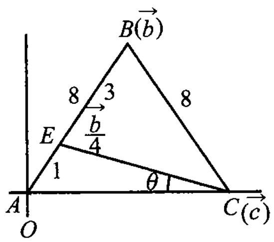


Fig. 2.10

Given $|\vec{c}| = 12$ and $|\vec{b}| = |\vec{b}-\vec{c}| = 8$

$\Rightarrow b^2 = b^2 + c^2 - 2\vec{b}\cdot\vec{c}$

$\Rightarrow \vec{b}\cdot\vec{c} = 72$

$$\cos\theta = \frac{\vec{c}\cdot\left(\vec{c}-\frac{\vec{b}}{4}\right)}{|\vec{c}|\left|\vec{c}-\frac{\vec{b}}{4}\right|} = \frac{\vec{c}\cdot\vec{c}-\frac{\vec{c}.\vec{b}}{4}}{12\left|\vec{c}-\frac{\vec{b}}{4}\right|} = \frac{144-18}{12\left|\vec{c}-\frac{\vec{b}}{4}\right|}$$

$$\text{Now } \left|\vec{c}-\frac{\vec{b}}{4}\right|^2 = |\vec{c}|^2 + \frac{|\vec{b}|^2}{16} - \frac{\vec{b}\cdot\vec{c}}{2} = 144+4-36 = 112$$

$$\Rightarrow \cos\theta = \frac{21}{2\times\sqrt{112}} = \frac{21}{2\times4\sqrt{7}} = \frac{3\sqrt{7}}{8}$$

Example 2.26 **Arc AC of a circle subtends a right angle at the centre O. Point B divides the arc in the ratio $1:2$. If $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$, then calculate $\overrightarrow{OC}$ in terms of $\vec{a}$ and $\vec{b}$.**

Sol. Vector $\vec{c}$ is coplanar with vectors $\vec{a}$ and $\vec{b}$. Therefore, $\vec{c} = x\vec{a} + y\vec{b}$ (i)

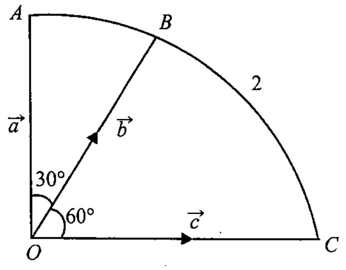


Fig. 2.11

Point B divides arc AC in the ratio 1 : 2 so that $\angle AOB = 30°$ and $\angle BOC = 60°$.

We have to find the values of x and y when we are given $|\vec{a}| = |\vec{b}| = |\vec{c}| = r$ (say).

$\vec{a}\cdot\vec{b} = r^2 \cos 30° = r^2 \dfrac{\sqrt{3}}{2}$ and $\vec{a}\cdot\vec{c} = 0$

$\vec{b}\cdot\vec{c} = r^2 \cos 60° = \dfrac{r^2}{2}$

Multiplying both sides of (i) scalarly by $\vec{c}$ and $\vec{a}$, $\vec{c}\cdot\vec{c} = x\vec{a}\cdot\vec{c} + y\vec{b}\cdot\vec{c}$

and $\vec{c}\cdot\vec{a} = x\vec{a}\cdot\vec{a} + y\vec{b}\cdot\vec{a}$

$r^2 = 0 + \dfrac{r^2}{2}y$, $y = 2$

and $0 = xr^2 + yr^2 \dfrac{\sqrt{3}}{2}$

Putting $y = 2$, $x = -\sqrt{3}$

$\vec{c} = -\sqrt{3}\vec{a} + 2\vec{b}$

Example 2.27 **Vector $\overrightarrow{OA} = \hat{i} + 2\hat{j} + 2\hat{k}$ turns through a right angle passing through the positive x-axis on the way. Show that the vector in its new position is $\dfrac{4\hat{i} - \hat{j} - \hat{k}}{\sqrt{2}}$.**

Sol. Let the new vector be $\overrightarrow{OB} = x\hat{i} + y\hat{j} + z\hat{k}$.

According to the given condition, we have

$|\overrightarrow{OB}| = |\overrightarrow{OA}| = 3 \Rightarrow x^2 + y^2 + z^2 = 9$ (i)

Also $\overrightarrow{OA} \perp \overrightarrow{OB} \Rightarrow x + 2y + 2z = 0$ (ii)

Since while turning $\overrightarrow{OA}$, it passes through the positive x-axis on the way,

Vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\lambda\hat{i}$ are coplanar.

$$\Rightarrow \begin{vmatrix} x & y & z \\ 1 & 2 & 2 \\ \lambda & 0 & 0 \end{vmatrix} = 0$$

$\Rightarrow y - z = 0$ (iii)

Solving (i), (ii) and (iii) for x, y and z, we have $x = -4y = -4z$

$\Rightarrow 16y^2 + y^2 + y^2 = 9$

$\Rightarrow y = \pm\frac{1}{\sqrt{2}}, z = \pm\frac{1}{\sqrt{2}}$ and $x = \mp 4\frac{1}{\sqrt{2}}$

$\Rightarrow \overrightarrow{OB} = \pm\left(\frac{4}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j} - \frac{1}{\sqrt{2}}\hat{k}\right)$

Since angle between $\overrightarrow{OB}$ and $\hat{i}$ is acute, $\overrightarrow{OB} = \frac{4}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j} - \frac{1}{\sqrt{2}}\hat{k}$

Concept Application Exercise 2.1

1. If $|\vec{a}| = 3, |\vec{b}| = 4$ and the angle between $\vec{a}$ and $\vec{b}$ is 120°, then find the value of $|4\vec{a}+3\vec{b}|$.
2. If vectors $\hat{i}-2x\hat{j}-3y\hat{k}$ and $\hat{i}+3x\hat{j}+2y\hat{k}$ are orthogonal to each other, then find the locus of the point (x, y).
3. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be pairwise mutually perpendicular vectors, such that $|\vec{a}| = 1$, $|\vec{b}| = 2$, $|\vec{c}| = 2$. Then find the length of $\vec{a}+\vec{b}+\vec{c}$.
4. If $\vec{a} + \vec{b} + \vec{c} = 0$, $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 7$, then find the angle between $\vec{a}$ and $\vec{b}$.
5. If the angle between unit vectors $\vec{a}$ and $\vec{b}$ is 60°, then find the value of $|\vec{a} - \vec{b}|$.
6. Let $\vec{u} = \hat{i} + \hat{j}$, $\vec{v} = \hat{i} - \hat{j}$ and $\vec{w} = \hat{i}+2\hat{j}+3\hat{k}$. If $\hat{n}$ is a unit vector such that $\vec{u}\cdot\hat{n} = 0$ and $\vec{v}\cdot\hat{n} = 0$, then find the value of $|\vec{w}\cdot\hat{n}|$.
7. A, B, C, D are any four points, prove that $\overrightarrow{AB}\cdot\overrightarrow{CD} + \overrightarrow{BC}\cdot\overrightarrow{AD} + \overrightarrow{CA}\cdot\overrightarrow{BD} = 0$.
8. $P(1, 0, -1)$, $Q(2, 0, -3)$, $R(-1, 2, 0)$ and $S(3, -2, -1)$, then find the projection length of $\overrightarrow{PQ}$ on $\overrightarrow{RS}$.
9. If the vectors $3\vec{p} + \vec{q}$; $5\vec{p} - 3\vec{q}$ and $2\vec{p}+\vec{q}$; $4\vec{p}-2\vec{q}$ are pairs of mutually perpendicular vectors, then find the angle between vectors $\vec{p}$ and $\vec{q}$.
10. Let $\vec{A}$ and $\vec{B}$ be two non-parallel unit vectors in a plane. If $(\alpha\vec{A} + \vec{B})$ bisects the internal angle between $\vec{A}$ and $\vec{B}$, then find the value of α.
11. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be unit vectors, such that $\vec{a} + \vec{b} + \vec{c} = \vec{x}$, $\vec{a}\cdot\vec{x} = 1$, $\vec{b}\cdot\vec{x} = \frac{3}{2}$, $|\vec{x}| = 2$. Then find the angle between $\vec{c}$ and $\vec{x}$.
12. If $\vec{a}$ and $\vec{b}$ are unit vectors, then find the greatest value of $|\vec{a} + \vec{b}| + |\vec{a} - \vec{b}|$.
13. Constant forces $P_1 = \hat{i} - \hat{j} + \hat{k}$, $P_2 = -\hat{i} + 2\hat{j} - \hat{k}$ and $P_3 = \hat{j} - \hat{k}$ act on a particle at a point A. Determine the work done when particle is displaced from position $A(4\hat{i} - 3\hat{j} - 2\hat{k})$ to $B(6\hat{i} + \hat{j} - 3\hat{k})$

VECTOR (OR CROSS) PRODUCT OF TWO VECTORS

The cross product is just a shorthand invented for the purpose of quickly writing down the angular momentum of an object. Here's how the cross product arises naturally from angular momentum. Recall that if we have a fixed axis and an object distance r away with velocity v and mass m is moving around the axis in a circle, the magnitude of the angular momentum is $m|r||v|$, where $|r|$ is the magnitude of vector r. But what direction should the angular momentum vector point in? Well, if you follow the path of the object, it lies in a plane, an infinite two-dimensional surface. One way to represent a plane is to write down two different vectors that lie in the plane.

Another method used by mathematicians to represent a plane is to write down a single vector that is normal to the plane (normal is a synonym for perpendicular). If a plane is a flat sheet, the normal vector points straight up. Now, for any plane, there are two vectors that are normal to it, since if a vector n is normal to a plane, $-n$ will be normal as well. So how do we determine whether to use n or $-n$?

A long time ago, physicists just made an arbitrary decision known today as the right-hand rule. Given vectors $\vec{a}$ and $\vec{b}$, just curl your fingers from $\vec{a}$ to $\vec{b}$ and the thumb points in the direction of the normal used.

The vector product of two vectors $\vec{a}$ and $\vec{b}$, written as $\vec{a} \times \vec{b}$, is the vector $\vec{c} = |a||b| \sin \theta\, \hat{n}$, where θ is the angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$, and $\hat{n}$ is a unit vector along the line perpendicular to both $\vec{a}$ and $\vec{b}$.

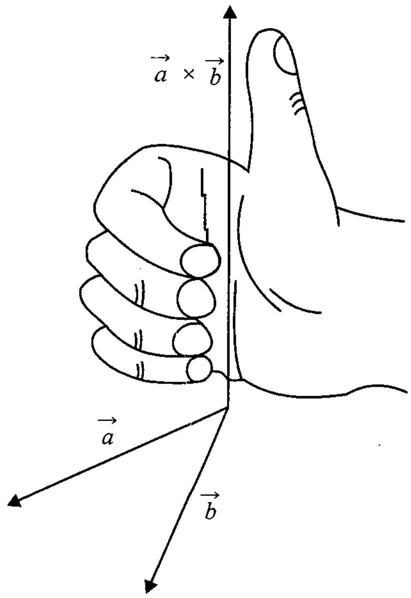


Fig. 2.12

Then direction of $\vec{c}$ is such that $\vec{a}$, $\vec{b}$ and $\vec{c}$ form a right-handed system.

We see that the direction of $\vec{b} \times \vec{a}$ is opposite to that of $\vec{a} \times \vec{b}$ as shown in Fig. 2.13.

$$\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$$

So the vector product is not commutative. In practice, this means that the order in which we do the calculation does matter.

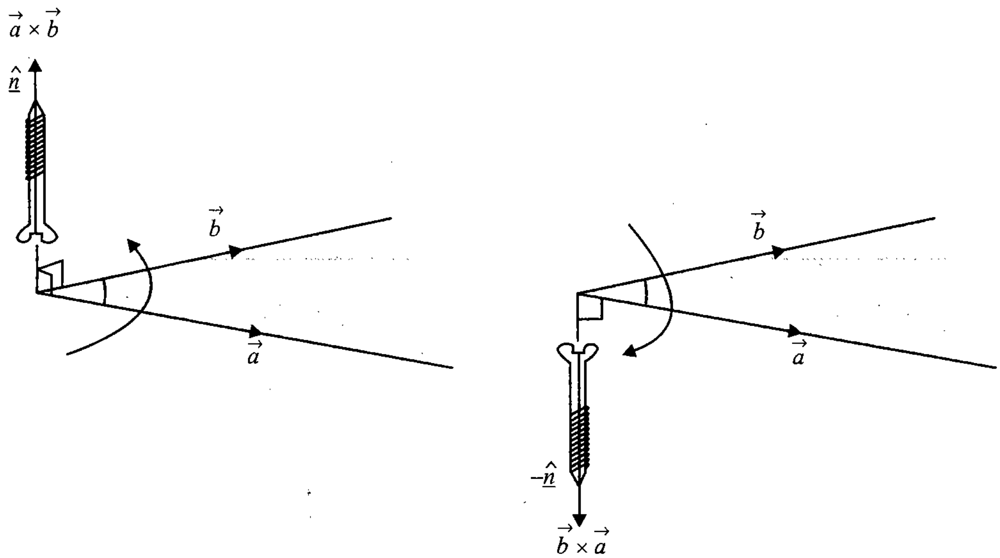


Fig. 2.13

Properties of Cross Product

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $\vec{a} \times \vec{a} = 0$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$ and $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$
5. Two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if $\vec{a} \times \vec{b} = \vec{0}$.
6. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

7. The unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\dfrac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$, and a vector of magnitude λ perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\pm \dfrac{\lambda(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$.

Physical Interpretation of Cross Product as a Moment of Force

Moment of force (often just *moment*) is the tendency of a force to twist or rotate an object. This is an important, basic concept in engineering and physics. A moment is valued mathematically as the product of the force and the moment arm. Moment arm is the perpendicular distance from the point of rotation to the *line of action* of the force. The moment may be thought of as a measure of the tendency of the force to cause rotation about an imaginary axis through a point.

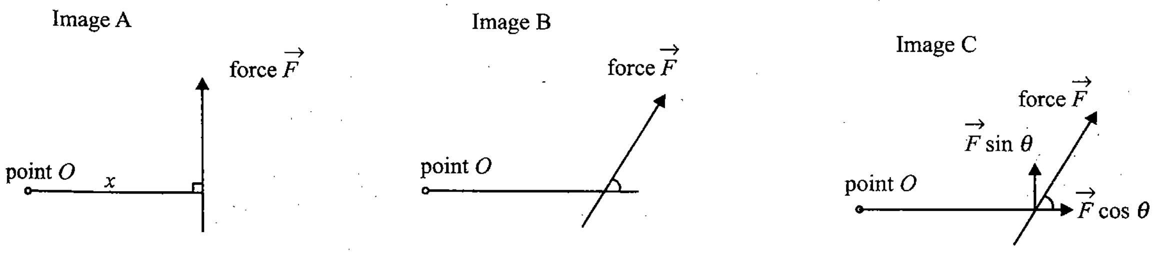


Fig. 2.14

The moment of a force can be calculated about any point and not just the points in which the line of action of the force is perpendicular.

Image A shows the components, the force F and the moment arm x when they are perpendicular to one another. When the force is not perpendicular to the point of interest, such as point O in Images B and C, the magnitude of moment $\vec{M}$ of a vector $\vec{F}$ about point O is

$\vec{M}_O = \vec{r}_{OF} \times \vec{F}$, where $\vec{r}_{OF}$ is the vector from point O to the position where quantity F is applied.

Image C represents the vector components of the force in Image B. In order to determine moment $\vec{M}$ of vector $\vec{F}$ about point O, when vector $\vec{F}$ is not perpendicular to point O, one must resolve the force $\vec{F}$ into its horizontal and vertical components. The sum of the moments of the two components of F about point O is

$$\vec{M}_{OF} = \vec{F}\sin\theta(x) + \vec{F}\cos\theta(0)$$

The moment arm to the vertical component of $\vec{F}$ is a distance x. The moment arm to the horizontal component of $\vec{F}$ does not exist. There is no rotational force about point O due to the horizontal component of $\vec{F}$. Thus, the moment arm distance is zero.

Thus $\vec{M}$ can be referred to as "moment $\vec{M}$ with respect to the axis that goes through point O", or simply "moment $\vec{M}$ about point O". If O is the origin, or informally, if the axis involved is clear from context, one often omits O and says simply *moment*, rather than *moment about O*. Therefore, the moment about *point O* is indeed the cross product, $\vec{M}_O = \vec{r}_{OF} \times \vec{F}$, since the cross product $= \vec{F}\sin\theta(x)$.

Geometric Interpretation of Cross Product

1.

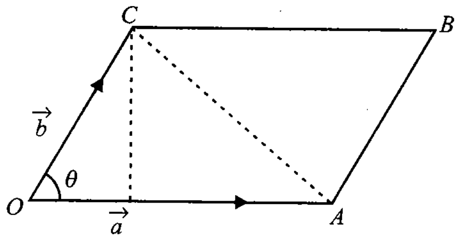


Fig. 2.15

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}$

$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$

$= 2\left(\frac{1}{2}|\vec{a}||\vec{b}| \sin\theta\right)$

$= 2$ (Area of triangle AOC)

$=$ Area of parallelogram

Area of the triangle OAB is $\frac{1}{2}|\vec{a} \times \vec{b}|$.

$\vec{a} \times \vec{b}$ is said to be the vector area of the parallelogram with adjacent sides OA and OB.

2. If $\vec{a}, \vec{b}$ are diagonals of a parallelogram, its area $= \frac{1}{2} |\vec{a} \times \vec{b}|$.

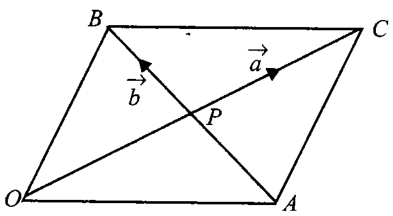


Fig. 2.16

In the above diagram $\overrightarrow{OC} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$

$\Rightarrow$ Area parallelogram $= 4 \times \frac{1}{2} |\overrightarrow{PC} \times \overrightarrow{PB}|$

$= 4 \times \frac{1}{2}\left|\frac{\vec{a}}{2} \times \frac{\vec{b}}{2}\right|$

$= \frac{1}{2}|\vec{a} \times \vec{b}|$

3. If AC and BD are the diagonals of a quadrilateral, then its vector area is $\frac{1}{2}\overrightarrow{AC}\times\overrightarrow{BD}$.

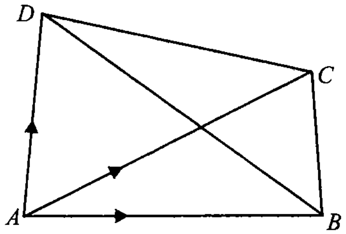


Fig. 2.17

Vector area of the quadrilateral $ABCD$ = vector area of ΔABC + vector area of ΔACD.

$$= \frac{1}{2}\overrightarrow{AB}\times\overrightarrow{AC}+\frac{1}{2}\overrightarrow{AC}\times\overrightarrow{AD}$$

$$= -\frac{1}{2}\overrightarrow{AC}\times\overrightarrow{AB}+\frac{1}{2}\overrightarrow{AC}\times\overrightarrow{AD}$$

$$= \frac{1}{2}\overrightarrow{AC}\times(\overrightarrow{AD}-\overrightarrow{AB})$$

$$= \frac{1}{2}\overrightarrow{AC}\times\overrightarrow{BD}$$

4. The area of a triangle whose vertices are $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$ is $\frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$

Area of triangle $= \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

$$= \frac{1}{2}|(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})|$$

$$= \frac{1}{2}|\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}+\vec{a}\times\vec{a}|$$

$$= \frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$$

Example 2.28 **If A, B and C are the vertices of a triangle ABC, prove sine rule $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$.**

Sol. Let $\overrightarrow{BC}=\vec{a}$, $\overrightarrow{CA}=\vec{b}$, $\overrightarrow{AB}=\vec{c}$, so that $\vec{a}+\vec{b}=-\vec{c}$

$\therefore\quad \vec{a}\times\vec{a}+\vec{a}\times\vec{b}=-\vec{a}\times\vec{c}$

$\vec{0}+\vec{a}\times\vec{b}=\vec{c}\times\vec{a}$

$|\vec{a}\times\vec{b}|=|\vec{c}\times\vec{a}|$

$ab\sin(180°-C)=ca\sin(180°-B)$

$ab\sin C=ca\sin B$

Dividing both sides by abc, we get

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\therefore \quad \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \text{(i)}$$

Similarly $\dfrac{c}{\sin C} = \dfrac{a}{\sin A}$ (ii)

From (i) and (ii), we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

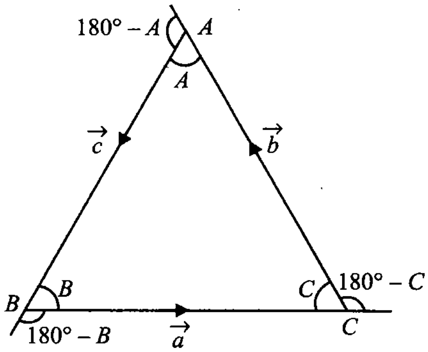


Fig. 2.18

Example 2.29 **Using cross product of vectors, prove that $\sin(A+B) = \sin A \cos B + \cos A \sin B$.**

Sol. Let OP and OQ be unit vectors making angles A and B with X-axis such that $\angle POQ = A + B$

$$\therefore \quad \vec{OP} = \hat{i}\cos A + \hat{j}\sin A$$

$$\vec{OQ} = \hat{i}\cos B - \hat{j}\sin B$$

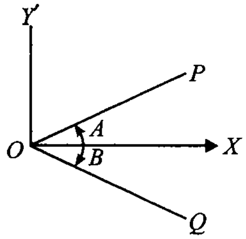


Fig. 2.19

Now $\vec{OP} \times \vec{OQ}$

$$= (1)\,(1)\sin(A+B)\,(-\hat{k})$$

$$= -\sin(A+B)\,\hat{k} \qquad \text{(i)}$$

Also $\vec{OP} \times \vec{OQ} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos A & \sin A & 0 \\ \cos B & -\sin B & 0 \end{vmatrix}$

$$= (-\cos A \sin B - \sin A \cos B)\,\hat{k}$$

$$\therefore \quad \vec{OP} \times \vec{OQ} = -(\sin A \cos B + \cos A \sin B)\,\hat{k} \qquad \text{(ii)}$$

From (i) and (ii), we get

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

Example 2.30 **Find a unit vector perpendicular to the plane determined by the points (1, –1, 2), (2, 0, –1) and (0, 2, 1).**

Sol. Given points are $A(1, -1, 2)$, $B(2, 0, -1)$ and $C(0, 2, 1)$

$$\Rightarrow \overline{AB} = \vec{a} = \hat{i} + \hat{j} - 3\hat{k},\ \overline{BC} = \vec{b} = -2\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\therefore \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -3 \\ -2 & 2 & 2 \end{vmatrix} = 8\hat{i} + 4\hat{j} + 4\hat{k}$$

Hence unit vector $= \pm \dfrac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{6}}$

Example 2.31 **If $\vec{a}$ and $\vec{b}$ are two vectors, then prove that $(\vec{a}\times\vec{b})^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} \end{vmatrix}$**

Sol.
$$(\vec{a}\times\vec{b})^2 = (ab\sin\theta\cdot\hat{n})^2$$
$$= a^2b^2\sin^2\theta$$
$$= a^2b^2 - a^2b^2\cos^2\theta$$
$$= (\vec{a}\cdot\vec{a})\,(\vec{b}\cdot\vec{b}) - (\vec{a}\cdot\vec{b})^2$$
$$= \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}$$

Example 2.32 **If $|\vec{a}| = 2$, then find the value of $|\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{j}|^2 + |\vec{a}\times\hat{k}|^2$.**

Sol.
$$|\vec{a}\times\hat{i}|^2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ 1 & 0 & 0 \end{vmatrix}^2 \quad (\text{since } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$$
$$= |a_3\hat{j} - a_2\hat{k}|^2 = a_3^2 + a_2^2$$

Similarly, $|\vec{a}\times\vec{j}|^2 = a_1^2 + a_3^2$ and $|\vec{a}\times\vec{k}|^2 = a_1^2 + a_2^2$

Hence the required result can be given as $2(a_1^2 + a_2^2 + a_3^2) = 2|\vec{a}|^2 = 8$

Example 2.33 **$\vec{r}\times\vec{a} = \vec{b}\times\vec{a}$; $\vec{r}\times\vec{b} = \vec{a}\times\vec{b}$; $\vec{a} \neq \vec{0}$; $\vec{b} \neq \vec{0}$; $\vec{a} \neq \lambda\vec{b}$, and $\vec{a}$ is not perpendicular to $\vec{b}$, then find $\vec{r}$ in terms of $\vec{a}$ and $\vec{b}$.**

Sol. $\vec{r}\times\vec{a} - \vec{b}\times\vec{a} = 0$ and $\vec{r}\times\vec{b} + \vec{b}\times\vec{a} = 0$

Adding, we get $\vec{r}\times(\vec{a} + \vec{b}) = 0$

But as we are given $\vec{a} \neq \lambda\vec{b}$, therefore

$\vec{r} = \mu(\vec{a} + \vec{b})$

Example 2.34 **A, B, C and D are any four points in the space, then prove that $|\overrightarrow{AB}\times\overrightarrow{CD} + \overrightarrow{BC}\times\overrightarrow{AD} + \overrightarrow{CA}\times\overrightarrow{BD}| = 4(\text{area of } \Delta ABC)$.**

Sol. Let P.V. of A, B, C and D be $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{0}$, respectively.

$\Rightarrow \overrightarrow{AB}\times\overrightarrow{CD} = (\vec{b} - \vec{a})\times(-\vec{c})$, $\overrightarrow{BC}\times\overrightarrow{AD} = (\vec{c} - \vec{b})\times(-\vec{a})$ and $\overrightarrow{CA}\times\overrightarrow{BD} = (\vec{a} - \vec{c})\times(-\vec{b})$

$$\overrightarrow{AB}\times\overrightarrow{CD} + \overrightarrow{BC}\times\overrightarrow{AD} + \overrightarrow{CA}\times\overrightarrow{BD} = \vec{c}\times\vec{b} + \vec{a}\times\vec{c} + \vec{a}\times\vec{c} + \vec{b}\times\vec{a} - \vec{a}\times\vec{b} + \vec{c}\times\vec{b}$$
$$= 2(\vec{c}\times\vec{b} + \vec{b}\times\vec{a} + \vec{a}\times\vec{c})$$
$$= 2(\vec{c}\times(\vec{b} - \vec{a}) - \vec{a}\times(\vec{b} - \vec{a}))$$

$$= 2((\vec{c}-\vec{a})\times(\vec{b}-\vec{a}))$$
$$= 2(\overrightarrow{AC}\times\overrightarrow{AB})$$

$$\Rightarrow |\overrightarrow{AB}\times\overrightarrow{CD}+\overrightarrow{BC}\times\overrightarrow{AD}+\overrightarrow{CA}\times\overrightarrow{BD}| = 4\left|\frac{1}{2}(\overrightarrow{AC}\times\overrightarrow{AB})\right| = 4\Delta ABC$$

Example 2.35 **If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vectors of the vertices A, B and C, respectively, of ΔABC, prove that the perpendicular distance of the vertex A from the base BC of the triangle ABC is $\dfrac{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}{|\vec{c}-\vec{b}|}$.**

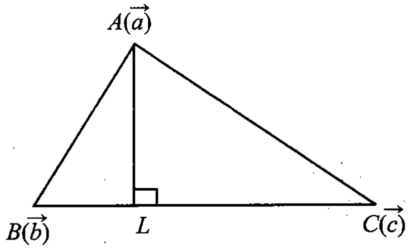


Fig. 2.20

Sol. $|\overrightarrow{BC}\times\overrightarrow{BA}| = |\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$

$$\Rightarrow |\overrightarrow{BC}||\overrightarrow{BA}|\sin B = |\vec{a}\times\vec{b}\times\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$$

$$\Rightarrow |\vec{c}-\vec{b}|(AB\sin B) = |\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$$

Therefore, the length of perpendicular from A on $BC = AL = AB\sin B = \dfrac{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}{|\vec{b}-\vec{c}|}$.

Example 2.36 **Find the area of the triangle whose vertices are $A(1,-1,2)$, $B(2,1,-1)$ and $C(3,-1,2)$.**

Sol. Here $\overrightarrow{OA} = \hat{i}-\hat{j}+2\hat{k}$ and $\overrightarrow{OB} = 2\hat{i}+\hat{j}-\hat{k}$ and $\overrightarrow{OC} = 3\hat{i}-\hat{j}+2\hat{k}$

$$\Rightarrow \overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA} = \hat{i}+2\hat{j}-3\hat{k} \text{ and } \overrightarrow{AC} = \overrightarrow{OC}-\overrightarrow{OA} = 2\hat{i}$$

Hence, the required area $= \dfrac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

Now, $$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -3\\ 2 & 0 & 0\end{vmatrix} = -2(3\hat{j}+2\hat{k})$$

$$\Rightarrow \text{Area of triangle} = \frac{1}{2}\times 2|3\hat{j}+2\hat{k}| = \sqrt{13}$$

Example 2.37 **Find the area of a parallelogram whose two adjacent sides are represented by vectors $3\hat{i}-\hat{k}$ and $\hat{i}+2\hat{j}$.**

Sol. The area of parallelogram is given by $= |\overrightarrow{AB}\times\overrightarrow{AD}|$

Here we are given adjacent sides. Therefore,

$$\overrightarrow{AB}\times\overrightarrow{AD} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 0 & -1\\ 1 & 2 & 0\end{vmatrix} = 2\hat{i}-\hat{j}+6\hat{k}$$

Hence the required area is $= |2\hat{i}-\hat{j}+6\hat{k}| = \sqrt{41}$

Example 2.38 **Find the area of a parallelogram whose diagonals are $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} - 3\hat{j} + 4\hat{k}$.**

Sol. $\Delta = \frac{1}{2}|\vec{a} \times \vec{b}|$

But $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = -2\hat{i} - 14\hat{j} - 10\hat{k}$

Hence $\Delta = \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}\sqrt{4+196+100} = 5\sqrt{3}$

Example 2.39 **Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that $\vec{a} \neq 0$, $|\vec{a}| = |\vec{c}| = 1$, $|\vec{b}| = 4$ and $|\vec{b} \times \vec{c}| = \sqrt{15}$. If $\vec{b} - 2\vec{c} = \lambda\vec{a}$, then find the value of λ.**

Sol. Let the angle between $\vec{b}$ and $\vec{c}$ be α

$|\vec{b} \times \vec{c}| = \sqrt{15}$

$\Rightarrow |\vec{b}||\vec{c}| \sin\alpha = \sqrt{15}$

$\Rightarrow \sin\alpha = \frac{\sqrt{15}}{4}$

$\Rightarrow \cos\alpha = \frac{1}{4}$

$\Rightarrow \vec{b} - 2\vec{c} = \lambda\vec{a}$

$\Rightarrow |\vec{b} - 2\vec{c}|^2 = \lambda^2 |\vec{a}|^2$

$\Rightarrow |\vec{b}|^2 + 4|\vec{c}|^2 - 4 \cdot \vec{b} \cdot \vec{c} = \lambda^2 |\vec{a}|^2$

$\Rightarrow 16 + 4 - 4\{|\vec{b}||\vec{c}| \cos\alpha\} = \lambda^2$

$\Rightarrow 16 + 4 - 4 \times 4 \times 1 \times \frac{1}{4} = \lambda^2$

$\Rightarrow \lambda^2 = 16 \Rightarrow \lambda = \pm 4$

Example 2.40 **Find the moment about (1, – 1, – 1) of the force $3\hat{i} + 4\hat{j} - 5\hat{k}$ acting at (1, 0, – 2).**

Sol. $\vec{F} = 3\hat{i} + 4\hat{j} - 5\hat{k}$

$\vec{PA}$ = P.V. of A – P.V. of P

$= (\hat{i} - 2\hat{j}) - (\hat{i} - \hat{j} - \hat{k})$

$= -\hat{j} + \hat{k}$

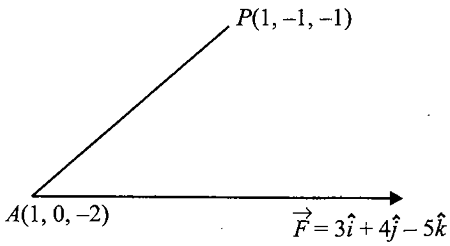


Fig. 2.21

Required vector moment $= \vec{PA} \times \vec{F}$

$$= (-\hat{j} + \hat{k}) \times (3\hat{i} + 4\hat{j} - 5\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 1 \\ 3 & 4 & -5 \end{vmatrix}$$

$$= \hat{i} + 3\hat{j} + 3\hat{k}$$

Example 2.41 **A rigid body is spinning about a fixed point (3, – 2, –1) with an angular velocity of 4 rad/s, the axis of rotation being in the direction of (1, 2, – 2). Find the velocity of the particle at point (4, 1, 1).**

Sol.

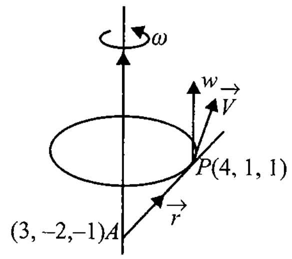


Fig. 2.22

$$\vec{\omega} = 4\left(\frac{\hat{i}+2\hat{j}-2\hat{k}}{\sqrt{1+4+4}}\right) = \frac{4}{3}(\hat{i} + 2\hat{j} - 2\hat{k})$$

$$\vec{r} = \overrightarrow{OP} - \overrightarrow{OA}$$

$$= (4\hat{i} + \hat{j} + \hat{k}) - (3\hat{i} - 2\hat{j} - \hat{k})$$

$$= \hat{i} + 3\hat{j} + 2\hat{k}$$

$$\vec{v} = \vec{\omega} \times \vec{r} = \frac{4}{3}(\hat{i} + 2\hat{j} - 2\hat{k}) \times (\hat{i} + 3\hat{j} + 2\hat{k})$$

$$= \frac{4}{3}(10\hat{i} - 4\hat{j} + \hat{k})$$

Example 2.42 **If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, then show that $\vec{a} - \vec{d}$ is parallel to $\vec{b} - \vec{c}$ provided $\vec{a} \neq \vec{d}$ and $\vec{b} \neq \vec{c}$.**

Sol. We have $\left.\begin{aligned} \vec{a} \times \vec{b} &= \vec{c} \times \vec{d} \\ \text{and } \vec{a} \times \vec{c} &= \vec{b} \times \vec{d} \end{aligned}\right]$ (i)

$\vec{a} - \vec{d}$ will be parallel to $\vec{b} - \vec{c}$

if $(\vec{a} - \vec{d}) \times (\vec{b} - \vec{c}) = \vec{0}$

i.e., if $\vec{a} \times \vec{b} - \vec{a} \times \vec{c} - \vec{d} \times \vec{b} + \vec{d} \times \vec{c} = \vec{0}$

i.e., if $(\vec{a} \times \vec{b} + \vec{d} \times \vec{c}) - (\vec{a} \times \vec{c} + \vec{d} \times \vec{b}) = \vec{0}$

i.e., if $(\vec{a} \times \vec{b} - \vec{c} \times \vec{d}) - (\vec{a} \times \vec{c} - \vec{b} \times \vec{d}) = \vec{0}$

i.e., if $\vec{0} - \vec{0} = \vec{0}$ [from (i)]

i.e., $\vec{0} = \vec{0}$, which is true

Hence the result

Example 2.43 **Show by a numerical example and geometrically also that $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ does not imply $\vec{b} = \vec{c}$.**

Sol. Let $\vec{a} = 3\hat{i} + 2\hat{j} + 5\hat{k}$, $\vec{b} = 6\hat{i} + 5\hat{j} + 8\hat{k}$, $\vec{c} = 3\hat{i} + 3\hat{j} + 3\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 5 \\ 6 & 5 & 8 \end{vmatrix}$$

$$= (16-25)\hat{i} - (24-30)\hat{j} + (15-12)\hat{k}$$

$$= -9\hat{i} + 6\hat{j} + 3\hat{k}$$

$$\vec{a} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 5 \\ 3 & 3 & 3 \end{vmatrix}$$

$$= (6-15)\hat{i} - (9-15)\hat{j} + (9-6)\hat{k} = -9\hat{i} + 6\hat{j} + 3\hat{k}$$

$\therefore$ $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, but $\vec{b} \neq \vec{c}$.

Geometrically

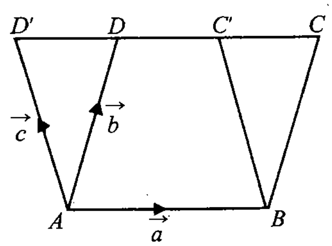


Fig. 2.23

Let $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{AD} = \vec{b}$, $\overrightarrow{AD'} = \vec{c}$

Vector area of parallelogram $ABCD = \vec{a} \times \vec{b}$

Vector area of parallelogram $ABC'D' = \vec{a} \times \vec{c}$

Now vector area of parallelogram $ABCD$ = vector area of parallelogram $ABC'D'$

($\because$ both parallelograms have same base and same height)

$\therefore\ \vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ but $\vec{b} \neq \vec{c}$

Example 2.44 **If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are the position vectors of the vertices of a cyclic quadrilateral $ABCD$, prove that** $\dfrac{|\vec{a} \times \vec{b} + \vec{b} \times \vec{d} + \vec{d} \times \vec{a}|}{(\vec{b} - \vec{a}) \cdot (\vec{d} - \vec{a})} + \dfrac{|\vec{b} \times \vec{c} + \vec{c} \times \vec{d} + \vec{d} \times \vec{b}|}{(\vec{b} - \vec{c}) \cdot (\vec{d} - \vec{c})} = 0.$

Sol.

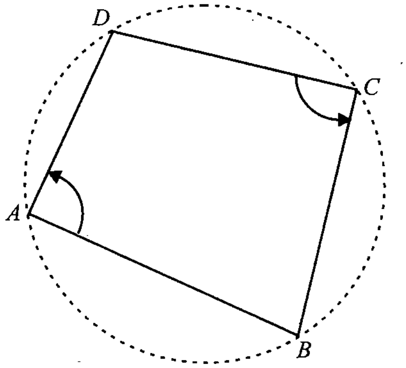


Fig. 2.24

Consider

$$\frac{|\vec{a} \times \vec{b} + \vec{b} \times \vec{d} + \vec{d} \times \vec{a}|}{(\vec{b} - \vec{a}) \cdot (\vec{d} - \vec{a})} = \frac{|(\vec{a} - \vec{d}) \times (\vec{b} - \vec{a})|}{(\vec{b} - \vec{a}) \cdot (\vec{d} - \vec{a})}$$

$$= \frac{|\vec{a} - \vec{d}||\vec{b} - \vec{a}| \sin A}{|\vec{b} - \vec{a}||\vec{d} - \vec{a}| \cos A}$$

$$= \tan A \qquad \text{(i)}$$

Also $$\frac{|\vec{b} \times \vec{c} + \vec{c} \times \vec{d} + \vec{d} \times \vec{b}|}{(\vec{b} - \vec{c}) \cdot (\vec{d} - \vec{c})} = \frac{|(\vec{b} - \vec{c}) \times (\vec{c} - \vec{d})|}{(\vec{b} - \vec{c}) \cdot (\vec{d} - \vec{c})}$$

$$= \frac{|\vec{b} - \vec{c}||\vec{c} - \vec{d}| \sin C}{|\vec{b} - \vec{c}| \cdot |\vec{d} - \vec{c}| \cos C}$$

$$= \tan C \qquad \text{(iii)}$$

As cyclic quadrilateral

$A = 180° - C$

$\Rightarrow \tan A = \tan(180° - C)$

$\Rightarrow \tan A + \tan C = 0$

$$\Rightarrow \frac{|\vec{a}\times\vec{b}+\vec{b}\times\vec{d}+\vec{d}\times\vec{a}|}{(\vec{b}-\vec{a})\cdot(\vec{d}-\vec{a})}+\frac{|\vec{b}\times\vec{c}+\vec{c}\times\vec{d}+\vec{d}\times\vec{b}|}{(\vec{b}-\vec{c})\cdot(\vec{d}-\vec{c})}=0$$

Example 2.45 **The position vectors of the vertices of a quadrilateral with A as origin are $B(\vec{b})$, $D(\vec{d})$ and $C(l\vec{b}+m\vec{d})$. Prove that the area of the quadrilateral is $\frac{1}{2}(l+m)|\vec{b}\times\vec{d}|$.**

Sol. Area of quadrilateral is $\frac{1}{2}|\overrightarrow{AC}\times\overrightarrow{BD}| = \frac{1}{2}|(l\vec{b}+m\vec{d})\times(\vec{d}-\vec{b})|$

$$= \frac{1}{2}|l\vec{b}\times\vec{d} - m\vec{d}\times\vec{b}|$$

$$= \frac{1}{2}(l+m)|\vec{b}\times\vec{d}|$$

Example 2.46 **Let $\vec{a}$ and $\vec{b}$ be unit vectors such that $|\vec{a}+\vec{b}| = \sqrt{3}$. Then find the value of $(2\vec{a}+5\vec{b})\cdot(3\vec{a}+\vec{b}+\vec{a}\times\vec{b})$.**

Sol. $(2\vec{a}+5\vec{b})\cdot(3\vec{a}+\vec{b}+\vec{a}\times\vec{b}) = 6\vec{a}\cdot\vec{a}+17\vec{a}\cdot\vec{b}+5\vec{b}\cdot\vec{b}$

($\because$ $\vec{a}\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\vec{a}\times\vec{b}) = 0$, as $\vec{a}$ and $\vec{b}$ are perpendicular to $\vec{a}\times\vec{b}$)

$= 11 + 17\vec{a}\cdot\vec{b}$

Now $|\vec{a}+\vec{b}| = \sqrt{3}$

$\Rightarrow |\vec{a}+\vec{b}|^2 = 3$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b} = 3$

$\Rightarrow \vec{a}\cdot\vec{b} = \frac{1}{2}$

$\Rightarrow (2\vec{a}+5\vec{b})\cdot(3\vec{a}+\vec{b}+\vec{a}\times\vec{b}) = 11+\frac{17}{2} = \frac{39}{2}$

Example 2.47 **$\hat{u}$ and $\hat{v}$ are two non-collinear unit vectors such that $\left|\frac{\hat{u}+\hat{v}}{2}+\hat{u}\times\hat{v}\right| = 1$. Prove that $|\hat{u}\times\hat{v}| = \left|\frac{\hat{u}-\hat{v}}{2}\right|$**

Sol. Given that $\left|\frac{\hat{u}+\hat{v}}{2}+\hat{u}\times\hat{v}\right| = 1$

$$\Rightarrow \left|\frac{\hat{u}+\hat{v}}{2}+\hat{u}\times\hat{v}\right|^2=1$$

$$\Rightarrow \frac{2+2\cos\theta}{4}+\sin^2\theta=1 \quad (\because \hat{u}\cdot(\hat{u}\times\hat{v})=\hat{v}\cdot(\hat{u}\times\hat{v})=0)$$

$$\Rightarrow \cos^2\frac{\theta}{2}=\cos^2\theta$$

$$\Rightarrow \theta = n\pi \pm \frac{\theta}{2},\ n\in Z$$

$$\Rightarrow \theta = \frac{2\pi}{3}$$

$$\Rightarrow |\hat{u}\times\hat{v}| = \sin\frac{2\pi}{3}=\sin\frac{\pi}{3}=\left|\frac{\hat{u}-\hat{v}}{2}\right|$$

Example 2.48 **In triangle *ABC*, points *D*, *E* and *F* are taken on the sides *BC*, *CA* and *AB*, respectively, such that $\frac{BD}{DC}=\frac{CE}{EA}=\frac{AF}{FB}=n$. Prove that $\Delta_{DEF}=\frac{n^2-n+1}{(n+1)^2}\Delta_{ABC}$.**

Sol. Take *A* as the origin and let the position vectors of points *B* and *C* be $\vec{b}$ and $\vec{c}$, respectively.

Therefore, the position vectors of *D*, *E* and *F* are, respectively, $\frac{n\vec{c}+\vec{b}}{n+1}$, $\frac{\vec{c}}{n+1}$ and $\frac{n\vec{b}}{n+1}$. Therefore,

$$\overrightarrow{ED}=\overrightarrow{AD}-\overrightarrow{AE}=\frac{(n-1)\vec{c}+\vec{b}}{n+1} \text{ and } \overrightarrow{EF}=\frac{n\vec{b}-\vec{c}}{n+1}$$

Now the vector area of $\Delta ABC=\frac{1}{2}(\vec{b}\times\vec{c})$

and the vector area of
$$\Delta DEF=\frac{1}{2}(\overrightarrow{EF}\times\overrightarrow{ED}) = \frac{1}{2(n+1)^2}[(n\vec{b}-\vec{c})\times\{(n-1)\vec{c}+\vec{b}\}]$$
$$=\frac{1}{2(n+1)^2}[(n^2-n)\,\vec{b}\times\vec{c}+\vec{b}\times\vec{c}]$$
$$=\frac{1}{2(n+1)^2}[(n^2-n+1)(\vec{b}\times\vec{c})]=\frac{n^2-n+1}{(n+1)^2}\Delta_{ABC}$$

Concept Application Exercise 2.2

1. If $\vec{a}=2\hat{i}+3\hat{j}-5\hat{k}$, $\vec{b}=m\hat{i}+n\hat{j}+12\hat{k}$ and $\vec{a}\times\vec{b}=\vec{0}$, then find (m, n).
2. If $|\vec{a}|=2, |\vec{b}|=5$ and $|\vec{a}\times\vec{b}|=8$, then find the value of $\vec{a}\cdot\vec{b}$.
3. If $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}\neq 0$, where $\vec{a},\vec{b}$ and $\vec{c}$ are coplanar vectors, then for some scalar *k* prove that $\vec{a}+\vec{c}=k\vec{b}$.
4. If $\vec{a}=2\vec{i}+3\vec{j}-\vec{k}$, $\vec{b}=-\vec{i}+2\vec{j}-4\vec{k}$ and $\vec{c}=\vec{i}+\vec{j}+\vec{k}$, then find the value of $(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{c})$.

5. If the vectors c, $a = xi + yj + zk$ and $b = j$ are such that a, c and b form a right-handed system, then find $\vec{c}$.

6. Given that $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$, $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$ and $\vec{a}$ is not a zero vector. Show that $\vec{b} = \vec{c}$.

7. Show that $(\vec{a} - \vec{b})\times(\vec{a} + \vec{b}) = 2\vec{a}\times\vec{b}$ and give a geometrical interpretation of it.

8. If $\vec{x}$ and $\vec{y}$ are unit vectors and $|\vec{z}| = \frac{2}{\sqrt{7}}$ such that $\vec{z} + \vec{z}\times\vec{x} = \vec{y}$, then find the angle θ between $\vec{x}$ and $\vec{z}$.

9. Prove that $(\vec{a}\cdot\hat{i})(\vec{a}\times\hat{i}) + (\vec{a}\cdot\hat{j})(\vec{a}\times\hat{j}) + (\vec{a}\cdot\hat{k})(\vec{a}\times\hat{k}) = \vec{0}$.

10. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-zero vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $\lambda\vec{b}\times\vec{a} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = \vec{0}$, then find the value of λ.

11. A particle has an angular speed of 3 rad/s and the axis of rotation passes through the points (1, 1, 2) and (1, 2, – 2). Find the velocity of the particle at point $P(3, 6, 4)$.

12. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be unit vectors such that $\vec{a}\cdot\vec{b} = 0 = \vec{a}\cdot\vec{c}$. If the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{6}$, then find $\vec{a}$.

13. If $(\vec{a}\times\vec{b})^2 + (\vec{a}\cdot\vec{b})^2 = 144$ and $|\vec{a}| = 4$, then find the value of $|\vec{b}|$.

14. Given $|\vec{a}| = |\vec{b}| = 1$ and $|\vec{a} + \vec{b}| = \sqrt{3}$. If $\vec{c}$ be a vector such that $\vec{c} - \vec{a} - 2\vec{b} = 3(\vec{a}\times\vec{b})$, then find the value of $\vec{c}\cdot\vec{b}$.

15. Find the moment of $\vec{F}$ about point (2, –1, 3), when force $\vec{F} = 3\hat{i} + 2\hat{j} - 4\hat{k}$ is acting on point (1, –1, 2).

SCALAR TRIPLE PRODUCT

The **scalar triple product** (also called the **mixed** or **box product**) is defined as the *dot product* of one of the vectors with the *cross product* of the other two.

Thus scalar triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is defined as $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$

We denote it by $[\vec{a}\ \vec{b}\ \vec{c}]$

The scalar triple product can be evaluated numerically using any one of the following equivalent characterizations:

$$\left(\vec{a}\times\vec{b}\right)\cdot\vec{c} = \vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \vec{b}\cdot\left(\vec{c}\times\vec{a}\right) = \vec{c}\cdot\left(\vec{a}\times\vec{b}\right)$$

(The parentheses may be omitted without causing ambiguity, since the *dot product* cannot be evaluated first. If it were, it would leave the cross product of a scalar and a vector, which is not defined.)

i.e., $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}] = -[\vec{b}\ \vec{a}\ \vec{c}] = -[\vec{c}\ \vec{b}\ \vec{a}]$

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}_2$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right]=\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\cdot(c_1\hat{i}+c_2\hat{j}+c_3\hat{k}).$$

$$=\begin{vmatrix}\hat{i}\cdot(c_1\hat{i}+c_2\hat{j}+c_3\hat{k}) & \hat{j}\cdot(c_1\hat{i}+c_2\hat{j}+c_3\hat{k}) & \hat{k}\cdot(c_1\hat{i}+c_2\hat{j}+c_3\hat{k})\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$=\begin{vmatrix}c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

Also $\left[\vec{a}\,\vec{b}\,\vec{c}\right]=\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$

$$=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

Geometrical Interpretation

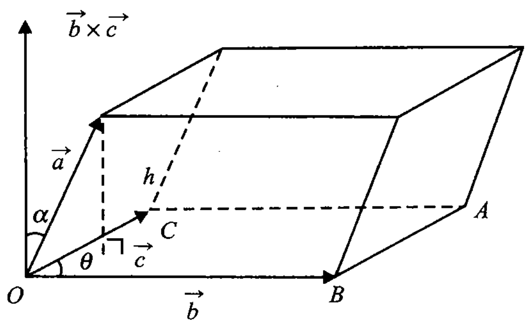


Fig. 2.25

Here $(\vec{a}\times\vec{b})\cdot\vec{c}$ represents (and is equal to) the volume of the parallelepiped whose adjacent sides are represented by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$.

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{a}\cdot(bc\sin\theta\,\hat{n})$$

$$= bc\sin\theta\,(\vec{a}\cdot\hat{n})$$
$$= bc\sin\theta\cdot a\cdot 1\cdot\cos\alpha$$
$$= (a\cos\alpha)(bc\sin\theta)$$
$$= \text{height}\times(\text{area of base})$$
$$= \text{volume of parallelepiped}$$

Also the volume of the tetrahedron $ABCD$ is equal to $\frac{1}{6}(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AD}$

Properties of Scalar Triple Product

1. $(\vec{a}\times\vec{b})\cdot\vec{c} = \vec{a}\cdot(\vec{b}\times\vec{c})$, i.e., position of the dot and the cross can be interchanged without altering the product.
2. $[k\vec{a}\vec{b}\vec{c}] = k[\vec{a}\vec{b}\vec{c}]$ (where k is scalar)
3. $[\vec{a}+\vec{b}\ \vec{c}\ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$
4. $\vec{a},\vec{b}$ and $\vec{c}$ in that order form a right-handed system if $[\vec{a}\ \vec{b}\ \vec{c}] > 0$

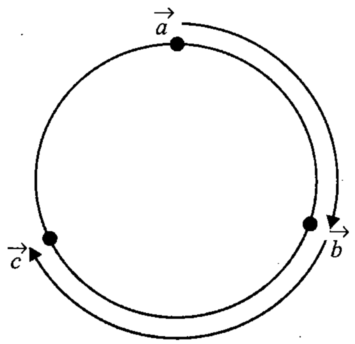


Fig. 2.26

$\vec{a},\vec{b}$ and $\vec{c}$ in that order form a left-handed system if $[\vec{a}\vec{b}\vec{c}] < 0$.

5. The necessary and sufficient condition for three non-zero, non-collinear vectors $\vec{a},\vec{b}$ and $\vec{c}$ to be coplanar is that $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.
6. $[\vec{a}\ \vec{a}\ \vec{b}] = 0$ ($\because$ $\vec{a}$ is $\perp$ to $\vec{a}\times\vec{b}$, $\vec{a}\cdot(\vec{a}\times\vec{b}) = 0$)

Example 2.49 **If $\vec{a},\vec{b}$ and $\vec{c}$ are three non-coplanar vectors, then find the value of** $\dfrac{\vec{a}\cdot(\vec{b}\times\vec{c})}{\vec{b}\cdot(\vec{c}\times\vec{a})} + \dfrac{\vec{b}\cdot(\vec{c}\times\vec{a})}{\vec{c}\cdot(\vec{a}\times\vec{b})} + \dfrac{\vec{c}\cdot(\vec{b}\times\vec{a})}{\vec{a}\cdot(\vec{b}\times\vec{c})}$.

Sol. Since, $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$

$$\frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{\vec{b}\cdot(\vec{c}\times\vec{a})}+\frac{\vec{b}\cdot(\vec{c}\times\vec{a})}{\vec{c}\cdot(\vec{a}\times\vec{b})}+\frac{\vec{c}\cdot(\vec{b}\times\vec{a})}{\vec{a}\cdot(\vec{b}\times\vec{c})}=\frac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{b}\,\vec{c}\,\vec{a}]}+\frac{[\vec{b}\,\vec{c}\,\vec{a}]}{[\vec{c}\,\vec{a}\,\vec{b}]}+\frac{[\vec{c}\,\vec{b}\,\vec{a}]}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

$$=\frac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}+\frac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}-\frac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

$$=1+1-1=1$$

Example 2.50 **If the vectors $2\hat{i}-3\hat{j},\hat{i}+\hat{j}-\hat{k}$ and $3\hat{i}-\hat{k}$ form three concurrent edges of a parallelopiped, then find the volume of the parallelopiped.**

Sol. Here, $\overrightarrow{OA}=2\hat{i}-3\hat{j}=\vec{a}$ (say),

$\overrightarrow{OB}=\hat{i}+\hat{j}-\hat{k}=\vec{b}$ (say),

and $\overrightarrow{OC}=3\hat{i}-\hat{k}=\vec{c}$ (say)

Hence, volume is $[\vec{a}\,\vec{b}\,\vec{c}]=\vec{a}\cdot(\vec{b}\times\vec{c})=\begin{vmatrix}2 & -3 & 0\\ 1 & 1 & -1\\ 3 & 0 & -1\end{vmatrix}=4$

Example 2.51 **Prove that $[\vec{a}+\vec{b}\;\;\vec{b}+\vec{c}\;\;\vec{c}+\vec{a}]=2\,[\vec{a}\;\;\vec{b}\;\;\vec{c}]$.**

Sol. $[\vec{a}+\vec{b}\;\;\vec{b}+\vec{c}\;\;\vec{c}+\vec{a}]=(\vec{a}+\vec{b})\cdot((\vec{b}+\vec{c})\times(\vec{c}+\vec{a}))$

$$=(\vec{a}+\vec{b})\cdot(\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a})$$

$$=[\vec{a}\,\vec{b}\,\vec{c}]+[\vec{b}\,\vec{c}\,\vec{a}]=2[\vec{a}\,\vec{b}\,\vec{c}]$$

Example 2.52 **Prove that $[\vec{l}\;\vec{m}\;\vec{n}][\vec{a}\;\vec{b}\;\vec{c}]=\begin{vmatrix}\vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} & \vec{l}\cdot\vec{c}\\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} & \vec{m}\cdot\vec{c}\\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} & \vec{n}\cdot\vec{c}\end{vmatrix}$.**

Sol. Let $\vec{l}=l_1\hat{i}+l_2\hat{j}+l_3\hat{k}$, $\vec{m}=m_1\hat{i}+m_2\hat{j}+m_3\hat{k}$ and $\vec{n}=n_1\hat{i}+n_2\hat{j}+n_3\hat{k}$

$\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$. Therefore,

$\vec{l}\cdot\vec{a}=l_1a_1+l_2a_2+l_3a_3=\Sigma l_1a_1$

Similarly, $\vec{l}\cdot\vec{b}=\Sigma l_1b_1$, etc.

Now $[\vec{l}\,\vec{m}\,\vec{n}][\vec{a}\,\vec{b}\,\vec{c}]=\begin{vmatrix}l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\\ n_1 & n_2 & n_3\end{vmatrix}\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$

$$= \begin{vmatrix} \Sigma l_1 a_1 & \Sigma l_1 b_1 & \Sigma l_1 c_1 \\ \Sigma m_1 a_1 & \Sigma m_1 b_1 & \Sigma m_1 c_1 \\ \Sigma n_1 a_1 & \Sigma n_1 b_1 & \Sigma n_1 c_1 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{l}.\vec{a} & \vec{l}.\vec{b} & \vec{l}.\vec{c} \\ \vec{m}.\vec{a} & \vec{m}.\vec{b} & \vec{m}.\vec{c} \\ \vec{n}.\vec{a} & \vec{n}.\vec{b} & \vec{n}.\vec{c} \end{vmatrix}$$

Example 2.53 **Find the value of a so that the volume of the parallelopiped formed by vectors $\hat{i}+a\hat{j}+\hat{k}, \hat{j}+a\hat{k}$ and $a\hat{i}+\hat{k}$ becomes minimum.**

Sol. $V = \begin{vmatrix} 1 & a & 1 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = 1 - a + a^3$

$\Rightarrow \dfrac{dV}{da} = 3a^2 - 1$

Sign scheme for $3a^2 - 1$ is as follows

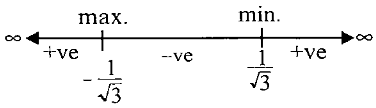


Fig. 2.27

V is minimum at $a = \dfrac{1}{\sqrt{3}}$

Example 2.54 **If $\vec{u}, \vec{v}$ and $\vec{w}$ are three non-coplanar vectors, then prove that $(\vec{u}+\vec{v}-\vec{w})\cdot(\vec{u}-\vec{v})\times(\vec{v}-\vec{w}) = \vec{u}\cdot\vec{v}\times\vec{w}$**

Sol. $(\vec{u}+\vec{v}-\vec{w})\cdot(\vec{u}-\vec{v})\times(\vec{v}-\vec{w}) = (\vec{u}+\vec{v}-\vec{w})\cdot(\vec{u}\times\vec{v}-\vec{u}\times\vec{w}-\vec{v}\times\vec{v}+\vec{v}\times\vec{w})$

$= (\vec{u}+\vec{v}-\vec{w})\cdot(\vec{u}\times\vec{v}-\vec{u}\times\vec{w}+\vec{v}\times\vec{w})$

$= 0 - 0 + \vec{u}\cdot(\vec{v}\times\vec{w}) + 0 - \vec{v}\cdot(\vec{u}\times\vec{w}) + 0 - \vec{w}\cdot(\vec{u}\times\vec{v}) + 0 - 0$

$= [\vec{u}\ \vec{v}\ \vec{w}] + [\vec{v}\ \vec{w}\ \vec{u}] - [\vec{w}\ \vec{u}\ \vec{v}] = \vec{u}\cdot(\vec{v}\times\vec{w})$

Example 2.55 **If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}\times\vec{b}| = 2$, then find the value of $[\vec{a}\ \vec{b}\ \vec{a}\times\vec{b}]$.**

Sol. $[\vec{a}\ \vec{b}\ \vec{a}\times\vec{b}] = (\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})$

$= |\vec{a}\times\vec{b}|^2$

$= 4$

Example 2.56 **Find the altitude of a parallelopiped whose three coterminous edges are vectors $\vec{A}=\hat{i}+\hat{j}+\hat{k}$, $\vec{B}=2\hat{i}+4\hat{j}-\hat{k}$ and $\vec{C}=\hat{i}+\hat{j}+3\hat{k}$ with $\vec{A}$ and $\vec{B}$ as the sides of the base of the parallelopiped.**

Sol. $h=\dfrac{\text{volume of parallelopiped}}{\text{area of base}}$

$$=\frac{[\vec{A}\,\vec{B}\,\vec{C}]}{|\vec{A}\times\vec{B}|}=\frac{\begin{vmatrix}1&1&1\\2&4&-1\\1&1&3\end{vmatrix}}{\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&1&1\\2&4&-1\end{vmatrix}}=\frac{4}{|-5\hat{i}+3\hat{j}+2\hat{k}|}=\frac{2\sqrt{38}}{19}$$

Example 2.57 **If $\vec{a},\vec{b}$ and $\vec{c}$ are mutually perpendicular vectors and $\vec{a}=\alpha(\vec{a}\times\vec{b})+\beta(\vec{b}\times\vec{c})+\gamma(\vec{c}\times\vec{a})$ and $[\vec{a}\,\vec{b}\,\vec{c}]=1$, then find the value of $\alpha+\beta+\gamma$.**

Sol. Taking dot product with $\vec{a}$, $\vec{b}$ and $\vec{c}$, respectively, we get

$|\vec{a}|^2=\beta\cdot[\vec{a}\,\vec{b}\,\vec{c}]=\beta$

$0=\gamma\cdot[\vec{a}\,\vec{b}\,\vec{c}]=\gamma$

and $0=\alpha\cdot[\vec{a}\,\vec{b}\,\vec{c}]=\alpha$

$\therefore\ \alpha+\beta+\gamma=|\vec{a}|^2$

Example 2.58 **If $\vec{a},\vec{b}$ and $\vec{c}$ are non-coplanar vectors, then prove that $|(\vec{a}\cdot\vec{d})(\vec{b}\times\vec{c})+(\vec{b}\cdot\vec{d})(\vec{c}\times\vec{a})+(\vec{c}\cdot\vec{d})(\vec{a}\times\vec{b})|$ is independent of $\vec{d}$, where $\vec{d}$ is a unit vector.**

Sol. Given $[\vec{a}\,\vec{b}\,\vec{c}]\neq 0$ as $\vec{a},\vec{b},\vec{c}$ are non-coplanar. Also there does not exist any linear relation between them because if any such relation exists, then they would be coplanar.

Let $A=x(\vec{b}\times\vec{c})+y(\vec{c}\times\vec{a})+z(\vec{a}\times\vec{b})$,

where $x=\vec{a}\cdot\vec{d}$, $y=\vec{b}\cdot\vec{d}$, $z=\vec{c}\cdot\vec{d}$

We have to find the value of modulus of $\vec{A}$, i.e., $|\vec{A}|$, which is independent of $\vec{d}$.

Multiplying both sides scalarly by $\vec{a}$, $\vec{b}$ and $\vec{c}$ and we know that scalar triple product is zero when two vectors are equal.

$\vec{A}\cdot\vec{a}=x[\vec{a}\,\vec{b}\,\vec{c}]+0$

Putting for x, we get

$(\vec{a}\cdot\vec{d})[\vec{a}\,\vec{b}\,\vec{c}]=\vec{A}\cdot\vec{a}$

Similarly, we have

$(\vec{b}\cdot\vec{d})[\vec{a}\ \vec{b}\ \vec{c}]=\vec{A}\cdot\vec{b}$

$(\vec{c}\cdot\vec{d}][\vec{a}\ \vec{b}\ \vec{c}]=\vec{A}\cdot\vec{c}$

Adding the above relations, we get

$[(\vec{a}+\vec{b}+\vec{c})\cdot\vec{d}]\ [\vec{a}\ \vec{b}\ \vec{c}\,]=\vec{A}\cdot(\vec{a}+\vec{b}+\vec{c})$

or $(\vec{a}+\vec{b}+\vec{c})\cdot[\vec{d}\,[\vec{a}\ \vec{b}\ \vec{c}]-\vec{A}\,]=0$

Since $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar, $\vec{a}+\vec{b}+\vec{c}\neq 0$ because otherwise any one is expressible as a linear combination of other two.

Hence $[\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d}=\vec{A}$

$|\vec{A}|=|[\vec{a}\ \vec{b}\ \vec{c}]|$ as $\vec{d}$ is a unit vector.

It is independent of $\vec{d}$.

Example 2.59 **Prove that vectors**

$$\vec{u}=(al+a_1l_1)\,\hat{i}+(am+a_1m_1)\,\hat{j}+(an+a_1n_1)\,\hat{k}$$
$$\vec{v}=(bl+b_1l_1)\,\hat{i}+(bm+b_1m_1)\,\hat{j}+(bn+b_1n_1)\,\hat{k}$$
$$\vec{w}=(cl+c_1l_1)\,\hat{i}+(cm+c_1m_1)\,\hat{j}+(cn+c_1n_1)\,\hat{k}$$

are coplanar.

Sol. $$[\vec{u}\ \vec{v}\ \vec{w}]=\begin{vmatrix} al+a_1l_1 & am+a_1m_1 & an+a_1n_1 \\ bl+b_1l_1 & bm+b_1m_1 & bn+b_1n_1 \\ cl+c_1l_1 & cm+c_1m_1 & cn+c_1n_1 \end{vmatrix}$$

$$\Rightarrow [\vec{u}\ \vec{v}\ \vec{w}]=\begin{vmatrix} a & a_1 & 0 \\ b & b_1 & 0 \\ c & c_1 & 0 \end{vmatrix}\begin{vmatrix} l & l_1 & 0 \\ m & m_1 & 0 \\ n & n_1 & 0 \end{vmatrix}=0$$

Therefore, the given vectors are coplanar.

Example 2.60 **Let G_1, G_2 and G_3 be the centroids of the trianglular faces *OBC*, *OCA* and *OAB*, respectively, of a tetrahedron *OABC*. If V_1 denotes the volume of the tetrahedron *OABC* and V_2 that of the parallelopiped with OG_1, OG_2 and OG_3 as three concurrent edges, then prove that $4V_1=9V_1$.**

Sol. Taking O as the origin, let the position vectors of A, B and C be $\vec{a}$, $\vec{b}$ and $\vec{c}$, respectively. Then the position vectors G_1, G_2 and G_3 are $\frac{\vec{b}+\vec{c}}{3}$, $\frac{\vec{c}+\vec{a}}{3}$ and $\frac{\vec{a}+\vec{b}}{3}$, respectively. Therefore,

$V_1=\frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$ and $V_2=[\overrightarrow{OG_1}\ \overrightarrow{OG_2}\ \overrightarrow{OG_3}]$

Now, $V_2 = [\overrightarrow{OG_1}\ \overrightarrow{OG_2}\ \overrightarrow{OG_3}]$

$\Rightarrow V_2 = \frac{1}{27}[\vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\ \ \vec{a}+\vec{b}]$

$\Rightarrow V_2 = \frac{2}{27}[\vec{a}\ \vec{b}\ \vec{c}]$

$\Rightarrow V_2 = \frac{2}{27} \times 6V_1 \Rightarrow 9V_2 = 4V_1$

VECTOR TRIPLE PRODUCT

The vector triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is the vector

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Also $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$

In general, $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

If $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$, then the vectors $\vec{a}$ and $\vec{c}$ are collinear.

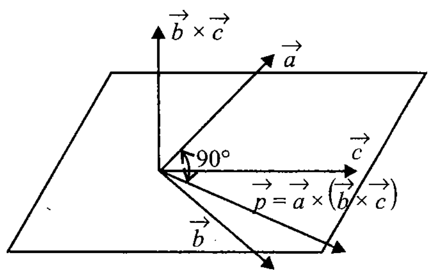


Fig. 2.28

$\vec{p} = \vec{a} \times (\vec{b} \times \vec{c})$ is a vector perpendicular to $\vec{a}$ and $\vec{b} \times \vec{c}$, but $\vec{b} \times \vec{c}$ is a vector perpendicular to the plane of $\vec{b}$ and $\vec{c}$.

$\Rightarrow$ Vector $\vec{p}$ must lie in the plane of $\vec{b}$ and $\vec{c}$.

$\Rightarrow\ \vec{p} = \vec{a} \times (\vec{b} \times \vec{c}) = x\vec{b} + y\vec{c}$ (i)

Multiplying (i) scalarly by $\vec{a}$, we have $\vec{p} \cdot \vec{a} = x(\vec{a} \cdot \vec{b}) + y(\vec{a} \cdot \vec{c})$ (ii)

But $\vec{p} \perp \vec{a} \Rightarrow \vec{p} \cdot \vec{a} = 0$. Therefore,

$x(\vec{a} \cdot \vec{b}) = -y(\vec{c} \cdot \vec{a})$, i.e., $= \frac{x}{\vec{c} \cdot \vec{a}} = \frac{-y}{\vec{a} \cdot \vec{b}} = \lambda$

$\therefore x = \lambda(\vec{c} \cdot \vec{a}),\ y = -\lambda(\vec{a} \cdot \vec{b})$ (iii)

Substituting x and y from (iii) in (i), $\vec{a} \times (\vec{b} \times \vec{c}) = \lambda[(\vec{c} \cdot \vec{a})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}]$ (iv)

The simplest way to determine λ is by taking specific vectors $\vec{a} = \hat{i}, \vec{b} = \hat{i}, \vec{c} = \hat{j}$

We have from (iv), $\hat{i}\times(\hat{i}\times\hat{j})=\lambda[(\hat{i}\cdot\hat{j})\hat{i}-(\hat{i}\cdot\hat{i})\hat{j}]$, i.e., $\hat{i}\times\hat{k}=\lambda[0\hat{i}-1\hat{j}]$, i.e., $-\hat{j}=-\lambda\hat{j}$

$\therefore \lambda=1$

Substituting λ in (iv), $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

Lagrange's Identity

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=\vec{a}\cdot[\vec{b}\times(\vec{c}\times\vec{d})]$$

$$=\vec{a}\cdot[(\vec{b}\cdot\vec{d})\vec{c}-(\vec{b}\cdot\vec{c})\vec{d}]$$

$$=(\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$$

$$=\begin{vmatrix}\vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d}\\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d}\end{vmatrix}$$

This is called Lagrange's identity.

Note:

$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[(\vec{a}\times\vec{b})\cdot\vec{d}]\vec{c}-[(\vec{a}\times\vec{b})\cdot\vec{c}]\vec{d}=[\vec{a}\,\vec{b}\,\vec{d}]\vec{c}-[\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$

Thus vector $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$ *lies in the plane of* $\vec{c}$ *and* $\vec{d}$; *otherwise*

$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=-(\vec{c}\times\vec{d})\times(\vec{a}\times\vec{b})=-[(\vec{c}\times\vec{d})\cdot\vec{b}]\vec{a}+[(\vec{c}\times\vec{d})\vec{a}]\vec{b}$

which shows that the vector lies in the plane of $\vec{a}$ and $\vec{b}$. *Thus the vector lies along the common section of the plane of* $\vec{c}$ and $\vec{d}$ *and the plane of* $\vec{a}$ and $\vec{b}$.

Example 2.61 **Prove that $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})=2\vec{a}$.**

Sol. $\hat{i}\times(\vec{a}\times\hat{i})=(\hat{i}\cdot\hat{i})\vec{a}-(\vec{a}\cdot\hat{i})\hat{i}=\vec{a}-(\vec{a}\cdot\hat{i})\hat{i}$

Similarly, $\hat{j}\times(\vec{a}\times\hat{j})=\vec{a}-(\vec{a}\cdot\hat{j})\hat{j}$ and $\hat{k}\times(\vec{a}\times\hat{k})=\vec{a}-(\vec{a}\cdot\hat{k})\hat{k}$. Therefore,

$\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})=3\vec{a}-((\vec{a}\cdot\hat{i})\hat{i}+(\vec{a}\cdot\hat{j})\hat{j}+(\vec{a}\times\hat{k})\hat{k})=2\vec{a}$

Example 2.62 **Let $\vec{a},\vec{b}$ and $\vec{c}$ be any three vectors, then prove that $[\vec{a}\times\vec{b}\;\;\vec{b}\times\vec{c}\;\;\vec{c}\times\vec{a}]=[\vec{a}\,\vec{b}\,\vec{c}]^2$.**

Sol. $[\vec{a}\times\vec{b}\;\;\vec{b}\times\vec{c}\;\;\vec{c}\times\vec{a}]=(\vec{a}\times\vec{b})\cdot((\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a}))$

$=(\vec{a}\times\vec{b})\cdot[[\vec{b}\,\vec{c}\,\vec{a}]\vec{c}-[\vec{b}\,\vec{c}\,\vec{c}]\vec{a}]$

$=[\vec{a}\,\vec{b}\,\vec{c}]^2$

Example 2.63 **For any four vectors, prove that $(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})+(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=0$.**

Sol. $(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d}) = (\vec{b}\cdot\vec{a})(\vec{c}\cdot\vec{d})-(\vec{b}\cdot\vec{d})(\vec{c}\cdot\vec{a})$

$(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d}) = (\vec{c}\cdot\vec{b})(\vec{a}\cdot\vec{d})-(\vec{c}\cdot\vec{d})(\vec{a}\cdot\vec{b})$

$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$

$\Rightarrow (\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})+(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=0$

Example 2.64 **Let $\hat{a},\hat{b}$ and $\hat{c}$ be the non-coplanar unit vectors. The angle between $\hat{b}$ and $\hat{c}$ is α, between $\hat{c}$ and $\hat{a}$ is β and between $\hat{a}$ and $\hat{b}$ is γ. If $A(\hat{a}\cos\alpha), B(\hat{b}\cos\beta)$ and $C(\hat{c}\cos\gamma)$, then show that in triangle ABC, $\dfrac{|\hat{a}\times(\hat{b}\times\hat{c})|}{\sin A}=\dfrac{|\hat{b}\times(\hat{c}\times\hat{a})|}{\sin B}=\dfrac{|\hat{c}\times(\hat{a}\times\hat{b})|}{\sin C}$**

$$=\frac{\prod|\hat{a}\times(\hat{b}\times\hat{c})|}{|\sum\sin\alpha.\cos\beta.\cos\gamma\,\hat{n}_1|}$$ **, where $\hat{n}_1=\dfrac{\hat{b}\times\hat{c}}{|\hat{b}\times\hat{c}|}$, $\hat{n}_2=\dfrac{\hat{c}\times\hat{a}}{|\hat{c}\times\hat{a}|}$ and $\hat{n}_3=\dfrac{\hat{a}\times\hat{b}}{|\hat{a}\times\hat{b}|}$.**

Sol. From the sine rule, we get

$$\frac{AB}{\sin C}=\frac{AC}{\sin B}=\frac{BC}{\sin A}=\frac{(AB)(BC)(CA)}{2\Delta ABC}$$

$BC=|\overrightarrow{BC}|=|\hat{c}\cos\gamma-\hat{b}\cos\beta|=|(\hat{a}\cdot\hat{b})\hat{c}-(\hat{c}\cdot\hat{a})\hat{b}|=|(\hat{a}\times(\hat{b}\times\hat{c}))|$

Similarly,

$AC=|\overrightarrow{AC}|=|\hat{b}\times(\hat{c}\times\hat{a})|$ and $AB=|\overrightarrow{AB}|=|\hat{c}\times(\hat{a}\times\hat{b})|$

Also,

$$\Delta ABC=\frac{1}{2}|\overrightarrow{BC}\times\overrightarrow{BA}|$$

$$=\frac{1}{2}|(\hat{c}\cos\gamma-\hat{b}\cos\beta)\times(\hat{a}\cos\alpha-\hat{b}\cos\beta)|$$

$$=\frac{1}{2}|(\hat{c}\times\hat{a})\cos\alpha\cos\gamma+(\hat{b}\times\hat{c})\cos\gamma\cos\beta+(\hat{a}\times\hat{b})\cos\beta\cos\alpha|$$

$\Rightarrow 2\Delta ABC=|\Sigma\hat{n}_1\sin\alpha\cos\beta\cos\gamma|$

$$\Rightarrow \frac{|\hat{a}\times(\hat{b}\times\hat{c})|}{\sin A}=\frac{|\hat{b}\times(\hat{c}\times\hat{a})|}{\sin B}=\frac{|\hat{c}\times(\hat{a}\times\hat{b})|}{\sin C}=\frac{\prod|\hat{a}\times(\hat{b}\times\hat{c})|}{|\Sigma\sin\alpha\cos\beta\cos\gamma\,\hat{n}_1|}$$

Example 2.65 **If $\vec{a},\vec{b}$ and $\vec{c}$ are three non-coplanar vectors, then prove that**

$$\vec{d}=\frac{\vec{a}\cdot\vec{d}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{b}\times\vec{c})+\frac{\vec{b}\cdot\vec{d}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{c}\times\vec{a})+\frac{\vec{c}\cdot\vec{d}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{a}\times\vec{b})$$

Sol. Since $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar, vectors $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$ are also non-coplanar. Let $\vec{d} = l(\vec{b}\times\vec{c}) + m(\vec{c}\times\vec{a}) + n(\vec{a}\times\vec{b})$ (i)

Now multiplying both sides of (i) scalarly by $\vec{a}$, we have

$\vec{a}\cdot\vec{d} = l\vec{a}\cdot(\vec{b}\times\vec{c}) + m\vec{a}\cdot(\vec{c}\times\vec{a}) + n\vec{a}\cdot(\vec{a}\times\vec{b}) = l[\vec{a}\,\vec{b}\,\vec{c}]$ $\quad\because [\vec{a}\,\vec{c}\,\vec{a}] = 0 = [\vec{a}\,\vec{a}\,\vec{b}]$

$\Rightarrow l = (\vec{a}\cdot\vec{d})/[\vec{a}\,\vec{b}\,\vec{c}]$

Similarly, multiplying (i) scalarly by $\vec{b}$ and $\vec{c}$ successively, we get

$m = (\vec{b}\cdot\vec{d})/[\vec{a}\,\vec{b}\,\vec{c}]$ and $n = (\vec{c}\cdot\vec{d})/[\vec{a}\,\vec{b}\,\vec{c}]$

Putting these values of l, m and n in (i), we get the required relation.

Example 2.66 **If $\vec{b}$ is not perpendicular to $\vec{c}$, then find the vector $\vec{r}$ satisfying the equation $\vec{r}\times\vec{b} = \vec{a}\times\vec{b}$ and $\vec{r}\cdot\vec{c} = 0$.**

Sol. Given $\vec{r}\times\vec{b} = \vec{a}\times\vec{b} \Rightarrow (\vec{r}-\vec{a})\times\vec{b} = 0$

Hence $(\vec{r}-\vec{a})$ and $\vec{b}$ are parallel.

$\Rightarrow \vec{r}-\vec{a} = t\vec{b}$ (i)

Also $\vec{r}\cdot\vec{c} = 0$

$\therefore$ Taking dot product of (i) by $\vec{c}$, we get $\vec{r}\cdot\vec{c} - \vec{a}\cdot\vec{c} = t(\vec{b}\cdot\vec{c})$

$$\Rightarrow 0 - \vec{a}\cdot\vec{c} = t(\vec{b}\cdot\vec{c}) \text{ or } t = -\left(\frac{\vec{a}\cdot\vec{c}}{\vec{b}\cdot\vec{c}}\right) \quad \text{(ii)}$$

From (i) and (ii), solution of $\vec{r}$ is $\vec{r} = \vec{a} - \left(\dfrac{\vec{a}\cdot\vec{c}}{\vec{b}\cdot\vec{c}}\right)\vec{b}$

Example 2.67 **If $\vec{a}$ and $\vec{b}$ are two given vectors and k is any scalar, then find the vector $\vec{r}$ satisfying $\vec{r}\times\vec{a} + k\vec{r} = \vec{b}$**

Sol. $\vec{r}\times\vec{a} + k\vec{r} = \vec{b}$ (i)

$\Rightarrow (\vec{r}\times\vec{a})\times\vec{a} + k\vec{r}\times\vec{a} = \vec{b}\times\vec{a}$

$\Rightarrow (\vec{r}\cdot\vec{a})\vec{a} - (\vec{a}\cdot\vec{a})\vec{r} + k(\vec{b} - k\vec{r}) = \vec{b}\times\vec{a}$

$\Rightarrow (\vec{r}\cdot\vec{a})\vec{a} + k\vec{b} - \vec{b}\times\vec{a} = (|\vec{a}|^2 + k^2)\vec{r}$

$$\Rightarrow \vec{r} = \frac{(\vec{r}\cdot\vec{a})\vec{a} + k\vec{b} - \vec{b}\times\vec{a}}{|\vec{a}|^2 + k^2}$$

Also in Eq. (i), taking dot product with $\vec{a}$, we have

$$(\vec{r}\times\vec{a})\cdot\vec{a}+k\vec{r}\cdot\vec{a}=\vec{b}\cdot\vec{a}$$

$$\Rightarrow \vec{r}\cdot\vec{a}=\frac{\vec{b}\cdot\vec{a}}{k}$$

$$\Rightarrow \vec{r}=\frac{1}{k^2+|\vec{a}|^2}\left[\frac{(\vec{a}\cdot\vec{b})\vec{a}}{k}+k\vec{b}+(\vec{a}\times\vec{b})\right]$$

Example 2.68 **If $\vec{r}\cdot\vec{a}=0$, $\vec{r}\cdot\vec{b}=1$ and $[\vec{r}\ \vec{a}\ \vec{b}]=1$, $\vec{a}\cdot\vec{b}\neq 0$, $(\vec{a}\cdot\vec{b})^2-|\vec{a}|^2|\vec{b}|^2=1$, then find $\vec{r}$ in terms of $\vec{a}$ and $\vec{b}$.**

Sol. Writing $\vec{r}$ as linear combination of $\vec{a}, \vec{b}$ and $\vec{a}\times\vec{b}$, we have

$$\vec{r}=x\vec{a}+y\vec{b}+z(\vec{a}\times\vec{b})$$

For scalars x, y and z

$0=\vec{r}\cdot\vec{a}=x|\vec{a}|^2+y\vec{a}\cdot\vec{b}$ (taking dot product with $\vec{a}$)

$1=\vec{r}\cdot\vec{b}=x\vec{a}\cdot\vec{b}+y|\vec{b}|^2$ (taking dot product with $\vec{b}$)

Solving, we get $y=\dfrac{|\vec{a}|^2}{|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2}=|\vec{a}|^2$

and $x=\dfrac{\vec{a}\cdot\vec{b}}{(\vec{a}\cdot\vec{b})^2-|\vec{a}|^2|\vec{b}|^2}=\vec{a}\cdot\vec{b}$

Also $1=[\vec{r}\,\vec{a}\,\vec{b}]=z|\vec{a}\times\vec{b}|^2$ (taking dot product with $\vec{a}\times\vec{b}$)

$$\Rightarrow z=\frac{1}{|a\times b|^2}$$

thus $\vec{r}=((\vec{a}\cdot\vec{b})\vec{a}-|\vec{a}|^2\vec{b})+\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|^2}$

$$=\vec{a}\times(\vec{a}\times\vec{b})+\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|^2}$$

Example 2.69 **If vector $\vec{x}$ satisfying $\vec{x}\times\vec{a}+(\vec{x}\cdot\vec{b})\vec{c}=\vec{d}$ is given by $\vec{x}=\lambda\vec{a}+\vec{a}\times\dfrac{\vec{a}\times(\vec{d}\times\vec{c})}{(\vec{a}\cdot\vec{c})|\vec{a}|^2}$, then find the value of λ.**

Sol. $\vec{x}\times\vec{a}+(\vec{x}\cdot\vec{b})\vec{c}=\vec{d}$

$\therefore \{\vec{x}\times\vec{a}+(\vec{x}\cdot\vec{b})\vec{c}\}\times\vec{c}=\vec{d}\times\vec{c}$

$\Rightarrow (\vec{x}\times\vec{a})\times\vec{c}+(\vec{x}\cdot\vec{b})(\vec{c}\times\vec{c})=\vec{d}\times\vec{c}$

$\Rightarrow (\vec{x}\cdot\vec{c})\vec{a}-(\vec{a}\cdot\vec{c})\vec{x}=(\vec{d}\times\vec{c})$

$\Rightarrow \vec{a}\times\{(\vec{x}\cdot\vec{c})\vec{a}-(\vec{a}\cdot\vec{c})\vec{x}\}=\vec{a}\times(\vec{d}\times\vec{c})$

$\Rightarrow -(\vec{a}\cdot\vec{c})(\vec{a}\times\vec{x})=\vec{a}\times(\vec{d}\times\vec{c})\quad (\because \vec{a}\times\vec{a}=0)$

$$\Rightarrow \vec{x}\times\vec{a}=\frac{\vec{a}\times(\vec{d}\times\vec{c})}{\vec{a}\cdot\vec{c}}$$

$$\Rightarrow \vec{a}\times(\vec{x}\times\vec{a})=\vec{a}\times\frac{\vec{a}\times(\vec{d}\times\vec{c})}{\vec{a}\cdot\vec{c}}$$

$$\Rightarrow (\vec{a}\cdot\vec{a})\vec{x}-(\vec{a}\cdot\vec{x})\vec{a}=\vec{a}\times\frac{\vec{a}\times(\vec{d}\times\vec{c})}{\vec{a}\cdot\vec{c}}$$

$$\Rightarrow (\vec{a}\cdot\vec{a})\vec{x}=(\vec{a}\cdot\vec{x})\vec{a}+\vec{a}\times\frac{\vec{a}\times(\vec{d}\times\vec{c})}{\vec{a}\cdot\vec{c}}$$

$$\Rightarrow \vec{x}=\frac{(\vec{a}\cdot\vec{x})\vec{a}}{|\vec{a}|^2}+\vec{a}\times\frac{\vec{a}\times(\vec{d}\times\vec{c})}{(\vec{a}\cdot\vec{c})|\vec{a}|^2}\quad \text{where } \lambda=\frac{\vec{a}\cdot\vec{x}}{|\vec{a}|^2}$$

Example 2.70 **$\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar vectors and $\vec{r}$ is any arbitrary vector. Prove that $[\vec{b}\,\vec{c}\,\vec{r}]\vec{a}+[\vec{c}\,\vec{a}\,\vec{r}]\vec{b}+[\vec{a}\,\vec{b}\,\vec{r}]\vec{c}=[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}$.**

Sol. Let $\vec{r}=x_1\vec{a}+x_2\vec{b}+x_3\vec{c}\Rightarrow \vec{r}\cdot(\vec{b}\times\vec{c})=x_1\vec{a}\cdot(\vec{b}\times\vec{c})\Rightarrow x_1=\dfrac{[\vec{r}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}$

Also, $\vec{r}\cdot(\vec{c}\times\vec{a}) = x_2\vec{b}\cdot(\vec{c}\times\vec{a})\Rightarrow x_2=\dfrac{[\vec{r}\,\vec{c}\,\vec{a}]}{[\vec{a}\,\vec{b}\,\vec{c}]}$ and $\vec{r}\cdot(\vec{a}\times\vec{b})=x_3\vec{c}\cdot(\vec{a}\times\vec{b})$

$$\Rightarrow x_3=\frac{[\vec{r}\,\vec{a}\,\vec{b}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\Rightarrow \vec{r}=\frac{[\vec{r}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{a}+\frac{[\vec{r}\,\vec{c}\,\vec{a}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{b}+\frac{[\vec{r}\,\vec{a}\,\vec{b}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{c}\Rightarrow[\vec{b}\,\vec{c}\,\vec{r}]\vec{a}+[\vec{c}\,\vec{a}\,\vec{r}]\vec{b}+[\vec{a}\,\vec{b}\,\vec{r}]\vec{c}=[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}$$

Example 2.71 **If $\vec{a},\vec{b}$ and $\vec{c}$ are non-coplanar unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c})=\dfrac{\vec{b}+\vec{c}}{\sqrt{2}}$, $\vec{b}$ and $\vec{c}$ are non-parallel, then prove that the angle between $\vec{a}$ and $\vec{b}$ is $3\pi/4$.**

Sol. $\vec{a}\times(\vec{b}\times\vec{c})=\dfrac{\vec{b}+\vec{c}}{\sqrt{2}}$

$$\Rightarrow (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}=\frac{1}{\sqrt{2}}\vec{b}+\frac{1}{\sqrt{2}}\vec{c}\qquad \text{(i)}$$

Since $\vec{b}$ and $\vec{c}$ are non-collinear, comparing coefficients of $\vec{c}$ on both sides of (i), we get

$$-\vec{a}.\vec{b}=\frac{1}{\sqrt{2}} \Rightarrow \vec{a}.\vec{b}=-\frac{1}{\sqrt{2}}$$

$$\Rightarrow (1)(1)\cos\theta=-\frac{1}{\sqrt{2}},$$

where θ is the angle between $\vec{a}$ and $\vec{b}$

$$\therefore \cos\theta=-\frac{1}{\sqrt{2}} \Rightarrow \cos\theta \cos 135°$$

$$\Rightarrow \theta=135°=3\pi/4$$

Example 2.72 **Prove that** $\vec{R}+\frac{[\vec{R}\cdot(\vec{\beta}\times(\vec{\beta}\times\vec{\alpha}))]\vec{\alpha}}{|\vec{\alpha}\times\vec{\beta}|^2}+\frac{[\vec{R}\cdot(\vec{\alpha}\times(\vec{\alpha}\times\vec{\beta}))]\vec{\beta}}{|\vec{\alpha}\times\vec{\beta}|^2}=\frac{[\vec{R}\vec{\alpha}\vec{\beta}](\vec{\alpha}\times\vec{\beta})}{|\vec{\alpha}\times\vec{\beta}|^2}$

Sol. $\vec{\alpha}, \vec{\beta}$ and $\vec{\alpha}\times\vec{\beta}$ are three non-coplanar vectors. Any vector $\vec{R}$ can be represented as a linear combination of these vectors.

$$\Rightarrow \vec{R}=k_1\vec{\alpha}+k_2\vec{\beta}+k_3(\vec{\alpha}\times\vec{\beta}) \quad \text{(i)}$$

Take dot product of (i) with $(\vec{\alpha}\times\vec{\beta})$

$$\Rightarrow \vec{R}\cdot(\vec{\alpha}\times\vec{\beta})=k_3(\vec{\alpha}\times\vec{\beta})\cdot(\vec{\alpha}\times\vec{\beta})=k_3|\vec{\alpha}\times\vec{\beta}|^2$$

$$\Rightarrow k_3=\frac{\vec{R}\cdot(\vec{\alpha}\times\vec{\beta})}{|\vec{\alpha}\times\vec{\beta}|^2}=\frac{[\vec{R}\vec{\alpha}\vec{\beta}]}{|\vec{\alpha}\times\vec{\beta}|^2}$$

Take dot product of (i) with $\vec{\alpha}\times(\vec{\alpha}\times\vec{\beta})$

$$\Rightarrow \vec{R}\cdot(\vec{\alpha}\times(\vec{\alpha}\times\vec{\beta}))=k_2(\vec{\alpha}\times(\vec{\alpha}\times\vec{\beta}))\cdot\vec{\beta}$$

$$=k_2[(\vec{\alpha}\cdot\vec{\beta})\vec{\alpha}-(\vec{\alpha}\cdot\vec{\alpha})\vec{\beta}]\cdot\vec{\beta}=k_2[(\vec{\alpha}\cdot\vec{\beta})^2-|\vec{\alpha}|^2|\vec{\beta}|^2]$$

$$=-k_2|\vec{\alpha}\times\vec{\beta}|^2$$

$$\Rightarrow k_2=\frac{-[\vec{R}\cdot(\vec{\alpha}\times(\vec{\alpha}\times\vec{\beta}))]}{|\vec{\alpha}\times\vec{\beta}|^2} \quad \text{Similarly, } k_1=-\frac{[\vec{R}\cdot(\vec{\beta}\times(\vec{\beta}\times\vec{\alpha}))]}{|\vec{\alpha}\times\vec{\beta}|^2}$$

$$\Rightarrow \vec{R}=\frac{-[\vec{R}.[\vec{\beta}\times(\vec{\beta}\times\vec{\alpha}))]\vec{\alpha}}{|\vec{\alpha}\times\vec{\beta}|^2}-\frac{[\vec{R}\cdot(\vec{\alpha}\times(\vec{\alpha}\times\vec{\beta}))]\vec{\beta}}{|\vec{\alpha}\times\vec{\beta}|^2}+\frac{[(\vec{R}\cdot(\vec{\alpha}\times\vec{\beta}))](\vec{\alpha}\times\vec{\beta})}{(\vec{\alpha}\times\vec{\beta})^2}$$

$$\Rightarrow \vec{R}+\frac{[\vec{R}\cdot(\vec{\beta}\times(\vec{\beta}\times\vec{\alpha}))]\vec{\alpha}}{|\vec{\alpha}\times\vec{\beta}|^2}+\frac{[\vec{R}\cdot(\vec{\alpha}\times(\vec{\alpha}\times\vec{\beta}))]\vec{\beta}}{|\vec{\alpha}\times\vec{\beta}|^2}=\frac{[\vec{R}\cdot(\vec{\alpha}\times\vec{\beta})](\vec{\alpha}\times\vec{\beta})}{|\vec{\alpha}\times\vec{\beta}|^2}$$

Example 2.73 **If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-coplanar non-zero vectors, then prove that $(\vec{a}\cdot\vec{a})\ \vec{b}\times\vec{c} + (\vec{a}\cdot\vec{b})\ \vec{c}\times\vec{a} + (\vec{a}\cdot\vec{c})\ \vec{a}\times\vec{b} = [\vec{b}\ \vec{c}\ \vec{a}]\vec{a}$.**

Sol. As $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar, $\vec{b}\times\vec{a}$, $\vec{c}\times\vec{a}$ and $\vec{a}\times\vec{b}$ are also non-coplanar.

So, any vector can be expressed as a linear combination of these vectors.

Let $\vec{a} = \lambda\,\vec{b}\times\vec{c} + \mu\,\vec{c}\times\vec{a} + \nu\,\vec{a}\times\vec{b}$

$\therefore\ \vec{a}\cdot\vec{a} = \lambda[\vec{b}\ \vec{c}\ \vec{a}],\ \vec{a}\cdot\vec{b} = \mu[\vec{c}\ \vec{a}\ \vec{b}],\ \vec{a}\cdot\vec{c} = \nu\ [\vec{a}\ \vec{b}\ \vec{c}]$

$$\therefore\ \vec{a} = \frac{(\vec{a}\cdot\vec{a})\vec{b}\times\vec{c}}{[\vec{b}\ \vec{c}\ \vec{a}]} + \frac{(\vec{a}\cdot\vec{b})\vec{c}\times\vec{a}}{[\vec{c}\ \vec{a}\ \vec{b}]} + \frac{(\vec{a}\cdot\vec{c})\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

RECIPROCAL SYSTEM OF VECTORS

Two systems of vectors are called reciprocal systems of vectors if by taking the dot product we get unity.

Thus if $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-coplanar vectors, and if

$\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $\vec{b}' = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ and $\vec{c}' = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, then $\vec{a}', \vec{b}', \vec{c}'$ are said to be the reciprocal systems of vectors for vectors $\vec{a}, \vec{b}$ and $\vec{c}$.

Properties

i. *If $\vec{a}, \vec{b}$ and $\vec{c}$ and $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ are reciprocal system of vectors, then* $\vec{a}\cdot\vec{a}' = \dfrac{\vec{a}\cdot(\vec{b}\times\vec{c})}{(\vec{a}\,\vec{b}\,\vec{c})} = \dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]} = 1.$
Similarly, $\vec{b}\cdot\vec{b}' = \vec{c}\cdot\vec{c}' = 1$.

Due to the above property, the two systems of vectors are called reciprocal systems.

ii. $\vec{a}\cdot\vec{b}' = \vec{a}\cdot\vec{c}' = \vec{b}\cdot\vec{a}' = \vec{b}\cdot\vec{c}' = \vec{c}\cdot\vec{a}' = \vec{c}\cdot\vec{b}' = 0$

iii. $[\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{a}'\vec{b}'\vec{c}'] = 1$

Proof:

We have $[\vec{a}'\vec{b}'\vec{c}'] = \left[\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}\ \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}\ \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}\right] = \dfrac{1}{[\vec{a}\,\vec{b}\,\vec{c}]^3}[\vec{b}\times\vec{c}\ \vec{c}\times\vec{a}\ \vec{a}\times\vec{b}] = \dfrac{1}{[\vec{a}\,\vec{b}\,\vec{c}]^3}[\vec{a}\,\vec{b}\,\vec{c}]^2 = \dfrac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$\Rightarrow [\vec{a}'\vec{b}'\vec{c}'][\vec{a}\,\vec{b}\,\vec{c}] = 1$

iv. *The orthogonal triad of vectors $\hat{i}, \hat{j}$ and $\hat{k}$ is self-reciprocal.*

Let $\hat{i}'$, $\hat{j}'$ and $\hat{k}'$ be the system of vectors reciprocal to the system $\hat{i}$, $\hat{j}$ and $\hat{k}$. Then, *we have* $\hat{i}' = \dfrac{\hat{j}\times\hat{k}}{[\hat{i}\,\hat{j}\,\hat{k}]} = \hat{i}$. *Similarly*, $\hat{j}' = \hat{j}$ *and* $\hat{k}' = \hat{k}$.

v. $\vec{a}, \vec{b}$ and $\vec{c}$ *are non-coplanar iff* $\vec{a}', \vec{b}'$ and $\vec{c}'$ *are non-coplanar.*

As $[\vec{a}\,\vec{b}\,\vec{c}]\cdot[\vec{a}'\,\vec{b}'\,\vec{c}'] = 1$ *and* $[\vec{a}\,\vec{b}\,\vec{c}] \neq 0$ *are non-coplanar* $\Leftrightarrow \dfrac{1}{[\vec{a}\,\vec{b}\,\vec{c}]} \neq 0 \Leftrightarrow [\vec{a}'\,\vec{b}'\,\vec{c}']$ *are non-coplanar.*

Example 2.74 **Find a set of vectors reciprocal to the set** $-\hat{i}+\hat{j}+\hat{k}, \hat{i}-\hat{j}+\hat{k}, \hat{i}+\hat{j}+\hat{k}$.

Sol. Let $\vec{a} = -\hat{i}+\hat{j}+\hat{k}$, $\vec{b} = \hat{i}-\hat{j}+\hat{k}$, $\vec{c} = \hat{i}+\hat{j}+\hat{k}$

Then $\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -2\hat{i}+2\hat{k}$, $\vec{c}\times\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = -2\hat{j}+2\hat{k}$, $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$

$= 2\hat{i}+2\hat{j}$

$$[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 4$$

If a', b', c' is the reciprocal system of vectors, then

$$\vec{a}' = (\vec{b}\times\vec{c})/[\vec{a}\,\vec{b}\,\vec{c}] = \frac{1}{2}(-\hat{i}+\hat{k}),\ \vec{b}' = (\vec{c}\times\vec{a})/[\vec{a}\,\vec{b}\,\vec{c}] = \frac{1}{2}(-\hat{j}+\hat{k}),$$

$$\vec{c}' = (\vec{a}\times\vec{b})/[\vec{a}\,\vec{b}\,\vec{c}] = \frac{1}{2}(\hat{i}+\hat{j})$$

Example 2.75 **Let $\vec{a}, \vec{b}$ and $\vec{c}$ be a set of non-coplanar vectors and $\vec{a}', \vec{b}'$ and $\vec{c}'$ be its reciprocal set. Prove that** $\vec{a} = \dfrac{b'\times c'}{[\vec{a}'\,\vec{b}'\,\vec{c}']}$, $\vec{b} = \dfrac{c'\times a'}{[\vec{a}'\,\vec{b}'\,\vec{c}']}$ **and** $\vec{c} = \dfrac{a'\times b'}{[\vec{a}'\,\vec{b}'\,\vec{c}']}$.

Sol. We have, $\vec{b}'\times\vec{c}' = \dfrac{(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})}{[\vec{a}\,\vec{b}\,\vec{c}]^2}$

$$= \frac{\{(\vec{c}\times\vec{a})\cdot\vec{b}\}\vec{a} - \{(\vec{c}\times\vec{a})\cdot\vec{a}\}\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]^2} = \frac{[\vec{c}\,\vec{a}\,\vec{b}]\vec{a} - [\vec{c}\,\vec{a}\,\vec{a}]\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]^2} = \frac{[\vec{a}\,\vec{b}\,\vec{c}]\vec{a} - 0}{[\vec{a}\,\vec{b}\,\vec{c}]^2} = \frac{\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Also, $[\vec{a}'\,\vec{b}'\,\vec{c}'] = \vec{a}'\cdot(\vec{b}'\times\vec{c}') = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}\cdot\dfrac{\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]} = \dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]^2} = \dfrac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$$\Rightarrow \frac{\vec{b}' \times \vec{c}'}{[\vec{a}' \vec{b}' \vec{c}']} = \vec{a}$$

Similarly, $\vec{b} = \dfrac{\vec{c}' \times \vec{a}'}{[\vec{a}' \vec{b}' \vec{c}']}$, $\vec{c} = \dfrac{\vec{a}' \times \vec{b}'}{[\vec{a}' \vec{b}' \vec{c}']}$

Example 2.76 **If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are reciprocal system of vectors, then prove that $\vec{a}' \times \vec{b}' + \vec{b}' \times \vec{c}' + \vec{c}' \times \vec{a}' = \dfrac{\vec{a} + \vec{b} + \vec{c}}{[\vec{a}\, \vec{b}\, \vec{c}]}$.**

Sol. $\vec{a}' \times \vec{b}' = \dfrac{(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a})}{[\vec{a}\, \vec{b}\, \vec{c}]^2} = \dfrac{\{(\vec{b} \times \vec{c}) \cdot \vec{a}\}\vec{c} - \{(\vec{b} \times \vec{c}) \cdot \vec{c}\}\vec{a}}{[\vec{a}\, \vec{b}\, \vec{c}]^2} = \dfrac{[\vec{b}\, \vec{c}\, \vec{a}]\vec{c}}{[\vec{a}\, \vec{b}\, \vec{c}]^2} = \dfrac{[\vec{a}\, \vec{b}\, \vec{c}]\vec{c}}{[\vec{a}\, \vec{b}\, \vec{c}]^2} = \dfrac{\vec{c}}{[\vec{a}\, \vec{b}\, \vec{c}]}$

Similarly, $\vec{b}' \times \vec{c}' = \dfrac{\vec{a}}{[\vec{a} \times \vec{b} \times \vec{c}]}$ and $\vec{c}' \times \vec{a}' = \dfrac{\vec{b}}{[\vec{a}\, \vec{b}\, \vec{c}]}$

Adding, $\vec{a}' \times \vec{b}' + \vec{b}' \times \vec{c}' + \vec{c}' \times \vec{a}' = \dfrac{\vec{a} + \vec{b} + \vec{c}}{[\vec{a}\, \vec{b}\, \vec{c}]}$

Example 2.77 **If $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-coplanar vectors and a', b' and c' constitute the reciprocal system of vectors, then prove that**

i. $\vec{r} = (\vec{r} \cdot \vec{a}')\vec{a} + (\vec{r} \cdot \vec{b}')\vec{b} + (\vec{r} \cdot \vec{c}')\vec{c}$

ii. $\vec{r} = (\vec{r} \cdot \vec{a})\vec{a}' + (\vec{r} \cdot \vec{b})\vec{b}' + (\vec{r} \cdot \vec{c})\vec{c}'$

Sol. i. Since a vector can be expressed as a linear combination of three non-coplanar vectors, therefore let $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$ (i)

where x, y and z are scalars.

Multiplying both sides of (i) scalarly by $\vec{a}'$, we get

$\vec{r} \cdot \vec{a}' = x\vec{a} \cdot \vec{a}' + y\vec{b} \cdot \vec{a}' + z\vec{c} \cdot \vec{a}' = x \cdot 1 = x$

$(\because \vec{a} \cdot \vec{a}' = 1,\ \vec{b} \cdot \vec{a}' = 0 = \vec{c} \cdot \vec{a}')$

Similarly multiplying both sides of (i) scalarly by $\vec{b}'$ and $\vec{c}'$, successively, we get

$y = \vec{r} \cdot \vec{b}'$ and $z = \vec{r} \cdot \vec{c}'$

Putting in (i), we get $\vec{r} = (\vec{r} \cdot \vec{a}')\vec{a} + (\vec{r} \cdot \vec{b}')\vec{b} + (\vec{r} \cdot \vec{c}')\vec{c}$

ii. Since $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ are three non-coplanar vectors, we can take $\vec{r} = x\vec{a}' + y\vec{b}' + z\vec{c}'$ (ii)

Multiplying both sides of (ii) scalarly by $\vec{a}$, we get $\vec{r} \cdot \vec{a} = x(\vec{a}' \cdot \vec{a}) + y(\vec{b}' \cdot \vec{a}) + z(\vec{c}' \cdot \vec{a}) = x$

$(\because \vec{a}' \cdot \vec{a} = 1,\ \vec{b}' \cdot \vec{a} = 0 = \vec{c}' \cdot \vec{a})$

Similarly, multiplying both sides of (i) scalarly by $\vec{b}$ and $\vec{c}$ successively, we get

$y = \vec{r} \cdot \vec{b}$ and $z = \vec{r} \cdot \vec{c}$

Putting in (ii), we get $\vec{r} = (\vec{r} \cdot \vec{a})\vec{a}' + (\vec{r} \cdot \vec{b})\vec{b}' + (\vec{r} \cdot \vec{c})\vec{c}'$.

Concept Application Exercise 2.3

1. If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are four non-coplanar unit vectors such that $\vec{d}$ makes equal angles with all the three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, then prove that $[\vec{d}\,\vec{a}\,\vec{b}]=[\vec{d}\,\vec{c}\,\vec{b}]=[\vec{d}\,\vec{c}\,\vec{a}]$.

2. Prove that if $[\vec{l}\,\vec{m}\,\vec{n}]$ are three non-coplanar vectors, then $[\vec{l}\,\vec{m}\,\vec{n}](\vec{a}\times\vec{b}) = \begin{vmatrix} \vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} & \vec{l} \\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} & \vec{m} \\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} & \vec{n} \end{vmatrix}$

3. If the volume of a parallelopiped whose adjacent edges are $\vec{a}=2\hat{i}+3\hat{j}+4\hat{k}$, $\vec{b}=\hat{i}+\alpha\hat{j}+2\hat{k}$, $\vec{c}=\hat{i}+2\hat{j}+\alpha\hat{k}$ is 15, then find the value of α if ($\alpha > 0$).

4. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$, then find vector $\vec{c}$ such that $\vec{a}\cdot\vec{c}=2$ and $\vec{a}\times\vec{c}=\vec{b}$.

5. If $\vec{x}\cdot\vec{a}=0$, $\vec{x}\cdot\vec{b}=0$ and $\vec{x}\cdot\vec{c}=0$ for some non-zero vector $\vec{x}$, then prove that $[\vec{a}\,\vec{b}\,\vec{c}]=0$.

6. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar vectors, show that

$$[\vec{a}\times\vec{b}\quad \vec{b}\times\vec{c}\quad \vec{c}\times\vec{d}] = [\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}.$$

7. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors such that $\vec{a}\times\vec{b}=\vec{c}, \vec{b}\times\vec{c}=\vec{a}, \vec{c}\times\vec{a}=\vec{b}$, then prove that $|\vec{a}|=|\vec{b}|=|\vec{c}|$.

8. If $\vec{a}=\vec{p}+\vec{q}, \vec{p}\times\vec{b}=\vec{0}$ and $\vec{q}\cdot\vec{b}=0$, then prove that $\dfrac{\vec{b}\times(\vec{a}\times\vec{b})}{\vec{b}\cdot\vec{b}}=\vec{q}$.

9. Prove that $(\vec{a}\cdot(\vec{b}\times\hat{i}))\,\hat{i}+(\vec{a}\cdot(\vec{b}\times\hat{j}))\,\hat{j}+(\vec{a}\cdot(\vec{b}\times\hat{k}))\,\hat{k} = \vec{a}\times\vec{b}$.

10. For any four vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, prove that $\vec{d}\cdot(\vec{a}\times(\vec{b}\times(\vec{c}\times\vec{d})))=(\vec{b}\cdot\vec{d})[\vec{a}\,\vec{c}\,\vec{d}]$.

11. If $\vec{a}$ and $\vec{b}$ be two non-collinear unit vectors such that $\vec{a}\times(\vec{a}\times\vec{b})=\frac{1}{2}\vec{b}$, then find the angle between $\vec{a}$ and $\vec{b}$.

12. Show that $(\vec{a}\times\vec{b})\times\vec{c}=\vec{a}\times(\vec{b}\times\vec{c})$ if and only if $\vec{a}$ and $\vec{c}$ are collinear or $(\vec{a}\times\vec{c})\times\vec{b}=\vec{0}$.

13. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be non-zero vectors such that no two are collinear and $(\vec{a}\times\vec{b})\times\vec{c}=\frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$. If θ is the acute angle between vectors $\vec{b}$ and $\vec{c}$, then find the value of sin θ.

14. If $\vec{p}, \vec{q}, \vec{r}$ denote vectors $\vec{b}\times\vec{c}, \vec{c}\times\vec{a}, \vec{a}\times\vec{b}$, respectively, show that $\vec{a}$ is parallel to $\vec{q}\times\vec{r}$, $\vec{b}$ is parallel to $\vec{r}\times\vec{p}$, $\vec{c}$ is parallel to $\vec{p}\times\vec{q}$.

Exercises

Subjective Type

Solutions on page 2.84

1. If $\begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-a)^2 \end{vmatrix} = 0$ and vectors $\vec{A}, \vec{B}$ and $\vec{C}$, where $\vec{A} = a^2\hat{i} + a\hat{j} + \hat{k}$, etc., are non-coplanar, then prove that vectors $\vec{X}, \vec{Y}$ and $\vec{Z}$, where $\vec{X} = x^2\hat{i} + x\hat{j} + \hat{k}$, etc. may be coplanar.
2. If $OABC$ is a tetrahedron where O is the origin and A, B and C are the other three vertices with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$, respectively, then prove that the centre of the sphere circumscribing the tetrahedron is given by position vector $\dfrac{a^2(\vec{b}\times\vec{c})+b^2(\vec{c}\times\vec{a})+c^2(\vec{a}\times\vec{b})}{2[\vec{a}\,\vec{b}\,\vec{c}]}$.
3. Let k be the length of any edge of a regular tetrahedron (a tetrahedron whose edges are equal in length is called a regular tetrahedron). Show that the angle between any edge and a face not containing the edge is $\cos^{-1}(1/\sqrt{3})$.
4. In ΔABC, a point P is taken on AB such that $AP/BP = 1/3$ and a point Q is taken on BC such that $CQ/BQ = 3/1$. If R is the point of intersection of the lines AQ and CP, using vector method, find the area of ΔABC if the area of ΔBRC is 1 unit.
5. Let O be an interior point of ΔABC such that $\overrightarrow{OA} + 2\overrightarrow{OB} + 3\overrightarrow{OC} = \vec{0}$. Then find the ratio of the area of ΔABC to the area of ΔAOC.
6. The lengths of two opposite edges of a tetrahedron are a and b; the shortest distance between these edges is d, and the angle between them is θ. Prove using vectors that the volume of the tetrahedron is $\dfrac{abd\sin\theta}{6}$.
7. Find the volume of a parallelopiped having three coterminus vectors of equal magnitude $|\vec{a}|$ and equal inclination θ with each other.
8. Let $\vec{p}$ and $\vec{q}$ be any two orthogonal vectors of equal magnitude 4 each. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors of lengths 7, $\sqrt{15}$ and $2\sqrt{33}$, mutually perpendicular to each other. Then find the distance of the vector $(\vec{a}\cdot\vec{p})\,\vec{p} + (\vec{a}\cdot\vec{q})\,\vec{q} + (\vec{a}\cdot(\vec{p}\times\vec{q}))\,(\vec{p}\times\vec{q}) + (\vec{b}\cdot\vec{p})\,\vec{p} + (\vec{b}\cdot\vec{q})\,\vec{q} + (\vec{b}\cdot(\vec{p}\times\vec{q}))\,(\vec{p}\times\vec{q}) + (\vec{c}\cdot\vec{p})\,\vec{p} + (\vec{c}\cdot\vec{q})\,\vec{q} + (\vec{c}\cdot(\vec{p}\times\vec{q}))\,(\vec{p}\times\vec{q})$ from the origin.
9. Given that vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ form a triangle such that $\vec{A} = \vec{B} + \vec{C}$. Find a, b, c and d such that the area of the triangle is $5\sqrt{6}$ where
$\vec{A} = a\hat{i} + b\hat{j} + c\hat{k}$
$\vec{B} = d\hat{i} + 3\hat{j} + 4\hat{k}$
$\vec{C} = 3\hat{i} + \hat{j} - 2\hat{k}$

10. A line l is passing through the point $\vec{b}$ and is parallel to vector $\vec{c}$. Determine the distance of point $A(\vec{a})$ from the line l in the form $\left|\vec{b}-\vec{a}+\frac{(\vec{a}-\vec{b})\cdot\vec{c}}{|\vec{c}|^2}\vec{c}\right|$ or $\frac{|(\vec{b}-\vec{a})\times\vec{c}|}{|\vec{c}|}$.

11. If $\vec{e}_1, \vec{e}_2, \vec{e}_3$ and $\vec{E}_1, \vec{E}_2, \vec{E}_3$ are two sets of vectors such that $\vec{e}_i\cdot\vec{E}_j = 1$, if $i = j$ and $\vec{e}_i\cdot\vec{E}_j = 0$ and if $i \neq j$, then prove that $[\vec{e}_1\ \vec{e}_2\ \vec{e}_3][\vec{E}_1\ \vec{E}_2\ \vec{E}_3] = 1$.

Objective Type

Solutions on page 2.90

Each question has four choices a, b, c and d, out of which *only one* answer is correct. Find the correct answer.

1. Two vectors in space are equal only if they have equal component in

a. a given direction
b. two given directions
c. three given directions
d. in any arbitrary direction

2. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the three vectors having magnitudes 1, 5 and 3, respectively, such that the angle between $\vec{a}$ and $\vec{b}$ is θ and $\vec{a}\times(\vec{a}\times\vec{b}) = \vec{c}$. Then $\tan\theta$ is equal to

a. 0
b. 2/3
c. 3/5
d. 3/4

3. $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors of equal magnitude. The angle between each pair of vectors is $\pi/3$ such that $|\vec{a}+\vec{b}+\vec{c}| = \sqrt{6}$. Then $|\vec{a}|$ is equal to

a. 2
b. −1
c. 1
d. $\sqrt{6}/3$

4. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors, then the vector which is equally inclined to these vectors is

a. $\vec{a}+\vec{b}+\vec{c}$
b. $\frac{\vec{a}}{|\vec{a}|}+\frac{\vec{b}}{|\vec{b}|}+\frac{\vec{c}}{|\vec{c}|}$
c. $\frac{\vec{a}}{|\vec{a}|^2}+\frac{\vec{b}}{|\vec{b}|^2}+\frac{\vec{c}}{|\vec{c}|^2}$
d. $|\vec{a}|\vec{a}-|\vec{b}|\vec{b}+|\vec{c}|\vec{c}$

5. Let $\vec{a} = \hat{i}+\hat{j}; \vec{b} = 2\hat{i}-\hat{k}$. Then vector $\vec{r}$ satisfying the equations $\vec{r}\times\vec{a} = \vec{b}\times\vec{a}$ and $\vec{r}\times\vec{b} = \vec{a}\times\vec{b}$ is

a. $\hat{i}-\hat{j}+\hat{k}$
b. $3\hat{i}-\hat{j}+\hat{k}$
c. $3\hat{i}+\hat{j}-\hat{k}$
d. $\hat{i}-\hat{j}-\hat{k}$

6. If $\vec{a}$ and $\vec{b}$ are two vectors, such that $\vec{a}\cdot\vec{b} < 0$ and $|\vec{a}\cdot\vec{b}| = |\vec{a}\times\vec{b}|$, then the angle between vectors $\vec{a}$ and $\vec{b}$ is

a. π
b. $7\pi/4$
c. $\pi/4$
d. $3\pi/4$

7. If $\hat{a}, \hat{b}$ and $\hat{c}$ are three unit vectors, such that $\hat{a}+\hat{b}+\hat{c}$ is also a unit vector and θ_1, θ_2 and θ_3 are angles between the vectors $\hat{a}, \hat{b}; \hat{b}, \hat{c}$ and $\hat{c}, \hat{a}$, respectively, then among θ_1, θ_2 and θ_3

a. all are acute angles
b. all are right angles
c. at least one is obtuse angle
d. none of these

8. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} \cdot \vec{b} = 0 = \vec{a} \cdot \vec{c}$ and the angle between $\vec{b}$ and $\vec{c}$ is $\pi/3$, then the value of $|\vec{a} \times \vec{b} - \vec{a} \times \vec{c}|$ is

a. 1/2 b. 1 c. 2 d. none of these

9. $P(\vec{p})$ and $Q(\vec{q})$ are the position vectors of two fixed points and $R(\vec{r})$ is the position vector of a variable point. If R moves such that $(\vec{r} - \vec{p}) \times (\vec{r} - \vec{q}) = \vec{0}$, then the locus of R is

a. a plane containing the origin O and parallel to two non-collinear vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$
b. the surface of a sphere described on PQ as its diameter
c. a line passing through points P and Q
d. a set of lines parallel to line PQ

10. Two adjacent sides of a parallelogram $ABCD$ are $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\hat{i} + 2\hat{j} + 3\hat{k}$. Then the value of $|\overrightarrow{AC} \times \overrightarrow{BD}|$ is

a. $20\sqrt{5}$ b. $22\sqrt{5}$ c. $24\sqrt{5}$ d. $26\sqrt{5}$

11. If $\hat{a}, \hat{b}$ and $\hat{c}$ are three unit vectors inclined to each other at an angle θ, then the maximum value of θ is

a. $\frac{\pi}{3}$ b. $\frac{\pi}{2}$ c. $\frac{2\pi}{3}$ a. $\frac{5\pi}{6}$

12. Let the pairs $\vec{a}, \vec{b}$ and $\vec{c}, \vec{d}$ each determine a plane. Then the planes are parallel if

a. $(\vec{a} \times \vec{c}) \times (\vec{b} \times \vec{d}) = \vec{0}$
b. $(\vec{a} \times \vec{c}) \cdot (\vec{b} \times \vec{d}) = \vec{0}$
c. $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{0}$
d. $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \vec{0}$

13. If $\vec{r} \cdot \vec{a} = \vec{r} \cdot \vec{b} = \vec{r} \cdot \vec{c} = 0$, where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar, then

a. $\vec{r} \perp (\vec{c} \times \vec{a})$ b. $\vec{r} \perp (\vec{a} \times \vec{b})$ c. $\vec{r} \perp (\vec{b} \times \vec{c})$ d. $\vec{r} = \vec{0}$

14. If $\vec{a}$ satisfies $\vec{a} \times (\hat{i} + 2\hat{j} + \hat{k}) = \hat{i} - \hat{k}$, then $\vec{a}$ is equal to

a. $\lambda\hat{i} + (2\lambda - 1)\hat{j} + \lambda\hat{k}, \lambda \in R$
b. $\lambda\hat{i} + (1 - 2\lambda)\hat{j} + \lambda\hat{k}, \lambda \in R$
c. $\lambda\hat{i} + (2\lambda + 1)\hat{j} + \lambda\hat{k}, \lambda \in R$
d. $\lambda\hat{i} - (1 + 2\lambda)\hat{j} + \lambda\hat{k}, \lambda \in R$

15. Vectors $3\vec{a} - 5\vec{b}$ and $2\vec{a} + \vec{b}$ are mutually perpendicular. If $\vec{a} + 4\vec{b}$ and $\vec{b} - \vec{a}$ are also mutually perpendicular, then the cosine of the angle between $\vec{a}$ and $\vec{b}$ is

a. $\frac{19}{5\sqrt{43}}$ b. $\frac{19}{3\sqrt{43}}$ c. $\frac{19}{2\sqrt{45}}$ d. $\frac{19}{6\sqrt{43}}$

16. The unit vector orthogonal to vector $-\hat{i} + 2\hat{j} + 2\hat{k}$ and making equal angles with the x- and y-axes is

a. $\pm\frac{1}{3}(2\hat{i} + 2\hat{j} - \hat{k})$ b. $\pm\frac{1}{3}(\hat{i} + \hat{j} - \hat{k})$ c. $\pm\frac{1}{3}(2\hat{i} - 2\hat{j} - \hat{k})$ d. None of these

17. The value of x for which the angle between $\vec{a} = 2x^2\hat{i} + 4x\hat{j} + \hat{k}$ and $\vec{b} = 7\hat{i} - 2\hat{j} + x\hat{k}$ is obtuse and the angle between $\vec{b}$ and the z-*axis* is acute and less than $\pi/6$, is

a. $a < x < 1/2$ **b.** $1/2 < x < 15$ **c.** $x > 1/2$ or $x < 0$ **d.** none of these

18. If vectors $\vec{a}$ and $\vec{b}$ are two adjacent sides of a parallelogram, then the vector representing the altitude of the parallelogram which is perpendicular to $\vec{a}$ is

a. $\vec{b} + \dfrac{\vec{b} \times \vec{a}}{|\vec{a}|^2}$ **b.** $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}$ **c.** $\vec{b} - \dfrac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2}\vec{a}$ **d.** $\dfrac{\vec{a} \times (\vec{b} \times \vec{a})}{|\vec{b}|^2}$

19. A parallelogram is constructed on $3\vec{a} + \vec{b}$ and $\vec{a} - 4\vec{b}$, where $|\vec{a}| = 6$ and $|\vec{b}| = 8$, and $\vec{a}$ and $\vec{b}$ are anti-parallel. Then the length of the longer diagonal is

a. 40 **b.** 64 **c.** 32 **d.** 48

20. Let $\vec{a} \cdot \vec{b} = 0$, where $\vec{a}$ and $\vec{b}$ are unit vectors and the unit vector $\vec{c}$ is inclined at an angle θ to both $\vec{a}$ and $\vec{b}$. If $\vec{c} = m\vec{a} + n\vec{b} + p(\vec{a} \times \vec{b})$, $(m, n, p \in R)$, then

a. $-\dfrac{\pi}{4} \le \theta \le \dfrac{\pi}{4}$ **b.** $\dfrac{\pi}{4} \le \theta \le \dfrac{3\pi}{4}$ **c.** $0 \le \theta \le \dfrac{\pi}{4}$ **d.** $0 \le \theta \le \dfrac{3\pi}{4}$

21. $\vec{a}$ and $\vec{c}$ are unit vectors and $|\vec{b}| = 4$. The angle between $\vec{a}$ and $\vec{c}$ is $\cos^{-1}(1/4)$ and $\vec{b} - 2\vec{c} = \lambda\vec{a}$. The value of λ is

a. $3, -4$ **b.** $1/4, 3/4$ **c.** $-3, 4$ **d.** $-1/4, 3/4$

22. Let the position vectors of the points P and Q be $4\hat{i} + \hat{j} + \lambda\hat{k}$ and $2\hat{i} - \hat{j} + \lambda\hat{k}$, respectively. Vector $\hat{i} - \hat{j} + 6\hat{k}$ is perpendicular to the plane containing the origin and the points P and Q. Then λ equals

a. $-1/2$ **b.** $1/2$ **c.** 1 **d.** none of these

23. A vector of magnitude $\sqrt{2}$ coplanar with the vectors $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$, and perpendicular to the vector $\vec{c} = \hat{i} + \hat{j} + \hat{k}$, is

a. $-\hat{j} + \hat{k}$ **b.** $\hat{i} - \hat{k}$ **c.** $\hat{i} - \hat{j}$ **d.** $\hat{i} - \hat{j}$

24. P be a point interior to the acute triangle ABC. If $\vec{PA} + \vec{PB} + \vec{PC}$ is a null vector then w.r.t. triangle ABC, point P is its

a. centroid **b.** orthocentre **c.** incentre **d.** circumcentre

25. G is the centroid of triangle ABC and A_1 and B_1 are the midpoints of sides AB and AC, respectively. If Δ_1 be the area of quadrilateral GA_1AB_1 and Δ be the area of triangle ABC, then Δ/Δ_1 is equal to

a. $\dfrac{3}{2}$ **b.** 3 **c.** $\dfrac{1}{3}$ **d.** none of these

26. Points $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are coplanar and $(\sin\alpha)\vec{a} + (2\sin 2\beta)\vec{b} + (3\sin 3\gamma)\vec{c} - \vec{d} = \vec{0}$. Then the least value of $\sin^2\alpha + \sin^2 2\beta + \sin^2 3\gamma$ is

a. 1/14 **b.** 14 **c.** 6 **d.** $1/\sqrt{6}$

27. If $\vec{a}$ and $\vec{b}$ are any two vectors of magnitudes 1 and 2, respectively, and $(1-3\vec{a}\cdot\vec{b})^2 + |2\vec{a}+\vec{b}+3(\vec{a}\times\vec{b})|^2 = 47$, then the angle between $\vec{a}$ and $\vec{b}$ is

a. $\pi/3$ b. $\pi - \cos^{-1}(1/4)$ c. $\frac{2\pi}{3}$ d. $\cos^{-1}(1/4)$

28. If $\vec{a}$ and $\vec{b}$ are any two vectors of magnitudes 2 and 3, respectively, such that $|2(\vec{a}\times\vec{b})|+|3(\vec{a}\cdot\vec{b})|= k$, then the maximum value of k is

a. $\sqrt{13}$ b. $2\sqrt{13}$ c. $6\sqrt{13}$ d. $10\sqrt{13}$

29. $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors such that $|\vec{a} + \vec{b} + 3\vec{c}| = 4$. Angle between $\vec{a}$ and $\vec{b}$ is θ_1, between $\vec{b}$ and $\vec{c}$ is θ_2 and between $\vec{a}$ and $\vec{c}$ varies $[\pi/6, 2\pi/3]$. Then the maximum value of $\cos\theta_1 + 3\cos\theta_2$ is

a. 3 b. 4 c. $2\sqrt{2}$ d. 6

30. If the vector product of a constant vector $\overrightarrow{OA}$ with a variable vector $\overrightarrow{OB}$ in a fixed plane OAB be a constant vector, then the locus of B is

a. a straight line perpendicular to $\overrightarrow{OA}$

b. a circle with centre O and radius equal to $|\overrightarrow{OA}|$

c. a straight line parallel to $\overrightarrow{OA}$

d. none of these

31. Let $\vec{u}, \vec{v}$ and $\vec{w}$ be such that $|\vec{u}|=1, |\vec{v}|=2$ and $|\vec{w}|=3$. If the projection of $\vec{v}$ along $\vec{u}$ is equal to that of $\vec{w}$ along $\vec{u}$ and vectors $\vec{v}$ and $\vec{w}$ are perpendicular to each other, then $|\vec{u} - \vec{v} + \vec{w}|$ equals

a. 2 b. $\sqrt{7}$ c. $\sqrt{14}$ d. 14

32. If the two adjacent sides of two rectangles are represented by vectors $\vec{p} = 5\vec{a} - 3\vec{b}$; $\vec{q} = -\vec{a} - 2\vec{b}$ and $\vec{r} = -4\vec{a} - \vec{b}$; $\vec{s} = -\vec{a} + \vec{b}$, respectivėly, then the angle between the vectors $\vec{x} = \frac{1}{3}(\vec{p}+\vec{r}+\vec{s})$ and $\vec{y} = \frac{1}{5}(\vec{r}+\vec{s})$ is

a. $-\cos^{-1}\left(\frac{19}{5\sqrt{43}}\right)$ b. $\cos^{-1}\left(\frac{19}{5\sqrt{43}}\right)$

c. $\pi\cos^{-1}\left(\frac{19}{5\sqrt{43}}\right)$ d. cannot be evaluated

33. If $\vec{\alpha} \parallel (\vec{\beta}\times\vec{\gamma})$, then $(\vec{\alpha}\times\vec{\beta})\cdot(\vec{\alpha}\times\vec{\gamma})$ equals to

a. $|\vec{\alpha}|^2(\vec{\beta}\cdot\vec{\gamma})$ b. $|\vec{\beta}|^2(\vec{\gamma}\cdot\vec{\alpha})$ c. $|\vec{\gamma}|^2(\vec{\alpha}\cdot\vec{\beta})$ d. $|\vec{\alpha}||\vec{\beta}||\vec{\gamma}|$

34. The position vectors of points A, B, and C are $\hat{i}+\hat{j}+\hat{k}$, $\hat{i}+5\hat{j}-\hat{k}$ and $2\hat{i}+3\hat{j}+5\hat{k}$, respectively. The greatest angle of triangle ABC is

a. 120° b. 90° c. $\cos^{-1}(3/4)$ d. none of these

35. Given three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, two of which are non-collinear. Further if $(\vec{a}+\vec{b})$ is collinear with $\vec{c}$, $(\vec{b}+\vec{c})$ is collinear with $\vec{a}$, $|\vec{a}| = |\vec{b}| = |\vec{c}| = \sqrt{2}$. Find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$.

a. 3
b. −3
c. 0
d. cannot be evaluated

36. If $\vec{a}$ and $\vec{b}$ are unit vectors such that $(\vec{a}+\vec{b})\cdot(2\vec{a}+3\vec{b})\times(3\vec{a}-2\vec{b}) = \vec{0}$, then angle between $\vec{a}$ and $\vec{b}$ is

a. 0
b. $\pi/2$
c. π
d. indeterminate

37. If in a right-angled triangle ABC, the hypotenuse $AB = p$, then $\overrightarrow{AB}\cdot\overrightarrow{AC}+\overrightarrow{BC}\cdot\overrightarrow{BA}+\overrightarrow{CA}\cdot\overrightarrow{CB}$ is equal to

a. $2p^2$
b. $\frac{p^2}{2}$
c. p^2
d. none of these

38. Resolved part of vector $\vec{a}$ and along vector $\vec{b}$ is $\vec{a}_1$ and that perpendicular to $\vec{b}$ is $\vec{a}_2$, then $\vec{a}_1\times\vec{a}_2$ is equal to

a. $\frac{(\vec{a}\times\vec{b})\cdot\vec{b}}{|\vec{b}|^2}$
b. $\frac{(\vec{a}\cdot\vec{b})\,\vec{a}}{|\vec{a}|^2}$
c. $\frac{(\vec{a}\cdot\vec{b})(\vec{b}\times\vec{a})}{|\vec{b}|^2}$
d. $\frac{(\vec{a}\cdot\vec{b})(\vec{b}\times\vec{a})}{|\vec{b}\times\vec{a}|}$

39. $\vec{a} = 2\hat{i}-\hat{j}+\hat{k}$, $\vec{b} = \hat{i}+2\hat{j}-\hat{k}$, $\vec{c} = \hat{i}+\hat{j}-2\hat{k}$. A vector coplanar with $\vec{b}$ and $\vec{c}$ whose projection on $\vec{a}$ is magnitude $\sqrt{\frac{2}{3}}$ is

a. $2\hat{i}+3\hat{j}-3\hat{k}$
b. $-2\hat{i}-\hat{j}+5\hat{k}$
c. $2\hat{i}+3\hat{j}+3\hat{k}$
d. $2\hat{i}+\hat{j}+5\hat{k}$

40. If 'P' is any arbitrary point on the circumcircle of the equilateral triangle of side length l units, then $|\overrightarrow{PA}|^2+|\overrightarrow{PB}|^2+|\overrightarrow{PC}|^2$ is always equal to

a. $2l^2$
b. $2\sqrt{3}\,l^2$
c. l^2
d. $3l^2$

41. If $\vec{r}$ and $\vec{s}$ are non-zero constant vectors and the scalar b is chosen such that $|\vec{r}+b\vec{s}|$ is minimum, then the the value of $|b\vec{s}|^2+|\vec{r}+b\vec{s}|^2$ is equal to

a. $2|\vec{r}|^2$
b. $|\vec{r}|^2/2$
c. $3|\vec{r}|^2$
d. $|\vec{r}|^2$

42. $\vec{a}$ and $\vec{b}$ are two unit vectors that are mutually perpendicular. A unit vector that is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{a}\times\vec{b}$ is

a. $\frac{1}{\sqrt{2}}(\vec{a}+\vec{b}+\vec{a}\times\vec{b})$
b. $\frac{1}{2}(\vec{a}\times\vec{b}+\vec{a}+\vec{b})$
c. $\frac{1}{\sqrt{3}}(\vec{a}+\vec{b}+\vec{a}\times\vec{b})$
d. $\frac{1}{3}(\vec{a}+\vec{b}+\vec{a}\times\vec{b})$

43. Given that $\vec{a}, \vec{b}, \vec{p}, \vec{q}$ are four vectors such that $\vec{a} + \vec{b} = \mu\vec{p}, \vec{b}\cdot\vec{q} = 0$ and $(\vec{b})^2 = 1$, where μ is a scalar. Then $|(\vec{a}\cdot\vec{q})\vec{p} - (\vec{p}\cdot\vec{q})\vec{a}|$ is equal to

a. $2|\vec{p}\cdot\vec{q}|$ b. $(1/2)|\vec{p}\cdot\vec{q}|$ c. $|\vec{p}\times\vec{q}|$ d. $|\vec{p}\cdot\vec{q}|$

44. The position vectors of the vertices A, B and C of a triangle are three unit vectors $\hat{a}, \hat{b}$ and $\hat{c}$, respectively. A vector $\vec{d}$ is such that $\vec{d}\cdot\hat{a} = \vec{d}\cdot\hat{b} = \vec{d}\cdot\hat{c}$ and $\vec{d} = \lambda(\hat{b} + \hat{c})$. Then triangle ABC is

a. acute angled b. obtuse angled c. right angled d. none of these

45. If a is a real constant and A, B and C are variable angles and $\sqrt{a^2 - 4}\tan A + a\tan B + \sqrt{a^2 + 4}\tan c = 6a$, then the least value of $\tan^2 A + \tan^2 B + \tan^2 C$ is

a. 6 b. 10 c. 12 d. 3

46. The vertex A of triangle ABC is on the line $\vec{r} = \hat{i} + \hat{j} + \lambda\hat{k}$ and the vertices B and C have respective position vectors $\hat{i}$ and $\hat{j}$. Let Δ be the area of the triangle and $\Delta \in [3/2, \sqrt{33}/2]$. Then the range of values of λ corresponding to A is

a. $[-8,-4]\cup[4,8]$ b. $[-4,4]$ c. $[-2,2]$ d. $[-4,-2]\cup[2,4]$

47. A non-zero vector $\vec{a}$ is such that its projections along vectors $\dfrac{\hat{i}+\hat{j}}{\sqrt{2}}, \dfrac{-\hat{i}+\hat{j}}{\sqrt{2}}$ and $\hat{k}$ are equal, then unit vector along $\vec{a}$ is

a. $\dfrac{\sqrt{2}\hat{j} - \hat{k}}{\sqrt{3}}$ b. $\dfrac{\hat{j} - \sqrt{2}\hat{k}}{\sqrt{3}}$ c. $\dfrac{\sqrt{2}}{\sqrt{3}}\hat{j} + \dfrac{\hat{k}}{\sqrt{3}}$ d. $\dfrac{\hat{j} - \hat{k}}{\sqrt{2}}$

48. Position vector $\hat{k}$ is rotated about origin by angle 135° in such a way that the plane made by it bisects the angle between $\hat{i}$ and $\hat{j}$. Then its new position is

a. $\pm\dfrac{\hat{i}}{\sqrt{2}} \pm \dfrac{\hat{j}}{\sqrt{2}}$ b. $\pm\dfrac{\hat{i}}{2} \pm \dfrac{\hat{j}}{2} - \dfrac{\hat{k}}{\sqrt{2}}$ c. $\dfrac{\hat{i}}{\sqrt{2}} - \dfrac{\hat{k}}{\sqrt{2}}$ d. none of these

49. In a quadrilateral $ABCD$, $\overrightarrow{AC}$ is the bisector of $\overrightarrow{AB}$ and $\overrightarrow{AD}$, angle between $\overrightarrow{AB}$ and $\overrightarrow{AD}$ is $2\pi/3$, $15|\overrightarrow{AC}| = 3|\overrightarrow{AB}| = 5|\overrightarrow{AD}|$. Then the angle between $\overrightarrow{BA}$ and $\overrightarrow{CD}$ is

a. $\cos^{-1}\dfrac{\sqrt{14}}{7\sqrt{2}}$ b. $\cos^{-1}\dfrac{\sqrt{21}}{7\sqrt{3}}$ c. $\cos^{-1}\dfrac{2}{\sqrt{7}}$ d. $\cos^{-1}\dfrac{2\sqrt{7}}{14}$

50. In the following figure, AB, DE and GF are parallel to each other and AD, BG and EF are parallel to each other. If $CD : CE = CG : CB = 2:1$, then the value of area (ΔAEG) : area (ΔABD) is equal to

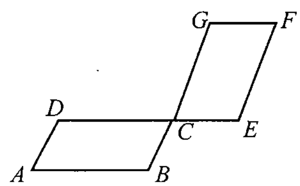


Fig. 2.29

a. 7/2 b. 3 c. 4 d. 9/2

51. Vector $\hat{a}$ in the plane of $\vec{b} = 2\hat{i} + \hat{j}$ and $\vec{c} = \hat{i} - \hat{j} + \hat{k}$ is such that it is equally inclined to $\vec{b}$ and $\vec{d}$ where $\vec{d} = \hat{j} + 2\hat{k}$. The value of $\hat{a}$ is

a. $\dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ b. $\dfrac{\hat{i}-\hat{j}+\hat{k}}{\sqrt{3}}$ c. $\dfrac{2\hat{i}+\hat{j}}{\sqrt{5}}$ d. $\dfrac{2\hat{i}+\hat{j}}{\sqrt{5}}$

52. Let $ABCD$ be a tetrahedron such that the edges AB, AC and AD are mutually perpendicular. Let the area of triangles ABC, ACD and ADB be 3, 4 and 5 sq. units, respectively. Then the area of triangle BCD is

a. $5\sqrt{2}$ b. 5   c. $\dfrac{\sqrt{5}}{2}$ d. $\dfrac{5}{2}$

53. Let $\vec{f}(t) = [t]\hat{i} + (t - [t])\hat{j} + [t+1]\hat{k}$, where [.] denotes the greatest integer function. Then the vectors $\vec{f}\left(\dfrac{5}{4}\right)$ and $\vec{f}(t), 0 < t < 1$, are

a. parallel to each other
b. perpendicular to each other
c. inclined at an angle $\cos^{-1}\dfrac{2}{\sqrt{7(1-t^2)}}$
d. inclined at $\cos^{-1}\dfrac{8+t}{9\sqrt{1+t^2}}$

54. If $\vec{a}$ is parallel to $\vec{b} \times \vec{c}$, then $(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c})$ is equal to

a. $|\vec{a}|^2 (\vec{b} \cdot \vec{c})$ b. $|\vec{b}|^2 (\vec{a} \cdot \vec{c})$ c. $|\vec{c}|^2 (\vec{a} \cdot \vec{b})$ d. none of these

55. Three vectors $\hat{i} + \hat{j}$, $\hat{j} + \hat{k}$ and $\hat{k} + \hat{i}$ taken two at a time form three planes. The three unit vectors drawn perpendicular to these three planes form a parallelepiped of volume

a. 1/3   b. 4   c. $(3\sqrt{3})/4$ d. $4\sqrt{3}$

56. If $\vec{d} = \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$ is a non-zero vector and $|(\vec{d} \cdot \vec{c})(\vec{a} \times \vec{b}) + (\vec{d} \cdot \vec{a})(\vec{b} \times \vec{c}) + (\vec{d} \cdot \vec{b})(\vec{c} \times \vec{a})| = 0$, then

a. $|\vec{a}| = |\vec{b}| = |\vec{c}|$
b. $|\vec{a}| + |\vec{b}| + |\vec{c}| = |\vec{d}|$
c. $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar
d. none of these

57. If $|\vec{a}| = 2$ and $|\vec{b}| = 3$ and $\vec{a} \cdot \vec{b} = 0$, then $(\vec{a} \times (\vec{a} \times (\vec{a} \times (\vec{a} \times \vec{b}))))$ is equal to

a. $48\hat{b}$ b. $-48\hat{b}$ c. $48\hat{a}$ d. $-48\hat{a}$

58. If the two diagonals of one of its faces are $6\hat{i} + 6\hat{k}$ and $4\hat{j} + 2\hat{k}$ and of the edges not containing the given diagonals is $\vec{c} = 4\hat{j} - 8\hat{k}$ then the volume of a parallelepiped is

a. 60   b. 80   c. 100   d. 120

59. The volume of a tetrahedron formed by the coterminus edges $\vec{a}, \vec{b}$ and $\vec{c}$ is 3. Then the volume of the parallelepiped formed by the coterminus edges $\vec{a} + \vec{b}, \vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ is

a. 6   b. 18   c. 36   d. 9

60. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three mutually orthogonal unit vectors, then the triple product $[\vec{a} + \vec{b} + \vec{c}\ \ \vec{a} + \vec{b}\ \ \vec{b} + \vec{c}]$ equals

a. 0   b. 1 or – 1   c. 1   d. 3

61. Vector $\vec{c}$ is perpendicular to vectors $\vec{a}$ = (2, −3, 1) and $\vec{b}$ = (1, −2, 3) and satisfies the condition $\vec{c}\cdot(\hat{i}+2\hat{j}-7\hat{k}) = 10$. Then vector $\vec{c}$ is equal to

a. (7, 5, 1) **b.** (−7, −5, −1) **c.** (1, 1, −1) **d.** none of these

62. Given $\vec{a} = x\hat{i} + y\hat{j} + 2\hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$, $\vec{c} = \hat{i} + 2\hat{j}$; $\vec{a} \perp \vec{b}$, $\vec{a}\cdot\vec{c} = 4$. Then

a. $[\vec{a}\,\vec{b}\,\vec{c}]^2 = |\vec{a}|$ **b.** $[\vec{a}\,\vec{b}\,\vec{c}] = |\vec{a}|$ **c.** $[\vec{a}\,\vec{b}\,\vec{c}] = 0$ **d.** $[\vec{a}\,\vec{b}\,\vec{c}] = |\vec{a}|^2$

63. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\pi/6$, then the value of $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$ is

a. 0 **b.** 1

c. $\frac{1}{4}(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$ **d.** $\frac{3}{4}(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$

64. Let $\vec{r}, \vec{a}, \vec{b}$ and $\vec{c}$ be four non-zero vectors such that $\vec{r}\cdot\vec{a} = 0$, $|\vec{r}\times\vec{b}| = |\vec{r}||\vec{b}|$ and $|\vec{r}\times\vec{c}| = |\vec{r}||\vec{c}|$. Then $[a\ b\ c]$ is equal to

a. $|a||b||c|$ **b.** $-|a||b||c|$ **c.** 0 **d.** none of these

65. If $\vec{a}, \vec{b}$ and $\vec{c}$ are such that $[\vec{a}\,\vec{b}\,\vec{c}] = 1$, $\vec{c} = \lambda\vec{a}\times\vec{b}$, angle between $\vec{a}$ and $\vec{b}$ is $2\pi/3$, $|\vec{a}| = \sqrt{2}$, $|\vec{b}| = \sqrt{3}$ and $|\vec{c}| = \frac{1}{\sqrt{3}}$, then the angle between $\vec{a}$ and $\vec{b}$ is

a. $\frac{\pi}{6}$ **b.** $\frac{\pi}{4}$ **c.** $\frac{\pi}{3}$ **d.** $\frac{\pi}{2}$

66. If $4\vec{a} + 5\vec{b} + 9\vec{c} = 0$, then $(\vec{a}\times\vec{b})\times[(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})]$ is equal to

a. a vector perpendicular to the plane of $\vec{a}, \vec{b}$ and $\vec{c}$

b. a scalar quantity

c. $\vec{0}$

d. none of these

67. Value of $[\vec{a}\times\vec{b}\ \ \vec{a}\times\vec{c}\ \ \vec{d}]$ is always equal to

a. $(\vec{a}\cdot\vec{d})[\vec{a}\,\vec{b}\,\vec{c}]$ **b.** $(\vec{a}\cdot\vec{c})[\vec{a}\,\vec{b}\,\vec{d}]$ **c.** $(\vec{a}\cdot\vec{b})[\vec{a}\,\vec{b}\,\vec{d}]$ **d.** none of these

68. Let $\hat{a}$ and $\hat{b}$ be mutually perpendicular unit vectors. Then for any arbitrary $\vec{r}$,

a. $\vec{r} = (\vec{r}\cdot\hat{a})\hat{a} + (\vec{r}\cdot\hat{b})\hat{b} + (\vec{r}\cdot(\hat{a}\times\hat{b}))(\hat{a}\times\hat{b})$

b. $\vec{r} = (\vec{r}\cdot\hat{a}) - (\vec{r}\cdot\hat{b})\hat{b} - (\vec{r}\cdot(\hat{a}\times\hat{b}))(\hat{a}\times\hat{b})$

c. $\vec{r} = (\vec{r}\cdot\hat{a})\hat{a} - (\vec{r}\cdot\hat{b})\hat{b} + (\vec{r}\cdot(\hat{a}\times\hat{b}))(\hat{a}\times\hat{b})$

d. none of these

69. Let $\vec{a}$ and $\vec{b}$ be unit vectors that are perpendicular to each other. Then

$[\vec{a} + (\vec{a} \times \vec{b}) \quad \vec{b} + (\vec{a} \times \vec{b}) \quad \vec{a} \times \vec{b}]$ will always be equal to

a. 1 **b.** 0 **c.** −1 **d.** none of these

70. $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}| = 1, |\vec{b}| = 4$ and $\vec{a} \cdot \vec{b} = 2$. If $\vec{c} = (2\vec{a} \times \vec{b}) - 3\vec{b}$, then find the angle between $\vec{b}$ and $\vec{c}$.

a. $\frac{\pi}{3}$ **b.** $\frac{\pi}{6}$ **c.** $\frac{3\pi}{4}$ **d.** $\frac{5\pi}{6}$

71. $\vec{b}$ and $\vec{c}$ are unit vectors. Then for any arbitrary vector $\vec{a}$, $(((\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})) \times (\vec{b} \times \vec{c})) \cdot (\vec{b} - \vec{c})$ is always equal to

a. $|\vec{a}|$ **b.** $\frac{1}{2}|\vec{a}|$ **c.** $\frac{1}{3}|\vec{a}|$ **d.** none of these

72. If $\vec{a}.\vec{b} = \beta$ and $\vec{a} \times \vec{b} = \vec{c}$, then $\vec{b}$ is

a. $\frac{(\beta\vec{a} - \vec{a} \times \vec{c})}{|\vec{a}|^2}$ **b.** $\frac{(\beta\vec{a} + \vec{a} \times \vec{c})}{|\vec{a}|^2}$

c. $\frac{(\beta\vec{c} - \vec{a} \times \vec{c})}{|\vec{a}|^2}$ **d.** $\frac{(\beta\vec{a} + \vec{a} \times \vec{c})}{|\vec{a}|^2}$

73. If $a(\vec{\alpha} \times \vec{\beta}) + b(\vec{\beta} \times \vec{\gamma}) + c(\vec{\gamma} \times \vec{\alpha}) = 0$ and at least one of a, b and c is non-zero, then vectors $\vec{\alpha}, \vec{\beta}$ and $\vec{\gamma}$ are

a. parallel **b.** coplanar
c. mutually perpendicular **d.** none of these

74. If $(\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c}) = \vec{b}$, where $\vec{a}, \vec{b}$ and $\vec{c}$ are non-zero vectors, then

a. $\vec{a}, \vec{b}$ and $\vec{c}$ can be coplanar **b.** $\vec{a}, \vec{b}$ and $\vec{c}$ must be coplanar

c. $\vec{a}, \vec{b}$ and $\vec{c}$ cannot be coplanar **d.** none of these

75. If $\vec{r}.\vec{a} = \vec{r}.\vec{b} = \vec{r}.\vec{c} = \frac{1}{2}$ for some non-zero vector $\vec{r}$, then the area of the triangle whose vertices are $A(\vec{a}), B(\vec{b})$ and $C(\vec{c})$ is ($\vec{a}, \vec{b}, \vec{c}$ are non-coplanar)

a. $|[\vec{a}\,\vec{b}\,\vec{c}]|$ **b.** $|\vec{r}|$ **c.** $|[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}|$ **d.** none of these

76. A vector of magnitude 10 along the normal to the curve $3x^2 + 8xy + 2y^2 - 3 = 0$ at its point $P(1, 0)$ can be

a. $6\hat{i} + 8\hat{j}$ **b.** $-8\hat{i} + 3\hat{j}$ **c.** $6\hat{i} - 8\hat{j}$ **d.** $8\hat{i} + 6\hat{j}$

77. If $\vec{a}$ and $\vec{b}$ are two unit vectors inclined at an angle $\pi/3$, then $\{\vec{a} \times (\vec{b} + \vec{a} \times \vec{b})\} \cdot \vec{b}$ is equal to

a. $\frac{-3}{4}$ b. $\frac{1}{4}$ c. $\frac{3}{4}$ d. $\frac{1}{2}$

78. If $\vec{a}$ and $\vec{b}$ are orthogonal unit vectors, then for a vector $\vec{r}$ non-coplanar with $\vec{a}$ and $\vec{b}$, vector $\vec{r} \times \vec{a}$ is equal to

a. $[\vec{r}\,\vec{a}\,\vec{b}]\vec{b} - (\vec{r}\cdot\vec{b})(\vec{b} \times \vec{a})$ b. $[\vec{r}\,\vec{a}\,\vec{b}](\vec{a} + \vec{b})$

c. $[\vec{r}\,\vec{a}\,\vec{b}]\vec{a} + (\vec{r}\cdot\vec{a})\vec{a} \times \vec{b}$ d. none of these

79. If $\vec{a}, \vec{b}, \vec{c}$ are any three non-coplanar vectors, then the equation $[\vec{b} \times \vec{c}\ \ \vec{c} \times \vec{a}\ \ \vec{a} \times \vec{b}]x^2 + [\vec{a} + \vec{b}\ \ \vec{b} + \vec{c}\ \ \vec{c} + \vec{a}]x + 1 + [\vec{b} - \vec{c}\ \ \vec{c} - \vec{a}\ \ \vec{a} - \vec{b}] = 0$ has roots

a. real and distinct b. real c. equal d. imaginary

80. If $\vec{x} + \vec{c} \times \vec{y} = \vec{a}$ and $\vec{y} + \vec{c} \times \vec{x} = \vec{b}$, where $\vec{c}$ is a non-zero vector, then which of the following is not correct.

a. $\vec{x} = \frac{\vec{b} \times \vec{c} + \vec{a} + (\vec{c}\cdot\vec{a})\vec{c}}{1 + \vec{c}\cdot\vec{c}}$ b. $\vec{x} = \frac{\vec{c} \times \vec{b} + \vec{b} + (\vec{c}\cdot\vec{a})\vec{c}}{1 + \vec{c}\cdot\vec{c}}$

c. $\vec{y} = \frac{\vec{a} \times \vec{c} + \vec{b} + (\vec{c}\cdot\vec{b})\vec{c}}{1 + \vec{c}\cdot\vec{c}}$ d. none of these

81. The condition for equations $\vec{r} \times \vec{a} = \vec{b}$ and $\vec{r} \times \vec{c} = \vec{d}$ to be consistent is

a. $\vec{b}\cdot\vec{c} = \vec{a}\cdot\vec{d}$ b. $\vec{a}\cdot\vec{b} = \vec{c}\cdot\vec{d}$ c. $\vec{b}\cdot\vec{c} + \vec{a}\cdot\vec{d} = 0$ d. $\vec{a}\cdot\vec{b} + \vec{c}\cdot\vec{d} = 0$

82. If $\vec{a}$ and $\vec{b}$ are non-zero non-collinear vectors, then $[\vec{a}\,\vec{b}\,\hat{i}]\hat{i} + [\vec{a}\,\vec{b}\,\hat{j}]\hat{j} + [\vec{a}\,\vec{b}\,\hat{k}]\hat{k}$ is equal to

a. $\vec{a} + \vec{b}$ b. $\vec{a} \times \vec{b}$ c. $\vec{a} - \vec{b}$ d. $\vec{b} \times \vec{a}$

83. If $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$, $\vec{c} = \hat{i} + \hat{j} + 2\hat{k}$ and $(1 + \alpha)\hat{i} + \beta(1 + \alpha)\hat{j} + \gamma(1 + \alpha)(1 + \beta)\hat{k} = \vec{a} \times (\vec{b} \times \vec{c})$, then α, β and γ are

a. $-2, -4, -\frac{2}{3}$ b. $2, -4, \frac{2}{3}$ c. $-2, 4, \frac{2}{3}$ d. $2, 4, -\frac{2}{3}$

84. Let $\vec{a}(x) = (\sin x)\hat{i} + (\cos x)\hat{j}$ and $\vec{b}(x) = (\cos 2x)\hat{i} + (\sin 2x)\hat{j}$ be two variable vectors ($x \in R$), then $\vec{a}(x)$ and $\vec{b}(x)$ are

a. collinear for unique value of x b. perpendicular for infinite values of x

c. zero vectors for unique value of x d. none of these

85. For any two vectors $\vec{a}$ and $\vec{b}$, $(\vec{a} \times \hat{i})\cdot(\vec{b} \times \hat{i}) + (\vec{a} \times \hat{j})\cdot(\vec{b} \times \hat{j}) + (\vec{a} \times \hat{k})\cdot(\vec{b} \times \hat{k})$ is always equal to

a. $\vec{a}\cdot\vec{b}$ b. $2\vec{a}\cdot\vec{b}$ c. zero d. none of these

86. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-coplanar vectors and $\vec{r}$ be any arbitrary vector. Then $(\vec{a} \times \vec{b}) \times (\vec{r} \times \vec{c}) + (\vec{b} \times \vec{c}) \times (\vec{r} \times \vec{a}) + (\vec{c} \times \vec{a}) \times (\vec{r} \times \vec{b})$ is always equal to

a. $[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}$ b. $2[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}$ c. $3[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}$ d. none of these

87. If $\vec{p}=\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $\vec{q}=\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ and $\vec{r}=\dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar vectors, then the value of the expression $(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{p}+\vec{q}+\vec{r})$ is

a. 3 b. 2 c. 1 d. 0

88. $A(\vec{a})$, $B(\vec{b})$ and $C(\vec{c})$ are the vertices of triangle ABC and $R(\vec{r})$ is any point in the plane of triangle ABC, then $\vec{r}.(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})$ is always equal to

a. zero b. $[\vec{a}\,\vec{b}\,\vec{c}]$ c. $-[\vec{a}\,\vec{b}\,\vec{c}]$ d. none of these

89. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar vectors and $\vec{a}\times\vec{c}$ is perpendicular to $\vec{a}\times(\vec{b}\times\vec{c})$, then the value of $[\vec{a}\times(\vec{b}\times\vec{c})]\times\vec{c}$ is equal to

a. $[\vec{a}\,\vec{b}\,\vec{c}]\,\vec{c}$ b. $[\vec{a}\,\vec{b}\,\vec{c}]\vec{b}$ c. $\vec{0}$ d. $[\vec{a}\,\vec{b}\,\vec{c}]\vec{a}$

90. If V be the volume of a tetrahedron and V' be the volume of another tetrahedran formed by the centroids of faces of the previous tetrahedron and $V=KV'$; then K is equal to

a. 9 b. 12 c. 27 d. 81

91. $[(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})\ \ (\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\ \ (\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})]$ is equal to (where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-zero non-coplanar vectors)

a. $[\vec{a}\,\vec{b}\,\vec{c}]^2$ b. $[\vec{a}\,\vec{b}\,\vec{c}]^3$ c. $[\vec{a}\,\vec{b}\,\vec{c}]^4$ d. $[\vec{a}\,\vec{b}\,\vec{c}]$

92. If $\vec{r}=x_1(\vec{a}\times\vec{b})+x_2(\vec{b}\times\vec{a})+x_3(\vec{c}\times\vec{d})$ and $4[\vec{a}\,\vec{b}\,\vec{c}]=1$, then $x_1+x_2+x_3$ is equal to

a. $\frac{1}{2}\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c})$ b. $\frac{1}{4}\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c})$ c. $2\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c})$ d. $4\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c})$

93. If $\vec{a}\perp\vec{b}$, then vector $\vec{v}$ in terms of $\vec{a}$ and $\vec{b}$ satisfying the equations $\vec{v}\cdot\vec{a}=0$ and $\vec{v}\cdot\vec{b}=1$ and $[\vec{v}\,\vec{a}\,\vec{b}]=1$ is

a. $\dfrac{\vec{b}}{|\vec{b}|^2}+\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|^2}$ b. $\dfrac{\vec{b}}{|\vec{b}|}+\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|^2}$ c. $\dfrac{\vec{b}}{|\vec{b}|^2}+\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ d. none of these

94. If $\vec{a}'=\hat{i}+\hat{j}$, $\vec{b}'=\hat{i}-\hat{j}+2\hat{k}$ and $\vec{c}'=2\hat{i}+\hat{j}-\hat{k}$, then the altitude of the parallelepiped formed by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ having base formed by $\vec{b}$ and $\vec{c}$ is (where $\vec{a}'$ is reciprocal vector $\vec{a}$, etc.)

a. 1 b. $3\sqrt{2}/2$ c. $1/\sqrt{6}$ d. $1/\sqrt{2}$

95. If $\vec{a}=\hat{i}+\hat{j}$, $\vec{b}=\hat{j}+\hat{k}$, $\vec{c}=\hat{k}+\hat{i}$, then in the reciprocal system of vectors $\vec{a},\vec{b},\vec{c}$ reciprocal $\vec{a}$ of vector $\vec{a}$ is

a. $\dfrac{\hat{i}+\hat{j}+\hat{k}}{2}$ b. $\dfrac{\hat{i}-\hat{j}+\hat{k}}{2}$ c. $\dfrac{-\hat{i}-\hat{j}+\hat{k}}{2}$ d. $\dfrac{\hat{i}+\hat{j}-\hat{k}}{2}$

Multiple Correct Answers Type

Solutions on page 2.114

Each question has four choices *a, b, c* and *d*, out of which *one or more* are correct.

1. If unit vectors $\vec{a}$ and $\vec{b}$ are inclined at an angle 2θ such that $|\vec{a} - \vec{b}| < 1$ and $0 \le \theta \le \pi$, then θ lies in the interval

   a. $[0, \pi/6)$ b. $(5\pi/6, \pi]$ c. $[\pi/6, \pi/2)$ d. $(\pi/2, 5\pi/6]$

2. $\vec{b}$ and $\vec{c}$ are non-collinear if $\vec{a} \times (\vec{b} \times \vec{c}) + (\vec{a} \cdot \vec{b})\vec{b} = (4 - 2x - \sin y)\vec{b} + (x^2 - 1)\vec{c}$ and $(\vec{c} \cdot \vec{c})\vec{a} = \vec{c}$. Then

   a. $x = 1$ b. $x = -1$

   c. $y = (4n+1)\dfrac{\pi}{2},\ n \in I$ d. $y = (2n+1)\dfrac{\pi}{2},\ n \in I$

3. Unit vectors $\vec{a}$ and $\vec{b}$ are perpendicular, and unit vector $\vec{c}$ is inclined at an angle θ to both $\vec{a}$ and $\vec{b}$. If $\vec{c} = \alpha\vec{a} + \beta\vec{b} + \gamma(\vec{a} \times \vec{b})$, then

   a. $a = \beta$ b. $\gamma^2 = 1 - 2\alpha^2$ c. $\gamma^2 = -\cos 2\theta$ d. $\beta^2 = \dfrac{1 + \cos 2\theta}{2}$

4. $\vec{a}$ and $\vec{b}$ are two given vectors. With these vectors as adjacent sides, a parallelogram is constructed. The vector which is the altitude of the parallelogram and which is perpendicular to $\vec{a}$ is

   a. $\dfrac{(\vec{a} \cdot \vec{b})}{|\vec{a}|^2}\vec{a} - \vec{b}$ b. $\dfrac{1}{|\vec{a}|^2}\{|\vec{a}|^2\vec{b} - (\vec{a} \cdot \vec{b})\vec{a}\}$

   c. $\dfrac{\vec{a} \times (\vec{a} \times \vec{b})}{|\vec{a}|^2}$ d. $\dfrac{\vec{a} \times (\vec{b} \times \vec{a})}{|\vec{b}|^2}$

5. If $\vec{a} \times (\vec{b} \times \vec{c})$ is perpendicular to $(\vec{a} \times \vec{b}) \times \vec{c}$, we may have

   a. $(\vec{a} \cdot \vec{c})|\vec{b}|^2 = (\vec{a} \cdot \vec{b})(\vec{b} \cdot \vec{c})$ b. $\vec{a} \cdot \vec{b} = 0$

   c. $\vec{a} \cdot \vec{c} = 0$ d. $\vec{b} \cdot \vec{c} = 0$

6. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be vectors forming right-hand triad. Let $\vec{p} = \dfrac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$, $\vec{q} = \dfrac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$ and $\vec{r} = \dfrac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$. If $x \in R^+$, then

   a. $x[\vec{a}\ \vec{b}\ \vec{c}] + \dfrac{[\vec{p}\ \vec{q}\ \vec{r}]}{x}$ has least value 2

   b. $x^4[\vec{a}\ \vec{b}\ \vec{c}]^2 + \dfrac{[\vec{p}\ \vec{q}\ \vec{r}]}{x^2}$ has least value $(3/2^{2/3})$

   c. $[\vec{p}\ \vec{q}\ \vec{r}] > 0$

   d. none of these

7. $a_1, a_2, a_3 \in R - \{0\}$ and $a_1 + a_2 \cos 2x + a_3 \sin^2 x = 0$ for all $x \in R$, then

a. vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = 4\hat{i} + 2\hat{j} + \hat{k}$ are perpendicular to each other

b. vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} + 2\hat{k}$ are parallel to each other

c. if vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is of length $\sqrt{6}$ units, then one of the ordered tripplet $(a_1, a_2, a_3) = (1, -1, -2)$

d. if $2a_1 + 3a_2 + 6a_3 = 26$, then $|a_1\hat{i} + a_2\hat{j} + a_3\hat{k}|$ is $2\sqrt{6}$

8. If $\vec{a}$ and $\vec{b}$ are two vectors and angle between them is θ, then

a. $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$

b. $|\vec{a} \times \vec{b}| = (\vec{a} \cdot \vec{b})$, if $\theta = \pi/4$

c. $\vec{a} \times \vec{b} = (\vec{a} \cdot \vec{b})\hat{n}$, ($\hat{n}$ is normal unit vector), if $\theta = \pi/4$

d. $(\vec{a} \times \vec{b}) \cdot (\vec{a} + \vec{b}) = 0$

9. Let $\vec{a}$ and $\vec{b}$ be two non-zero perpendicular vectors. A vector $\vec{r}$ satisfying the equation $\vec{r} \times \vec{b} = \vec{a}$ can be

a. $\vec{b} - \dfrac{\vec{a} \times \vec{b}}{|\vec{b}|^2}$ b. $2\vec{b} - \dfrac{\vec{a} \times \vec{b}}{|\vec{b}|^2}$ c. $|\vec{a}|\vec{b} - \dfrac{\vec{a} \times \vec{b}}{|\vec{b}|^2}$ d. $|\vec{b}|\vec{b} - \dfrac{\vec{a} \times \vec{b}}{|\vec{b}|^2}$

10. If vectors $\vec{b} = (\tan\alpha, -1, 2\sqrt{\sin\alpha/2})$ and $\vec{c} = \left(\tan\alpha, \tan\alpha, -\dfrac{3}{\sqrt{\sin\alpha/2}}\right)$ are orthogonal and vector $\vec{a} = (1, 3, \sin 2\alpha)$ makes an obtuse angle with the z-axis, then the value of α is

a. $\alpha = (4n + 1)\pi + \tan^{-1} 2$

b. $\alpha = (4n + 1)\pi - \tan^{-1} 2$

c. $\alpha = (4n + 2)\pi + \tan^{-1} 2$

d. $\alpha = (4n + 2)\pi - \tan^{-1} 2$

11. Let $\vec{r}$ be a unit vector satisfying $\vec{r} \times \vec{a} = \vec{b}$, where $|\vec{a}| = \sqrt{3}$ and $|\vec{b}| = \sqrt{2}$. Then

a. $\vec{r} = \dfrac{2}{3}(\vec{a} + \vec{a} \times \vec{b})$

b. $\vec{r} = \dfrac{1}{3}(\vec{a} + \vec{a} \times \vec{b})$

c. $\vec{r} = \dfrac{2}{3}(\vec{a} - \vec{a} \times \vec{b})$

d. $\vec{r} = \dfrac{1}{3}(-\vec{a} + \vec{a} \times \vec{b})$

12. If $\vec{a}$ and $\vec{b}$ are unequal unit vectors such that $(\vec{a} - \vec{b}) \times [(\vec{b} + \vec{a}) \times (2\vec{a} + \vec{b})] = \vec{a} + \vec{b}$, then angle θ between $\vec{a}$ and $\vec{b}$ is

a. 0 b. $\pi/2$ c. $\pi/4$ d. π

13. If $\vec{a}$ and $\vec{b}$ are two unit vectors perpendicular to each other and $\vec{c} = \lambda_1\vec{a} + \lambda_2\vec{b} + \lambda_3(\vec{a} \times \vec{b})$, then which of the following is (are) true?

a. $\lambda_1 = \vec{a} \cdot \vec{c}$

b. $\lambda_2 = |\vec{b} \times \vec{c}|$

c. $\lambda_3 = |(\vec{a} \times \vec{b}) \times \vec{c}|$

d. $\lambda_1 + \lambda_2 + \lambda_3 = (\vec{a} + \vec{b} + \vec{a} \times \vec{b}) \cdot \vec{c}$

14. If vectors $\vec{a}$ and $\vec{b}$ are non-collinear, then $\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}$ is

a. a unit vector

b. in the plane of $\vec{a}$ and $\vec{b}$

c. equally inclined to $\vec{a}$ and $\vec{b}$

d. perpendicular to $\vec{a} \times \vec{b}$

15. If $\vec{a}$ and $\vec{b}$ are non zero vectors such that $|\vec{a} + \vec{b}| = |\vec{a} - 2\vec{b}|$, then

a. $2\vec{a}.\vec{b} = |\vec{b}|^2$

b. $\vec{a}.\vec{b} = |\vec{b}|^2$

c. least value of $\vec{a}\cdot\vec{b} + \frac{1}{|\vec{b}|^2 + 2}$ is $\sqrt{2}$

d. least value of $\vec{a}\cdot\vec{b} + \frac{1}{|\vec{b}| + 2}$ is $\sqrt{2} - 1$

16. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be non-zero vectors and $\vec{V_1} = \vec{a} \times (\vec{b} \times \vec{c})$ and $\vec{V_2} = (\vec{a} \times \vec{b}) \times \vec{c}$. Vectors $\vec{V_1}$ and $\vec{V_2}$ are equal. Then

a. $\vec{a}$ and $\vec{b}$ are orthogonal

b. $\vec{a}$ and $\vec{c}$ are collinear

c. $\vec{b}$ and $\vec{c}$ are orthogonal

d. $\vec{b} = \lambda(\vec{a} \times \vec{c})$ when λ is a scalar

17. Vectors $\vec{A}$ and $\vec{B}$ satisfying the vector equation $\vec{A} + \vec{B} = \vec{a}$, $\vec{A} \times \vec{B} = \vec{b}$ and $\vec{A}\cdot\vec{a} = 1$, where $\vec{a}$ and $\vec{b}$ are given vectors, are

a. $\vec{A} = \frac{(\vec{a} \times \vec{b}) - \vec{a}}{a^2}$

b. $\vec{B} = \frac{(\vec{b} \times \vec{a}) + \vec{a}(a^2 - 1)}{a^2}$

c. $\vec{A} = \frac{(\vec{a} \times \vec{b}) + \vec{a}}{a^2}$

d. $\vec{B} = \frac{(\vec{b} \times \vec{a}) - \vec{a}(a^2 - 1)}{a^2}$

18. A vector $\vec{d}$ is equally inclined to three vectors $\vec{a} = \hat{i} - \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + \hat{j}$ and $\vec{c} = 3\hat{j} - 2\hat{k}$. Let $\vec{x}, \vec{y}$ and $\vec{z}$ be three vectors in the plane of $\vec{a}, \vec{b}$; $\vec{b}, \vec{c}$; $\vec{c}, \vec{a}$, respectively. Then

a. $\vec{x}\cdot\vec{d} = -1$

b. $\vec{y}\cdot\vec{d} = 1$

c. $\vec{z}\cdot\vec{d} = 0$

d. $\vec{r}\cdot\vec{d} = 0$, where $\vec{r} = \lambda\vec{x} + \mu\vec{y} + \delta\vec{z}$

19. Vectors perpendicular to $\hat{i} - \hat{j} - \hat{k}$ and in the plane of $\hat{i} + \hat{j} + \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ are

a. $\hat{i} + \hat{k}$

b. $2\hat{i} + \hat{j} + \hat{k}$

c. $3\hat{i} + 2\hat{j} + \hat{k}$

d. $-4\hat{i} - 2\hat{j} - 2\hat{k}$

20. If side $\overrightarrow{AB}$ of an equilateral triangle ABC lying in the x–y plane is $3\hat{i}$, then side $\overrightarrow{CB}$ can be

a. $-\frac{3}{2}(\hat{i} - \sqrt{3}\hat{j})$

b. $\frac{3}{2}(\hat{i} - \sqrt{3}\hat{j})$

c. $-\frac{3}{2}(\hat{i} + \sqrt{3}\hat{j})$

d. $\frac{3}{2}(\hat{i} + \sqrt{3}\hat{j})$

21. The angles of a triangle, two of whose sides are represented by vectors $\sqrt{3}(\hat{a} \times \vec{b})$ and $\hat{b} - (\hat{a} \cdot \vec{b})\hat{a}$, where $\vec{b}$ is a non-zero vector and $\hat{a}$ is a unit vector in the direction of $\vec{a}$, are

a. $\tan^{-1}(\sqrt{3})$ b. $\tan^{-1}(1/\sqrt{3})$ c. $\cot^{-1}(0)$ d. $\tan^{-1}(1)$

22. $\vec{a}, \vec{b}$ and $\vec{c}$ are unimodular and coplanar. A unit vector $\vec{d}$ is perpendicular to them. If $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \frac{1}{6}\hat{i} - \frac{1}{3}\hat{j} + \frac{1}{3}\hat{k}$, and the angle between $\vec{a}$ and $\vec{b}$ is 30°, then $\vec{c}$ is

a. $(\hat{i} - 2\hat{j} + 2\hat{k})/3$ b. $(-\hat{i} + 2\hat{j} - 2\hat{k})/3$ c. $(2\hat{i} + 2\hat{j} - \hat{k})/3$ d. $(-2\hat{i} - 2\hat{j} + \hat{k})/3$

23. If $\vec{a} + 2\vec{b} + 3\vec{c} = \vec{0}$, then $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} =$

a. $2(\vec{a} \times \vec{b})$ b. $6(\vec{b} \times \vec{c})$ c. $3(\vec{c} \times \vec{a})$ d. $\vec{0}$

24. $\vec{a}$ and $\vec{b}$ are two non-collinear unit vectors, and $\vec{u} = \vec{a} - (\vec{a} \cdot \vec{b})\vec{b}$ and $\vec{v} = \vec{a} \times \vec{b}$. Then $|\vec{v}|$ is

a. $|\vec{u}|$ b. $|\vec{u}| + |\vec{u} \cdot \vec{b}|$ c. $|\vec{u}| + |\vec{u} \cdot \vec{a}|$ d. none of these

25. If $\vec{a} \times \vec{b} = \vec{c}$, $\vec{b} \times \vec{c} = \vec{a}$, where $\vec{c} \neq \vec{0}$, then

a. $|\vec{a}| = |\vec{c}|$ b. $|\vec{a}| = |\vec{b}|$

c. $|\vec{b}| = 1$ d. $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

26. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-coplanar vectors and $\vec{d}$ be a non-zero vector, which is perpendicular to $(\vec{a} + \vec{b} + \vec{c})$. Now $\vec{d} = (\vec{a} \times \vec{b}) \sin x + (\vec{b} \times \vec{c}) \cos y + 2(\vec{c} \times \vec{a})$. Then

a. $\dfrac{\vec{d} \cdot (\vec{a} + \vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]} = 2$ b. $\dfrac{\vec{d} \cdot (\vec{a} + \vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]} = -2$

c. minimum value of $x^2 + y^2$ is $\pi^2/4$ d. minimum value of $x^2 + y^2$ is $5\pi^2/4$

27. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three unit vectors such that $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{1}{2}\vec{b}$, then ($\vec{b}$ and $\vec{c}$ being non-parallel)

a. angle between $\vec{a}$ and $\vec{b}$ is $\pi/3$ b. angle between $\vec{a}$ and $\vec{c}$ is $\pi/3$

c. angle between $\vec{a}$ and $\vec{b}$ is $\pi/2$ d. angle between $\vec{a}$ and $\vec{c}$ is $\pi/2$

28. If in triangle ABC, $\overrightarrow{AB} = \dfrac{\vec{u}}{|\vec{u}|} - \dfrac{\vec{v}}{|\vec{v}|}$ and $\overrightarrow{AC} = \dfrac{2\vec{u}}{|\vec{u}|}$, where $|\vec{u}| \neq |\vec{v}|$, then

a. $1 + \cos 2A + \cos 2B + \cos 2C = 0$ b. $\sin A = \cos C$

c. projection of AC on BC is equal to BC d. projection of AB on BC is equal to AB

29. $[\vec{a} \times \vec{b} \;\; \vec{c} \times \vec{d} \;\; \vec{e} \times \vec{f}]$ is equal to

a. $[\vec{a}\,\vec{b}\,\vec{d}][\vec{c}\,\vec{e}\,\vec{f}] - [\vec{a}\,\vec{b}\,\vec{c}][\vec{d}\,\vec{e}\,\vec{f}]$ b. $[\vec{a}\,\vec{b}\,\vec{e}][\vec{f}\,\vec{c}\,\vec{d}] - [\vec{a}\,\vec{b}\,\vec{f}][\vec{e}\,\vec{c}\,\vec{d}]$

c. $[\vec{c}\,\vec{d}\,\vec{a}][\vec{b}\,\vec{e}\,\vec{f}] - [\vec{a}\,\vec{d}\,\vec{b}][\vec{a}\,\vec{e}\,\vec{f}]$ d. $[\vec{a}\,\vec{c}\,\vec{e}][\vec{b}\,\vec{d}\,\vec{f}]$

30. The scalars l and m such that $l\vec{a} + m\vec{b} = \vec{c}$, where $\vec{a}, \vec{b}$ and $\vec{c}$ are given vectors, are equal to

a. $l = \dfrac{(\vec{c} \times \vec{b}) \cdot (\vec{a} \times \vec{b})}{(\vec{a} \times \vec{b})^2}$

b. $l = \dfrac{(\vec{c} \times \vec{a}) \cdot (\vec{b} \times \vec{a})}{(\vec{b} \times \vec{a})}$

c. $m = \dfrac{(\vec{c} \times \vec{a}) \cdot (\vec{b} \times \vec{a})}{(\vec{b} \times \vec{a})^2}$

d. $m = \dfrac{(\vec{c} \times \vec{a}) \cdot (\vec{b} \times \vec{a})}{(\vec{b} \times \vec{a})}$

31. If $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) \cdot (\vec{a} \times \vec{d}) = 0$, then which of the following may be true?

a. $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are necessarily coplanar

b. $\vec{a}$ lies in the plane of $\vec{c}$ and $\vec{d}$

c. $\vec{b}$ lies in the plane of $\vec{a}$ and $\vec{d}$

d. $\vec{c}$ lies in the plane of $\vec{a}$ and $\vec{d}$

32. A, B, C and D are four points such that $\overrightarrow{AB} = m(2\hat{i} - 6\hat{j} + 2\hat{k})$, $\overrightarrow{BC} = (\hat{i} - 2\hat{j})$ and $\overrightarrow{CD} = n(-6\hat{i} + 15\hat{j} - 3\hat{k})$. If CD intersects AB at some point E, then

a. $m \ge 1/2$

b. $n \ge 1/3$

c. $m = n$

d. $m < n$

33. If vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar and l, m and n are distinct scalars, then $[(l\vec{a} + m\vec{b} + n\vec{c})(l\vec{b} + m\vec{c} + n\vec{a})(l\vec{c} + m\vec{a} + n\vec{b})] = 0$ implies

a. $l + m + n = 0$

b. roots of the equation $lx^2 + mx + n = 0$ are real

c. $l^2 + m^2 + n^2 = 0$

d. $l^3 + m^3 + n^3 = 3lmn$

34. Let $\vec{\alpha} = a\hat{i} + b\hat{j} + c\hat{k}$, $\vec{\beta} = b\hat{i} + c\hat{j} + a\hat{k}$ and $\vec{\gamma} = c\hat{i} + a\hat{j} + b\hat{k}$ be three coplanar vectors with $a \ne b$, and $\vec{v} = \hat{i} + \hat{j} + \hat{k}$. Then $\vec{v}$ is perpendicular to

a. $\vec{\alpha}$

b. $\vec{\beta}$

c. $\vec{\gamma}$

d. none of these

35. If vectors $\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, $\vec{B} = \hat{i} + \hat{j} + 5\hat{k}$ and $\vec{C}$ form a left-handed system, then $\vec{C}$ is

a. $11\hat{i} - 6\hat{j} - \hat{k}$

b. $-11\hat{i} + 6\hat{j} + \hat{k}$

c. $11\hat{i} - 6\hat{j} + \hat{k}$

d. $-11\hat{i} + 6\hat{j} - \hat{k}$

Reasoning Type

Solutions on page 2.126

Each question has four choices *a, b, c* and *d*, out of which *only one* is correct. Each equation contains Statement 1 and Statement 2.

a. Both the statements are true and Statement 2 is the correct explanation for Statement 1.

b. Both the statements are true but Statement 2 is not the correct explanation for Statement 1.

c. Statement 1 is true and Statement 2 is false.

d. Statement 1 is false and Statement 2 is true.

1. **Statement 1:** Vector $\vec{c} = -5\hat{i} + 7\hat{j} + 2\hat{k}$ is along the bisector of angle between $\vec{a} = \hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = -8\hat{i} + \hat{j} - 4\hat{k}$.

   Statement 2: $\vec{c}$ is equally inclined to $\vec{a}$ and $\vec{b}$.

2. **Statement 1:** A component of vector $\vec{b} = 4\hat{i} + 2\hat{j} + 3\hat{k}$ in the direction perpendicular to the direction of vector $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ is $\hat{i} - \hat{j}$.

   Statement 2: A component of vector in the direction of $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ is $2\hat{i} + 2\hat{j} + 2\hat{k}$.

3. **Statement 1:** Distance of point $D(1, 0, -1)$ from the plane of points $A(1, -2, 0)$, $B(3, 1, 2)$ and $C(-1, 1, -1)$ is $\frac{8}{\sqrt{229}}$.

   Statement 2: Volume of tetrahedron formed by the points A, B, C and D is $\frac{\sqrt{229}}{2}$.

4. Let $\vec{r}$ be a non-zero vector satisfying $\vec{r}\cdot\vec{a} = \vec{r}\cdot\vec{b} = \vec{r}\cdot\vec{c} = 0$ for given non-zero vectors $\vec{a}, \vec{b}$ and $\vec{c}$.

   Statement 1: $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}] = 0$

   Statement 2: $[\vec{a}\,\vec{b}\,\vec{c}] = 0$

5. **Statement 1:** If $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ are three mutually perpendicular unit vectors, then $a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$, $a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ and $a_3\hat{i} + b_3\hat{j} + c_3\hat{k}$ may be mutually perpendicular unit vectors.

   Statement 2: Value of determinant and its transpose are the same.

6. **Statement 1:** If $\vec{A} = 2\hat{i} + 3\hat{j} + 6\hat{k}$, $\vec{B} = \hat{i} + \hat{j} - 2\hat{k}$ and $\vec{C} = \hat{i} + 2\hat{j} + \hat{k}$, then $|\vec{A}\times(\vec{A}\times(\vec{A}\times\vec{B}))\cdot\vec{C}| = 243$.

   Statement 2: $|\vec{A}\times(\vec{A}\times(\vec{A}\times\vec{B}))\cdot\vec{C}| = |\vec{A}|^2|[\vec{A}\vec{B}\vec{C}]|$

7. **Statement 1:** $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three mutually perpendicular unit vectors and $\vec{d}$ is a vector such that $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are non-coplanar. If $[\vec{d}\,\vec{b}\,\vec{c}] = [\vec{d}\,\vec{a}\,\vec{b}] = [\vec{d}\,\vec{c}\,\vec{a}] = 1$, then $\vec{d} = \vec{a} + \vec{b} + \vec{c}$.

   Statement 2: $[\vec{d}\,\vec{b}\,\vec{c}] = [\vec{d}\,\vec{a}\,\vec{b}] = [\vec{d}\,\vec{c}\,\vec{a}] \Rightarrow \vec{d}$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$.

8. Consider three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

   Statement 1: $\vec{a}\times\vec{b} = ((\hat{i}\times\vec{a})\cdot\vec{b})\hat{i} + ((\hat{j}\times\vec{a})\cdot\vec{b})\hat{j} + ((\hat{k}\times\vec{a})\cdot\vec{b})\hat{k}$

   Statement 2: $\vec{c} = (\hat{i}\cdot\vec{c})\hat{i} + (\hat{j}\cdot\vec{c})\hat{j} + (\hat{k}\cdot\vec{c})\hat{k}$

Linked Comprehension Type

Solutions on page 2.128

Based on each paragraph, three multiple choice questions have to be answered. Each question has four choices *a*, *b*, *c* and *d*, out of which *only one* is correct.

For Problems 1–3

Let $\vec{u}, \vec{v}$ and $\vec{w}$ be three unit vectors such that $\vec{u} + \vec{v} + \vec{w} = \vec{a}$, $\vec{u}\times(\vec{v}\times\vec{w}) = \vec{b}$, $(\vec{u}\times\vec{v})\times\vec{w} = \vec{c}$, $\vec{a}\cdot\vec{u} = 3/2$, $\vec{a}\cdot\vec{v} = 7/4$ and $|\vec{a}| = 2$.

1. Vector $\vec{u}$ is

a. $\vec{a}-\frac{2}{3}\vec{b}+\vec{c}$ b. $\vec{a}+\frac{4}{3}\vec{b}+\frac{8}{3}\vec{c}$ c. $2\vec{a}-\vec{b}+\frac{1}{3}\vec{c}$ d. $\frac{4}{3}\vec{a}-\vec{b}+\frac{2}{3}\vec{c}$

2. Vector $\vec{v}$ is

a. $2\vec{a}-3\vec{c}$ b. $3\vec{b}-4c$ c. $-4\vec{c}$ d. $\vec{a}+\vec{b}+2\vec{c}$

3. Vector $\vec{w}$ is

a. $\frac{2}{3}(2\vec{c}-\vec{b})$ b. $\frac{1}{3}(\vec{a}-\vec{b}-\vec{c})$ c. $\frac{1}{3}\vec{a}-\frac{2}{3}\vec{b}-2\vec{c}$ d. $\frac{4}{3}(\vec{c}-\vec{b})$

For Problems 4–6

Vectors $\vec{x},\vec{y}$ and $\vec{z}$, each of magnitude $\sqrt{2}$, make an angle of 60° with each other. $\vec{x}\times(\vec{y}\times\vec{z})=\vec{a}$, $\vec{y}\times(\vec{z}\times\vec{x})=\vec{b}$ and $\vec{x}\times\vec{y}=\vec{c}$.

4. Vector $\vec{x}$ is

a. $\frac{1}{2}[(\vec{a}-\vec{b})\times\vec{c}+(\vec{a}+\vec{b})]$ b. $\frac{1}{2}[(\vec{a}+\vec{b})\times\vec{c}+(\vec{a}-\vec{b})]$

c. $\frac{1}{2}[-(\vec{a}+\vec{b})\times\vec{c}+(\vec{a}+\vec{b})]$ d. $\frac{1}{2}[(\vec{a}+\vec{b})\times\vec{c}-(\vec{a}+\vec{b})]$

5. Vector $\vec{y}$ is

a. $\frac{1}{2}[(\vec{a}+\vec{c})\times\vec{b}-\vec{b}-\vec{a}]$ b. $\frac{1}{2}[(\vec{a}-\vec{c})\times\vec{c}+\vec{b}+\vec{a}]$

c. $\frac{1}{2}[(\vec{a}+\vec{b})\times\vec{c}+\vec{b}+\vec{a}]$ d. $\frac{1}{2}[(\vec{a}-\vec{c})\times\vec{a}+\vec{b}-\vec{a}]$

6. Vector $\vec{z}$ is

a. $\frac{1}{2}[(\vec{a}-\vec{c})\times\vec{c}-\vec{b}+\vec{a}]$ b. $\frac{1}{2}[(\vec{a}+\vec{b})\times\vec{c}+\vec{b}-\vec{a}]$

c. $\frac{1}{2}[\vec{c}\times(\vec{a}-\vec{b})+\vec{b}+\vec{a}]$ d. none of these

For Problems 7–9

If $\vec{x}\times\vec{y}=\vec{a}$, $\vec{y}\times\vec{z}=\vec{b}$, $\vec{x}\cdot\vec{b}=\gamma$, $\vec{x}\cdot\vec{y}=1$ and $\vec{y}\cdot\vec{z}=1$

7. Vector $\vec{x}$ is

a. $\frac{1}{|\vec{a}\times\vec{b}|^2}[\vec{a}\times(\vec{a}\times\vec{b})]$ b. $\frac{\gamma}{|\vec{a}\times\vec{b}|^2}[\vec{a}\times\vec{b}-\vec{a}\times(\vec{a}\times\vec{b})]$

c. $\frac{\gamma}{|\vec{a}\times\vec{b}|^2}[\vec{a}\times\vec{b}+\vec{b}\times(\vec{a}\times\vec{b})]$ d. none of these

8. Vector $\vec{y}$ is

a. $\dfrac{\vec{a}\times\vec{b}}{\gamma}$ b. $\vec{a}+\dfrac{\vec{a}\times\vec{b}}{\gamma}$ c. $\vec{a}+\vec{b}+\dfrac{\vec{a}\times\vec{b}}{\gamma}$ d. none of these

9. Vector $\vec{z}$ is

a. $\dfrac{\gamma}{|\vec{a}\times\vec{b}|^2}[\vec{a}+\vec{b}\times(\vec{a}\times\vec{b})]$

b. $\dfrac{\gamma}{|\vec{a}\times\vec{b}|^2}[\vec{a}+\vec{b}-\vec{a}\times(\vec{a}\times\vec{b})]$

c. $\dfrac{\gamma}{|\vec{a}\times\vec{b}|^2}[\vec{a}\times\vec{b}+\vec{b}\times(\vec{a}\times\vec{b})]$

d. none of these

For Problems 10–12

Given two orthogonal vectors $\vec{A}$ and $\vec{B}$ each of length unity. Let $\vec{P}$ be the vector satisfying the equation $\vec{P}\times\vec{B}=\vec{A}-\vec{P}$. Then

10. $(\vec{P}\times\vec{B})\times\vec{B}$ is equal to

a. $\vec{P}$ b. $-\vec{P}$ c. $2\vec{B}$ d. $\vec{A}$

11. $\vec{P}$ is equal to

a. $\dfrac{\vec{A}}{2}+\dfrac{\vec{A}\times\vec{B}}{2}$ b. $\dfrac{\vec{A}}{2}+\dfrac{\vec{B}\times\vec{A}}{2}$ c. $\dfrac{\vec{A}\times\vec{B}}{2}-\dfrac{\vec{A}}{2}$ d. $\vec{A}\times\vec{B}$

12. Which of the following statements is false?

a. vectors $\vec{P}$, $\vec{A}$ and $\vec{P}\times\vec{B}$ are linearly dependent.

b. vectors $\vec{P}$, $\vec{B}$ and $\vec{P}\times\vec{B}$ are linearly independent.

c. $\vec{P}$ is orthogonal to $\vec{B}$ and has length $1/\sqrt{2}$.

d. none of the above.

For Problems 13–15

Let $\vec{a}=2\hat{i}+3\hat{j}-6\hat{k}$, $\vec{b}=2\hat{i}-3\hat{j}+6\hat{k}$ and $\vec{c}=-2\hat{i}+3\hat{j}+6\hat{k}$. Let $\vec{a}_1$ be the projection of $\vec{a}$ on $\vec{b}$ and $\vec{a}_2$ be the projection of $\vec{a}_1$ on $\vec{c}$. Then

13. $\vec{a}_2$ is equal to

a. $\dfrac{943}{49}(2\hat{i}-3\hat{j}-6\hat{k})$

b. $\dfrac{943}{49^2}(2\hat{i}-3\hat{j}-6\hat{k})$

c. $\dfrac{943}{49}(-2\hat{i}+3\hat{j}+6\hat{k})$

d. $\dfrac{943}{49^2}(-2\hat{i}+3\hat{j}+6\hat{k})$

14. $\vec{a}_1 \cdot \vec{b}$ is equal to

a. -41 b. $-41/7$ c. 41 d. 287

15. Which of the following is true?

a. $\vec{a}$ and $\vec{a}_2$ are collinear
b. $\vec{a}_1$ and $\vec{c}$ are collinear
c. $\vec{a}$, $\vec{a}_1$ and $\vec{b}$ are coplanar
d. $\vec{a}$, $\vec{a}_1$ and $\vec{a}_2$ are coplanar

For Problems 16–18

Consider a triangular pyramid $ABCD$ the position vectors of whose angular points are $A(3, 0, 1)$, $B(-1, 4, 1)$, $C(5, 2, 3)$ and $D(0, -5, 4)$. Let G be the point of intersection of the medians of triangle BCD.

16. The length of vector $\overrightarrow{AG}$ is

a. $\sqrt{17}$ b. $\sqrt{51}/3$ c. $3/\sqrt{6}$ d. $\sqrt{59}/4$

17. Area of triangle ABC in sq. units is

a. 24 b. $8\sqrt{6}$ c. $4\sqrt{6}$ d. none of these

18. The length of the perpendicular from vertex D on the opposite face is

a. $14/\sqrt{6}$ b. $2/\sqrt{6}$ c. $3/\sqrt{6}$ d. none of these

For Problems 19–21

Vertices of a parallelogram taken in order are $A(2, -1, 4)$; $B(1, 0 -1)$; $C(1, 2, 3)$ and D.

19. The distance between the parallel lines AB and CD is

a. $\sqrt{6}$ b. $3\sqrt{6/5}$ c. $2\sqrt{2}$ d. 3

20. Distance of the point $P(8, 2, -12)$ from the plane of the parallelogram is

a. $\frac{4\sqrt{6}}{9}$ b. $\frac{32\sqrt{6}}{9}$ c. $\frac{16\sqrt{6}}{9}$ d. none

21. The orthogonal projections of the parallelogram on the three coordinate planes xy, yz and zx, respectively, are

a. 14, 4, 2 b. 2, 4, 14 c. 4, 2, 14 d. 2, 14, 4

For Problems 22–24

Let $\vec{r}$ be a position vector of a variable point in Cartesian OXY plane such that $\vec{r} \cdot (10\hat{j} - 8\hat{i} - \vec{r}) = 40$ and $p_1 = \max\{|\vec{r} + 2\hat{i} - 3\hat{j}|^2\}$, $p_2 = \min\{|\vec{r} + 2\hat{i} - 3\hat{j}|^2\}$. A tangent line is drawn to the curve $y = 8/x^2$ at point A with abscissa 2. The drawn line cuts the x-axis at a point B.

22. p_2 is equal to

a. 9 b. $2\sqrt{2} - 1$ c. $6\sqrt{2} + 3$ d. $9 - 4\sqrt{2}$

23. $p_1 + p_2$ is equal to

a. 2 b. 10 c. 18 d. 5

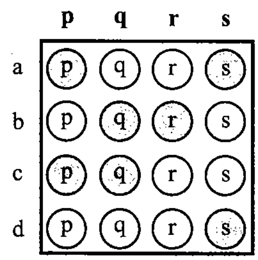
24. $\overrightarrow{AB} \cdot \overrightarrow{OB}$ is equal to

a. 1 b. 2 c. 3 d. 4

Matrix-Match Type

Solutions on page 2.134

Each question contains statements given in two columns which have to be matched. Statements (a, b, c, d) in Column I have to be matched with statements (p, q, r, s) in Column II. If the correct matches are a → p, s; b → q, r; c → p, q and d → s, then the correctly bubbled 4 × 4 matrix should be as follows:



1.

Column I	Column II
a. The possible value of a if $\vec{r} = (\hat{i} + \hat{j}) + \lambda(\hat{i} + 2\hat{j} - \hat{k})$ and $\vec{r} = (\hat{i} + 2\hat{j}) + \mu(-\hat{i} + \hat{j} + a\hat{k})$ are not consistent, where λ and μ are scalars, is	**p.** -4
b. The angle between vectors $\vec{a} = \lambda\hat{i} - 3\hat{j} - \hat{k}$ and $\vec{b} = 2\lambda\hat{i} + \lambda\hat{j} - \hat{k}$ is acute, whereas vector $\vec{b}$ makes an obtuse angle with the axes of coordinates. Then λ may be	**q.** -2
c. The possible value of a such that $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} + (1 + a)\hat{k}$ and $3\hat{i} + a\hat{j} + 5\hat{k}$ are coplanar is	**r.** 2
d. If $\vec{A} = 2\hat{i} + \lambda\hat{j} + 3\hat{k}$, $\vec{B} = 2\hat{i} + \lambda\hat{j} + \hat{k}$, $\vec{C} = 3\hat{i} + \hat{j}$ and $\vec{A} + \lambda\vec{B}$ is perpendicular to $\vec{C}$, then $\|2\lambda\|$ is	**s.** 3

2.

Column I	Column II
a. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors where $\|\vec{a}\| = \|\vec{b}\| = 2, \|\vec{c}\| = 1$, then $[\vec{a}\times\vec{b} \quad \vec{b}\times\vec{c} \quad \vec{c}\times\vec{a}]$ is	**p.** -12
b. If $\vec{a}$ and $\vec{b}$ are two unit vectors inclined at $\pi/3$, then $16[\vec{a} \quad \vec{b}+\vec{a}\times\vec{b} \quad \vec{b}]$ is	**q.** 0
c. If $\vec{b}$ and $\vec{c}$ are orthogonal unit vectors and $\vec{b}\times\vec{c} = \vec{a}$, then $[\vec{a}+\vec{b}+\vec{c} \quad \vec{a}+\vec{b} \quad \vec{b}+\vec{c}]$ is	**r.** 16
d. If $[\vec{x}\ \vec{y}\ \vec{a}] = [\vec{x}\ \vec{y}\ \vec{b}] = [\vec{a}\ \vec{b}\ \vec{c}] = 0$, each vector being a non-zero vector, then $[\vec{x}\ \vec{y}\ \vec{c}]$ is	**s.** 1

3.

Column I	Column II
a. If $\|\vec{a}\| = \|\vec{b}\| = \|\vec{c}\|$, angle between each pair of vectors is $\frac{\pi}{3}$ and $\|\vec{a}+\vec{b}+\vec{c}\| = \sqrt{6}$, then $2\|\vec{a}\|$ is equal to	**p.** 3
b. If $\vec{a}$ is perpendicular to $\vec{b}+\vec{c}$, $\vec{b}$ is perpendicular to $\vec{c}+\vec{a}$, $\vec{c}$ is perpendicular to $\vec{a}+\vec{b}$, $\|\vec{a}\| = 2$, $\|\vec{b}\| = 3$ and $\|\vec{c}\| = 6$, then $\|\vec{a}+\vec{b}+\vec{c}\| - 2$ is equal to	**q.** 2
c. $\vec{a} = 2\hat{i}+3\hat{j}-\hat{k}$, $\vec{b} = -\hat{i}+2\hat{j}-4\hat{k}$, $\vec{c} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{d} = 3\hat{i}+2\hat{j}+\hat{k}$, then $\frac{1}{7}(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})$ is equal to	**r.** 4
d. If $\|\vec{a}\| = \|\vec{b}\| = \|\vec{c}\| = 2$ and $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 2$, then $[\vec{a}\ \vec{b}\ \vec{c}]\cos 45^\circ$ is equal to	**s.** 5

4. Given two vectors $\vec{a} = -\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b} = -\hat{i}-2\hat{j}-\hat{k}$.

Column I	Column II
a. Area of triangle formed by $\vec{a}$ and $\vec{b}$	**p.** 3
b. Area of parallelogram having sides $\vec{a}$ and $\vec{b}$	**q.** $12\sqrt{3}$
c. Area of parallelogram having diagonals $2\vec{a}$ and $4\vec{b}$	**r.** $3\sqrt{3}$
d. Volume of parallelepiped formed by $\vec{a}$, $\vec{b}$ and $\vec{c} = \hat{i}+\hat{j}+\hat{k}$	**s.** $\frac{3\sqrt{3}}{2}$

5. Given two vectors $\vec{a} = -\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} + 2\hat{k}$.

Column I	Column II
a. A vector coplanar with $\vec{a}$ and $\vec{b}$	**p.** $-3\hat{i} + 3\hat{j} + 4\hat{k}$
b. A vector which is perpendicular to both $\vec{a}$ and $\vec{b}$	**q.** $2\hat{i} - 2\hat{j} + 3\hat{k}$
c. A vector which is equally inclined to $\vec{a}$ and $\vec{b}$	**r.** $\hat{i} + \hat{j}$
d. A vector which forms a triangle with $\vec{a}$ and $\vec{b}$	**s.** $\hat{i} - \hat{j} + 5\hat{k}$

6.

Column I	Column II
a. If $\lvert\vec{a} + \vec{b}\rvert = \lvert\vec{a} + 2\vec{b}\rvert$, then angle between $\vec{a}$ and $\vec{b}$ is	**p.** 90°
b. If $\lvert\vec{a} + \vec{b}\rvert = \lvert\vec{a} - 2\vec{b}\rvert$, then angle between $\vec{a}$ and $\vec{b}$ is	**q.** obtuse
c. If $\lvert\vec{a} + \vec{b}\rvert = \lvert\vec{a} - \vec{b}\rvert$, then angle between $\vec{a}$ and $\vec{b}$ is	**r.** 0°
d. Angle between $\vec{a} \times \vec{b}$ and a vector perpendicular to the vector $\vec{c} \times (\vec{a} \times \vec{b})$ is	**s.** acute

7. Volume of parallelepiped formed by vectors $\vec{a} \times \vec{b}$, $\vec{b} \times \vec{c}$ and $\vec{c} \times \vec{a}$ is 36 sq. units.

Column I	Column II
a. Volume of parallelepiped formed by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is	**p.** 0 sq. units
b. Volume of tetrahedron formed by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is	**q.** 12 sq. units
c. Volume of parallelepiped formed by vectors $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ is	**r.** 6 sq. units
d. Volume of parallelepiped formed by vectors $\vec{a} - \vec{b}$, $\vec{b} - \vec{c}$ and $\vec{c} - \vec{a}$ is	**s.** 1 sq. units

Integer Answer Type

Solutions on page 2.138

1. If $\vec{a}$ and $\vec{b}$ are any two unit vectors, then find the greatest positive integer in the range of $\frac{3|\vec{a}+\vec{b}|}{2} + 2\,|\vec{a}-\vec{b}|$.

2. Let $\vec{u}$ be a vector on rectangular coordinate system with sloping angle 60°. Suppose that $|\vec{u}-\hat{i}|$ is geometric mean of $|\vec{u}|$ and $|\vec{u}-2\hat{i}|$, where $\hat{i}$ is the unit vector along x-axis. Then find the value of $(\sqrt{2}+1)|\vec{u}|$.

3. Find the absolute value of parameter t for which the area of the triangle whose vertices are $A(-1, 1, 2)$; $B(1, 2, 3)$ and $C(t, 1, 1)$ is minimum.

4. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ and $[3\vec{a}+\vec{b}\;\; 3\vec{b}+\vec{c}\;\; 3\vec{c}+\vec{a}] = \lambda\begin{vmatrix}\vec{a}\cdot\hat{i} & \vec{a}\cdot\hat{j} & \vec{a}\cdot\hat{k}\\ \vec{b}\cdot\hat{i} & \vec{b}\cdot\hat{j} & \vec{b}\cdot\hat{k}\\ \vec{c}\cdot\hat{i} & \vec{c}\cdot\hat{j} & \vec{c}\cdot\hat{k}\end{vmatrix}$, then find the value of $\frac{\lambda}{4}$.

5. Let $\vec{a} = \alpha\hat{i} + 2\hat{j} - 3\hat{k}$, $\vec{b} = \hat{i} + 2\alpha\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \alpha\hat{j} + \hat{k}$. Find the value of 6α, such that $\{(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})\}\times(\vec{c}\times\vec{a}) = 0$.

6. If $\vec{x}, \vec{y}$ are two non-zero and non-collinear vectors satisfying $[(a-2)\alpha^2+(b-3)\alpha+c]\,\vec{x} + [(a-2)\beta^2+(b-3)\beta+c]\,\vec{y} + [(a-2)\gamma^2+(b-3)\gamma+c]\,(\vec{x}\times\vec{y}) = 0$, where α, β, γ are three distinct real numbers, then find the value of $(a^2+b^2+c^2-4)$.

7. Let $\vec{u}$ and $\vec{v}$ are unit vectors such that $\vec{u}\times\vec{v}+\vec{u} = \vec{w}$ and $\vec{w}\times\vec{u} = \vec{v}$. Find the value of $[\vec{u}\,\vec{v}\,\vec{w}]$.

8. Find the value of λ if the volume of a tetrahedron whose vertices are with position vectors $\hat{i} - 6\hat{j} + 10\hat{k}, -\hat{i} - 3\hat{j} + 7\hat{k}, 5\hat{i} - \hat{j} + \lambda\hat{k}$ and $7\hat{i} - 4\hat{j} + 7\hat{k}$ is 11 cubic unit.

9. Given that $\vec{u} = \hat{i} - 2\hat{j} + 3\hat{k}$; $\vec{v} = 2\hat{i} + \hat{j} + 4\hat{k}$; $\vec{w} = \hat{i} + 3\hat{j} + 3\hat{k}$ and $(\vec{u}\cdot\vec{R} - 15)\hat{i} + (\vec{v}\cdot\vec{R} - 30)\hat{j} + (\vec{w}\cdot\vec{R} - 20)\hat{k} = \vec{0}$. Then find the greatest integer less than or equal to $|\vec{R}|$.

10. Let a three-dimensional vector $\vec{V}$ satisfies the condition, $2\vec{V} + \vec{V}\times(\hat{i}+2\hat{j}) = 2\hat{i}+\hat{k}$. If $3|\vec{V}| = \sqrt{m}$, then find the value of m.

11. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a}\cdot\vec{b} = 0 = \vec{a}\cdot\vec{c}$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{3}$, then find the value of $|\vec{a}\times\vec{b} - \vec{a}\times\vec{c}|$.

12. Let $\vec{OA} = \vec{a}$, $\vec{OB} = 10\vec{a} + 2\vec{b}$ and $\vec{OC} = \vec{b}$, where O, A and C are non-collinear points. Let p denote the area of quadrilateral $OACB$, and let q denote the area of parallelogram with OA and OC as adjacent sides. If $p = k\,q$, then find k.

13. Find the work done by the force $F = 3\hat{i} - \hat{j} - 2\hat{k}$ acting on a particle such that the particle is displaced from point A (– 3, –4, 1) to point B (– 1, – 1, – 2).

Archives

Solutions on page 2.144

Subjective Type

1. From a point O inside a triangle ABC, perpendiculars OD, OE and OF are drawn to the sides BC, CA and AB, respectively. Prove that the perpendiculars from A, B and C to the sides EF, FD and DE are concurrent. **(IIT-JEE, 1978)**

2. $A_1, A_2, \ldots, A_n$ are the vertices of a regular plane polygon with n sides and O as its centre. Show that $\sum_{i=1}^{n-1} (\overrightarrow{OA_i} \times \overrightarrow{OA_{i+1}}) = (1-n)(\overrightarrow{OA_2} \times \overrightarrow{OA_1})$. **(IIT-JEE,1998)**

3. If c be a given non-zero scalar, and $\vec{A}$ and $\vec{B}$ be given non-zero vectors such that $\vec{A} \perp \vec{B}$, find the vector $\vec{X}$ which satisfies the equations $\vec{A} \cdot \vec{X} = c$ and $\vec{A} \times \vec{X} = \vec{B}$. **(IIT-JEE, 1983)**

4. If A, B, C, D are any four points in space, prove that $|\overrightarrow{AB} \times \overrightarrow{CD} + \overrightarrow{BC} \times \overrightarrow{AD} + \overrightarrow{CA} \times \overrightarrow{BD}| = 4$ (area of triangle ABC). **(IIT-JEE, 1986)**

5. If vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, show that $\begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \end{vmatrix} = \vec{0}$. **(IIT-JEE, 1989)**

6. Let $\vec{A} = 2\vec{i} + \vec{k}, \vec{B} = \vec{i} + \vec{j} + \vec{k}$ and $\vec{C} = 4\vec{i} - 3\vec{j} + 7\vec{k}$. Determine a vector $\vec{R}$ satisfying $\vec{R} \times \vec{B} = \vec{C} \times \vec{B}$ and $\vec{R} \cdot \vec{A} = 0$. **(IIT-JEE, 1990)**

7. Determine the value of c so that for all real x, vectors $cx\hat{i} - 6\hat{j} - 3\hat{k}$ and $x\hat{i} + 2\hat{j} + 2cx\hat{k}$ make an obtuse angle with each other. **(IIT-JEE, 1991)**

8. If vectors $\vec{b}, \vec{c}$ and $\vec{d}$, are not coplanar, then prove that vector $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) + (\vec{a} \times \vec{c}) \times (\vec{d} \times \vec{b}) + (\vec{a} \times \vec{d}) \times (\vec{b} \times \vec{c})$ is parallel to $\vec{a}$. **(IIT-JEE, 1994)**

9. The position vectors of the vertices A, B and C of a tetrahedron $ABCD$ are $\hat{i} + \hat{j} + \hat{k}$, $\hat{i}$ and $3\hat{i}$, respectively. The altitude from vertex D to the opposite face ABC meets the median line through A of triangle ABC at a point E. If the length of the side AD is 4 and the volume of the tetrahedron is $2\sqrt{2}/3$, find the position vectors of the point E for all its possible positions. **(IIT-JEE, 1996)**

10. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be non-coplanar unit vectors, equally inclined to one another at an angle θ. If $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\vec{a} + q\vec{b} + r\vec{c}$, find scalars p, q and r in terms of θ. **(IIT-JEE 1997)**

11. If $\vec{A}, \vec{B}$ and $\vec{C}$ are vectors such that $|\vec{B}| = |\vec{C}|$. Prove that $[(\vec{A} + \vec{B}) \times (\vec{A} + \vec{C})] \times (\vec{B} + \vec{C}) \cdot (\vec{B} + \vec{C}) = 0$. **(IIT-JEE, 1997)**

12. For any two vectors $\vec{u}$ and $\vec{v}$, prove that

a. $(\vec{u}.\vec{v})^2 + |\vec{u}\times\vec{v}|^2 = |\vec{u}|^2|\vec{v}|^2$ and

b. $(\vec{1} + |\vec{u}|^2)(\vec{1} + |\vec{v}|^2) = (1 - \vec{u}.\vec{v})^2 + |\vec{u} + \vec{v} + (\vec{u}\times\vec{v})|^2$ **(IIT-JEE, 1998)**

13. Let $\vec{u}$ and $\vec{v}$ be unit vectors. If $\vec{w}$ is a vector such that $\vec{w} + (\vec{w}\times\vec{u}) = \vec{v}$, then prove that $|(\vec{u}\times\vec{v})\cdot\vec{w}| \le 1/2$ and that the equality holds if and only if $\vec{u}$ is perpendicular to $\vec{v}$. **(IIT-JEE, 1999)**

14. Find three-dimensional vectors $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ satisfying $\vec{v}_1\cdot\vec{v}_1 = 4, \vec{v}_1\cdot\vec{v}_2 = -2, \vec{v}_1\cdot\vec{v}_3 = 6, \vec{v}_2\cdot\vec{v}_2 = 2,$ $\vec{v}_2\cdot\vec{v}_3 = -5, \vec{v}_3\cdot\vec{v}_3 = 29.$ **(IIT-JEE, 2001)**

15. Let V be the volume of the parallelepiped formed by the vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. If a_r, b_r and c_r, where $r = 1, 2, 3$, are non-negative real numbers and $\sum_{r=1}^{3}(a_r + b_r + c_r) = 3L$, show that $V \le L^3$. **(IIT-JEE, 2002)**

16. $\vec{u}, \vec{v}$ and $\vec{w}$ are three non-coplanar unit vectors and α, β and γ are the angles between $\vec{u}$ and $\vec{v}$, $\vec{v}$ and $\vec{w}$, and $\vec{w}$ and $\vec{u}$, respectively, and $\vec{x}, \vec{y}$ and $\vec{z}$ are unit vectors along the bisectors of the angles α, β and γ, respectively. Prove that $[\vec{x}\times\vec{y}\ \ \vec{y}\times\vec{z}\ \ \vec{z}\times\vec{x}] = \frac{1}{16}[\vec{u}\ \vec{v}\ \vec{w}]^2 \sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}.$

(IIT-JEE, 2003)

17. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are distinct vectors such that $\vec{a}\times\vec{c} = \vec{b}\times\vec{d}$ and $\vec{a}\times\vec{b} = \vec{c}\times\vec{d}$, prove that $(\vec{a} - \vec{d})\cdot(\vec{b} - \vec{c}) \ne 0$, i.e., $\vec{a}\cdot\vec{b} + \vec{d}\cdot\vec{c} \ne \vec{d}\cdot\vec{b} + \vec{a}\cdot\vec{c}$. **(IIT-JEE, 2004)**

18. P_1 and P_2 are planes passing through origin. L_1 and L_2 are two lines on P_1 and P_2, respectively, such that their intersection is the origin. Show that there exist points A, B and C, whose permutation A', B' and C', respectively, can be chosen such that (i) A is on L_1, B on P_1 but not on L_1 and C not on P_1 (ii) A' is on L_2, B' on P_2 but not on L_2 and C' not on P_2. **(IIT-JEE, 2004)**

19. If the incident ray on a surface is along the unit vector $\hat{v}$, the reflected ray is along the unit vector $\hat{w}$ and the normal is along the unit vector $\hat{a}$ outwards, express $\hat{w}$ in terms of $\hat{a}$ and $\hat{v}$.

(IIT-JEE, 2005)

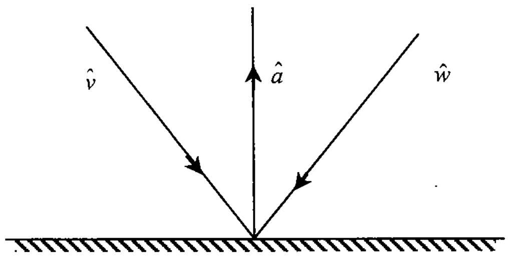


Fig. 2.30

Objective Type

Fill in the blanks

1. Let $\vec{A}, \vec{B}$ and $\vec{C}$ be vectors of length, 3, 4 and 5, respectively. Let $\vec{A}$ be perpendicular to $\vec{B}+\vec{C}$, $\vec{B}$ to $\vec{C}+\vec{A}$ and $\vec{C}$ to $\vec{A}+\vec{B}$. Then the length of vector $\vec{A}+\vec{B}+\vec{C}$ is____________.

(IIT-JEE, 1981)

2. The unit vector perpendicular to the plane determined by P (1, – 1, 2), Q (2, 0, – 1) and R (0, 2, 1) is____________.

(IIT-JEE, 1983)

3. The area of the triangle whose vertices are A (1, – 1, 2), B (2, 1, – 1), C(3, – 1, 2) is____________.

(IIT-JEE, 1983)

4. If $\vec{A}, \vec{B}$ and $\vec{C}$ are the three non-coplanar vectors, then $\dfrac{\vec{A}\cdot\vec{B}\times\vec{C}}{\vec{C}\times\vec{A}\cdot\vec{B}}+\dfrac{\vec{B}\cdot\vec{A}\times\vec{C}}{\vec{C}\cdot\vec{A}\times\vec{B}}=$____________.

(IIT-JEE, 1985)

5. If $\vec{A}=(1, 1, 1)$ and $\vec{C}=(0, 1, -1)$ are given vectors, then vector $\vec{B}$ satisfying the equations $\vec{A}\times\vec{B}=\vec{C}$ and $\vec{A}.\vec{B}=3$ is____________.

(IIT-JEE, 1985)

6. Let $\vec{b}=4\hat{i}+3\hat{j}$ and $\vec{c}$ be two vectors perpendicular to each other in the xy-plane. All vectors in the same plane having projections 1 and 2 along $\vec{b}$ and $\vec{c}$, respectively, are given by____________.

(IIT-JEE, 1987)

7. The components of a vector $\vec{a}$ along and perpendicular to a non-zero vector $\vec{b}$ are____________ and____________, respectively.

(IIT-JEE, 1988)

8. A unit vector coplanar with $\vec{i}+\vec{j}+2\vec{k}$ and $\vec{i}+2\vec{j}+\vec{k}$ and perpendicular to $\vec{i}+\vec{j}+\vec{k}$ is____________.

(IIT-JEE, 1992)

9. A non-zero vector $\vec{a}$ is parallel to the line of intersection of the plane determined by vectors $\hat{i}$ and $\hat{i}+\hat{j}$ and the plane determined by vectors $\hat{i}-\hat{j}$ and $\hat{i}+\hat{k}$. The angle between $\vec{a}$ and vector $\hat{i}-2\hat{j}+2\hat{k}$ is____________.

(IIT-JEE, 1996)

10. If $\vec{b}$ and $\vec{c}$ are mutually perpendicular unit vectors and $\vec{a}$ is any vector, then $(\vec{a}\cdot\vec{b})\,\vec{b}+(\vec{a}\cdot\vec{c})\,\vec{c}+\dfrac{\vec{a}\cdot(\vec{b}\times\vec{c})}{|\vec{b}\times\vec{c}|}\,(\vec{b}\times\vec{c})=$____________.

(IIT-JEE, 1996)

11. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors having magnitudes 1, 1 and 2, respectively. If $\vec{a}\times(\vec{a}\times\vec{c})+\vec{b}=\vec{0}$, then the acute angle between $\vec{a}$ and $\vec{c}$ is____________.

(IIT-JEE, 1997)

12. A, B, C and D are four points in a plane with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, respectively, such that $(\vec{a}-\vec{d})\cdot(\vec{b}-\vec{c})=(\vec{b}-\vec{d})\cdot(\vec{c}-\vec{a})=0$. Then point D is the____________of triangle ABC.

(IIT-JEE, 1984)

13. Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = 10\vec{a} + 2\vec{b}$ and $\overrightarrow{OC} = \vec{b}$, where O, A and C are non-collinear points. Let p denote the area of the quadrilateral $OABC$, and let q denote the area of the parallelogram with OA and OC as adjacent sides. If $p = kq$, then $k=$ ____________. **(IIT-JEE, 1997)**

14. If $\vec{a} = \hat{j} + \sqrt{3}\hat{k}$, $\vec{b} = -\hat{j} + \sqrt{3}\hat{k}$ and $\vec{c} = 2\sqrt{3}\hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is ____________. **(IIT-JEE, 2011)**

True or false

1. Let $\vec{A}, \vec{B}$ and $\vec{C}$ be unit vectors such that $\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{C} = 0$ and the angle between $\vec{B}$ and $\vec{C}$ is $\pi/3$. Then $\vec{A} = \pm 2(\vec{B}\times\vec{C})$. **(IIT-JEE, 1981)**

2. If $\vec{X}\cdot\vec{A} = 0$, $\vec{X}\cdot\vec{B} = 0$ and $\vec{X}\cdot\vec{C} = 0$ for some non-zero vector $\vec{X}$, then $[\vec{A}\,\vec{B}\,\vec{C}] = 0$.

(IIT-JEE, 1983)

3. For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$, $(\vec{a}-\vec{b})\cdot(\vec{b}-\vec{c})\times(\vec{c}-\vec{a}) = 2\vec{a}\cdot\vec{b}\times\vec{c}$. **(IIT-JEE, 1989)**

Multiple choice questions with one correct answer

1. The scalar $\vec{A}\cdot(\vec{B}+\vec{C})\times(\vec{A}+\vec{B}+\vec{C})$ equals

   a. 0 **b.** $[\vec{A}\,\vec{B}\,\vec{C}]+[\vec{B}\,\vec{C}\,\vec{A}]$ **c.** $[\vec{A}\,\vec{B}\,\vec{C}]$ **d.** none of these

   (IIT-JEE, 1981)

2. For non-zero vectors $\vec{a}, \vec{b}$ and $\vec{c}$, $|(\vec{a}\times\vec{b})\cdot\vec{c}| = |\vec{a}||\vec{b}||\vec{c}|$ holds if and only if

   a. $\vec{a}\cdot\vec{b} = 0, \vec{b}\cdot\vec{c} = 0$ **b.** $\vec{b}.\vec{c} = 0, \vec{c}\cdot\vec{a} = 0$

   c. $\vec{c}\cdot\vec{a} = 0, \vec{a}\cdot\vec{b} = 0$ **d.** $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0$ **(IIT-JEE, 1982)**

3. The volume of the parallelopiped whose sides are given by $\overrightarrow{OA} = 2i - 2j$, $\overrightarrow{OB} = i + j - k$ and $\overrightarrow{OC} = 3i - k$ is

   a. 4/13 **b.** 4 **c.** 2/7 **d.** 2

   (IIT-JEE, 1983)

4. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-coplanar vectors and $\vec{p}, \vec{q}$ and $\vec{r}$ the vectors defined by the relations $\vec{p} = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $\vec{q} = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ and $\vec{r} = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$. Then the value of the expression $(\vec{a}+\vec{b})\cdot\vec{p} + (\vec{b}+\vec{c})\cdot\vec{q} + (\vec{c}+\vec{a})\cdot\vec{r}$ is

   a. 0 **b.** 1 **c.** 2 **d.** 3

   (IIT-JEE, 1988)

5. Let $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = \hat{j} - \hat{k}$ and $\vec{c} = \hat{k} - \hat{i}$. If $\vec{d}$ is a unit vector such that $\vec{a}\cdot\vec{d} = 0 = [\vec{b}\,\vec{c}\,\vec{d}]$, then $\vec{d}$ equals

   a. $\pm\dfrac{\hat{i}+\hat{j}-2\hat{k}}{\sqrt{6}}$ **b.** $\pm\dfrac{\hat{i}+\hat{j}-\hat{k}}{\sqrt{3}}$ **c.** $\pm\dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ **d.** $\pm\hat{k}$

   (IIT-JEE, 1995)

6. If $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar unit vectors such that $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\vec{b} + \vec{c}}{\sqrt{2}}$, then the angle between $\vec{a}$ and $\vec{b}$ is

a. $3\pi/4$ b. $\pi/4$ c. $\pi/2$ d. π

(IIT-JEE, 1995)

7. Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vectors such that $\vec{u} + \vec{v} + \vec{w} = 0$. If $|\vec{u}| = 3$, $|\vec{v}| = 4$ and $|\vec{w}| = 5$, then $\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{u}$ is

a. 47 b. -25 c. 0 d. 25

(IIT-JEE, 1995)

8. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-coplanar vectors, then $(\vec{a} + \vec{b} + \vec{c}) \cdot [(\vec{a} + \vec{b}) \times (\vec{a} + \vec{c})]$ equals

a. 0 b. $[\vec{a}\ \vec{b}\ \vec{c}]$ c. $2\,[\vec{a}\ \vec{b}\ \vec{c}]$ d. $-[\vec{a}\ \vec{b}\ \vec{c}]$

(IIT-JEE, 1995)

9. $\vec{p}, \vec{q}$ and $\vec{r}$ are three mutually perpendicular vectors of the same magnitude. If vector $\vec{x}$ satisfies the equation $\vec{p} \times ((\vec{x} - \vec{q}) \times \vec{p}) + \vec{q} \times ((\vec{x} - \vec{r}) \times \vec{q}) + \vec{r} \times ((\vec{x} - \vec{p}) \times \vec{r}) = \vec{0}$, then $\vec{x}$ is given by

a. $\frac{1}{2}(\vec{p} + \vec{q} - 2\vec{r})$ b. $\frac{1}{2}(\vec{p} + \vec{q} + \vec{r})$ c. $\frac{1}{3}(\vec{p} + \vec{q} + \vec{r})$ d. $\frac{1}{3}(2\vec{p} + \vec{q} - \vec{r})$

(IIT-JEE, 1997)

10. Let $\vec{a} = 2i + j - 2k$ and $b = i + j$. If c is a vector such that $\vec{a} \cdot \vec{c} = |\vec{c}|$, $|\vec{c} - \vec{a}| = 2\sqrt{2}$ and the angle between $\vec{a} \times \vec{b}$ and $\vec{c}$ is 30°, then $|(\vec{a} \times \vec{b}) \times \vec{c}|$ is equal to

a. 2/3 b. 3/2 c. 2 d. 3

(IIT-JEE, 1999)

11. Let $\vec{a} = 2i + j + k$, $\vec{b} = i + 2j - k$ and a unit vector $\vec{c}$ be coplanar. If $\vec{c}$ is perpendicular to $\vec{a}$, then $\vec{c}$ is

a. $\frac{1}{\sqrt{2}}(-j + k)$ b. $\frac{1}{\sqrt{3}}(-i - j - k)$ c. $\frac{1}{\sqrt{5}}(i - 2j)$ d. $\frac{1}{\sqrt{3}}(i - j - k)$

12. If the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ form the sides BC, CA and AB, respectively, of triangle ABC, then

a. $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = 0$ b. $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

c. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a}$ d. $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}$

(IIT-JEE, 2000)

13. Let vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be such that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{0}$. Let P_1 and P_2 be planes determined by the pairs of vectors $\vec{a}, \vec{b}$ and $\vec{c}, \vec{d}$, respectively. Then the angle between P_1 and P_2 is

a. 0 b. $\pi/4$ c. $\pi/3$ d. $\pi/2$

(IIT-JEE, 2000)

14. If $\vec{a}, \vec{b}$ and $\vec{c}$ are unit coplanar vectors, then the scalar triple product $[2\vec{a} - \vec{b}\ \ 2\vec{b} - \vec{c}\ \ 2\vec{c} - \vec{a}]$ is

a. 0 b. 1 c. $-\sqrt{3}$ d. $\sqrt{3}$

(IIT-JEE, 2000)

15. If $\hat{a}, \hat{b}$ and $\hat{c}$ are unit vectors, then $|\hat{a}-\hat{b}|^2+|\hat{b}-\hat{c}|^2+|\hat{c}-\hat{a}|^2$ does not exceed

a. 4 **b.** 9 **c.** 8 **d.** 6

(IIT-JEE, 2001)

16. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a}+2\vec{b}$ and $5\vec{a}-4\vec{b}$ are perpendicular to each other, then the angle between $\vec{a}$ and $\vec{b}$ is

a. 45° **b.** 60° **c.** $\cos^{-1}(1/3)$ **d.** $\cos^{-1}(2/7)$

(IIT-JEE, 2002)

17. Let $\vec{V}=2\hat{i}+\hat{j}-\hat{k}$ and $\vec{W}=\hat{i}+3\hat{k}$. If $\vec{U}$ is a unit vector, then the maximum value of the scalar triple product $[\vec{U}\,\vec{V}\,\vec{W}]$ is

a. -1 **b.** $\sqrt{10}+\sqrt{6}$ **c.** $\sqrt{59}$ **d.** $\sqrt{60}$

(IIT-JEE, 2002)

18. The value of a so that the volume of parallelopiped formed by $\hat{i}+a\hat{j}+\hat{k}, \hat{j}+a\hat{k}$ and $a\hat{i}+\hat{k}$ is minimum is

a. -3 **b.** 3 **c.** $1/\sqrt{3}$ **d.** $\sqrt{3}$

(IIT-JEE, 2003)

19. If $\vec{a}=(\hat{i}+\hat{j}+\hat{k}), \vec{a}\cdot\vec{b}=1$ and $\vec{a}\times\vec{b}=\hat{j}-\hat{k}$, then $\vec{b}$ is

a. $\hat{i}-\hat{j}+\hat{k}$ **b.** $2\hat{j}-\hat{k}$ **c.** $\hat{i}$ **d.** $2\hat{i}$

(IIT-JEE, 2004)

20. The unit vector which is orthogonal to the vector $5\hat{j}+2\hat{j}+6\hat{k}$ and is coplanar with vectors $2\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}-\hat{j}+\hat{k}$ is

a. $\dfrac{2\hat{i}-6\hat{j}+\hat{k}}{\sqrt{41}}$ **b.** $\dfrac{2\hat{i}-3\hat{j}}{\sqrt{13}}$ **c.** $\dfrac{3\hat{i}-\hat{k}}{\sqrt{10}}$ **d.** $\dfrac{4\hat{i}+3\hat{j}-3\hat{k}}{\sqrt{34}}$

(IIT-JEE, 2004)

21. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-zero, non-coplanar vectors and $\vec{b}_1=\vec{b}-\dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}, \vec{b}_2=\vec{b}+\dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}$,

$$\vec{c}_1=\vec{c}-\frac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}+\frac{\vec{b}\cdot\vec{c}}{|\vec{c}|^2}\vec{b}_1,\quad \vec{c}_2=\vec{c}-\frac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}-\frac{\vec{b}\cdot\vec{c}}{|\vec{b}_1|^2}\vec{b}_1,\quad \vec{c}_3=\vec{c}-\frac{\vec{c}\cdot\vec{a}}{|\vec{c}|^2}\vec{a}+\frac{\vec{b}\cdot\vec{c}}{|\vec{c}|^2}\vec{b}_1,$$

$\vec{c}_4=\vec{c}-\dfrac{\vec{c}\cdot\vec{a}}{|\vec{c}|^2}\vec{a}=\dfrac{\vec{b}\cdot\vec{c}}{|\vec{b}|^2}\vec{b}_1$, then the set of orthogonal vectors is

a. $(\vec{a}, \vec{b}_1, \vec{c}_3)$ **b.** $(\vec{a}, \vec{b}_1, \vec{c}_2)$ **c.** $(\vec{a}, \vec{b}_1, \vec{c}_1)$ **d.** $(\vec{a}, \vec{b}_2, \vec{c}_2)$

(IIT-JEE, 2005)

22. Let $\vec{a}=\hat{i}+2\hat{j}+\hat{k}, \vec{b}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}-\hat{k}$. A vector in the plane of $\vec{a}$ and $\vec{b}$ whose projection on $\vec{c}$ is $1/\sqrt{3}$, is

a. $4\hat{i}-\hat{j}+4\hat{k}$ b. $3\hat{i}+\hat{j}-3\hat{k}$ c. $2\hat{i}+\hat{j}-2\hat{k}$ d. $4\hat{i}+\hat{j}-4\hat{k}$

(IIT-JEE, 2006)

23. Let two non-collinear unit vectors $\hat{a}$ and $\hat{b}$ form an acute angle. A point P moves so that at any time t, the position vector $\overrightarrow{OP}$ (where O is the origin) is given by $\hat{a}\cot t+\hat{b}\sin t$. When P is farthest from origin O, let M be the length of $\overrightarrow{OP}$ and $\hat{u}$ be the unit vector along $\overrightarrow{OP}$. Then

a. $\hat{u}=\dfrac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$ and $M=(1+\hat{a}\cdot\hat{b})^{1/2}$

b. $\hat{u}=\dfrac{\hat{a}-\hat{b}}{|\hat{a}-\hat{b}|}$ and $M=(1+\hat{a}\cdot\hat{b})^{1/2}$

c. $\hat{u}=\dfrac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$ and $M=(1+2\hat{a}\cdot\hat{b})^{1/2}$

d. $\hat{u}=\dfrac{\hat{a}-\hat{b}}{|\hat{a}-\hat{b}|}$ and $M=(1+2\hat{a}\cdot\hat{b})^{1/2}$ **(IIT-JEE, 2008)**

24. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are unit vectors such that $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=1$ and $\vec{a}\cdot\vec{c}=\frac{1}{2}$, then

a. $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar

b. $\vec{b}, \vec{c}$ and $\vec{d}$ are non-coplanar

c. $\vec{b}$ and $\vec{d}$ are non-parallel

d. $\vec{a}$ and $\vec{d}$ are parallel and $\vec{b}$ and $\vec{c}$ are parallel **(IIT-JEE, 2009)**

25. Two adjacent sides of a parallelogram $ABCD$ are given by $\overrightarrow{AB}=2\hat{i}+10\hat{j}+11\hat{k}$ and $\overrightarrow{AD}=-\hat{i}+2\hat{j}+2\hat{k}$. The side AD is rotated by an acute angle α in the plane of the parallelogram so that AD becomes AD'. If AD' makes a right angle with the side AB, then the cosine of the angle α is given by

a. $\dfrac{8}{9}$ b. $\dfrac{\sqrt{17}}{9}$ c. $\dfrac{1}{9}$ d. $\dfrac{4\sqrt{5}}{9}$

(IIT-JEE, 2010)

26. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{i}-\hat{j}, 4\hat{i}, 3\hat{i}+3\hat{j}$ and $-3\hat{j}+2\hat{j}$ respectively. The quadrilateral $PQRS$ must be a
 a. parallelogram, which is neither a rhombus nor a rectangle
 b. square
 c. rectangle, but not a square
 d. rhombus, but not a square **(IIT-JEE, 2010)**

27. Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}-\hat{j}-\hat{k}$ be three vectors. A vector $\vec{v}$ in the plane of $\vec{a}$ and $\vec{b}$, whose projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$, is given by
 a. $\hat{i}-3\hat{j}+3\hat{k}$ b. $-3\hat{i}-3\hat{j}+\hat{k}$ c. $3\hat{i}-\hat{j}+3\hat{k}$ d. $\hat{i}+3\hat{j}-3\hat{k}$

(IIT-JEE, 2011)

Multiple choice questions with one or more than one correct answer

1. Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}, \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both vectors $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\pi/6$, then $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$ is equal to
 a. 0
 b. 1
 c. $\frac{1}{4}(a_1^2+a_2^2+a_2^2)(b_1^2+b_2^2+b_3^2)$
 d. $\frac{3}{4}(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)(c_1^2+c_2^2+c_3^2)$ **(IIT-JEE, 1986)**

2. The number of vectors of unit length perpendicular to vectors $\vec{a}=(1, 1, 0)$ and $\vec{b}=(0, 1, 1)$ is
 a. one b. two c. three d. infinite

(IIT-JEE, 1987)

3. Let $\vec{a}=2\hat{i}-\hat{j}+\hat{k}, \vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}-2\hat{k}$ be three vectors. A vector in the plane of $\vec{b}$ and $\vec{c}$, whose projection on $\vec{a}$ is of magnitude $\sqrt{2/3}$, is
 a. $2\hat{i}+3\hat{j}-3\hat{k}$ b. $2\hat{i}+3\hat{j}+3\hat{k}$ c. $-2\hat{i}-\hat{j}+5\hat{k}$ d. $2\hat{i}+\hat{j}+5\hat{k}$

(IIT-JEE, 1993)

4. For three vectors $\vec{u}, \vec{v}$ and $\vec{w}$ which of the following expressions is not equal to any of the remaining three?
 a. $\vec{u}\cdot(\vec{v}\times\vec{w})$ b. $(\vec{v}\times\vec{w})\cdot\vec{u}$
 c. $\vec{v}\cdot(\vec{u}\times\vec{w})$ d. $(\vec{u}\times\vec{v})\cdot\vec{w}$ **(IIT-JEE, 1998)**

5. Which of the following expressions are meaningful?

a. $\vec{u}\cdot(\vec{v}\times\vec{w})$ b. $(\vec{u}\cdot\vec{v})\cdot\vec{w}$

c. $(\vec{u}\cdot\vec{v})\,\vec{w}$ d. $\vec{u}\times(\vec{v}\cdot\vec{w})$ **(IIT-JEE, 1998)**

6. Let $\vec{a}$ and $\vec{b}$ be two non-collinear unit vectors. If $\vec{u} = \vec{a}-(\vec{a}\cdot\vec{b})\,\vec{b}$ and $\vec{v}=\vec{a}\times\vec{b}$, then $|\vec{v}|$ is

a. $|\vec{u}|$ b. $|\vec{u}|+|\vec{u}\cdot\vec{a}|$ c. $|\vec{u}|+|\vec{u}\cdot\vec{b}|$ d. $|\vec{u}|+\vec{u}\cdot(\vec{a}+\vec{b})$

(IIT-JEE, 1999)

7. Vector $\frac{1}{3}(2\hat{i}-2\hat{j}+\hat{k})$ is

a. a unit vector

b. makes an angle $\pi/3$ with vector $(2\hat{i}-4\hat{j}+3\hat{k})$

c. parallel to vector $\left(-\hat{i}+\hat{j}-\frac{1}{2}\hat{k}\right)$

d. perpendicular to vector $3\hat{i}+2\hat{j}-2\hat{k}$ **(IIT-JEE, 1994)**

8. Let $\vec{A}$ be a vector parallel to the line of intersection of planes P_1 and P_2. Plane P_1 is parallel to vectors $2\hat{j}+3\hat{k}$ and $4\hat{j}-3\hat{k}$ and P_2 is parallel to $\hat{j}-\hat{k}$ and $3\hat{i}+3\hat{j}$. Then the angle between vector $\vec{A}$ and a given vector $2\hat{i}+\hat{j}-2\hat{k}$ is

a. $\pi/2$ b. $\pi/4$ c. $\pi/6$ d. $3\pi/4$

(IIT-JEE, 2006)

9. The vector(s) which is/are coplanar with vectors $\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$, and perpendicular to vector $\hat{i}+\hat{j}+\hat{k}$ is/are

a. $\hat{j}-\hat{k}$ b. $-\hat{i}+\hat{j}$ c. $\hat{i}-\hat{j}$ d. $-\hat{j}+\hat{k}$

(IIT-JEE, 2011)

Integer Answer Type

1. If $\vec{a}$ and $\vec{b}$ are vectors in space given by $\vec{a}=\frac{\hat{i}-2\hat{j}}{\sqrt{5}}$ and $\vec{b}=\frac{2\hat{i}+\hat{j}+3\hat{k}}{\sqrt{14}}$, then find the value of $(2\vec{a}+\vec{b})\cdot[(\vec{a}\times\vec{b})\times(\vec{a}-2\vec{b})]$. **(IIT-JEE, 2010)**

2. Let $\vec{a}=-\hat{i}-\hat{k}, \vec{b}=-\hat{i}+\hat{j}$ and $\vec{c}=\hat{i}+2\hat{j}+3\hat{k}$ be three given vectors. If $\vec{r}$ is a vector such that $\vec{r}\times\vec{b}=\vec{c}\times\vec{d}$ and $\vec{r}\cdot\vec{a}=0$, then find the value of $\vec{r}\cdot\vec{b}$. **(IIT-JEE, 2011)**

ANSWERS AND SOLUTIONS

Subjective Type

1. $D = D_1 D_2$ (see determinants)

$$= 2\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$

Since $\vec{A}$, $\vec{B}$ and $\vec{C}$ are non-coplanar, $D_1 \neq 0$,

$$D_2 = 0 \text{ or } \begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} = 0$$

or $\vec{X}, \vec{Y}$ and $\vec{Z}$ are coplanar.

2.

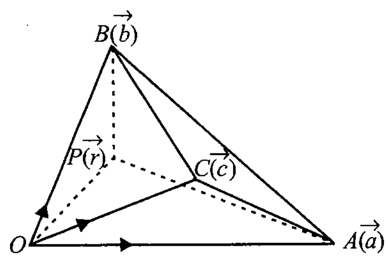


Fig. 2.31

If the centre P is with position vector $\vec{r}$, then

$\vec{a} - \vec{r} = \overrightarrow{PA}$, $\vec{b} - \vec{r} = \overrightarrow{PB}$, $\vec{c} - \vec{r} = \overrightarrow{PC}$,

where $|\overrightarrow{PA}| = |\overrightarrow{PB}| = |\overrightarrow{PC}| = |\overrightarrow{OP}| = |\vec{r}|$

Consider $|\vec{a} - \vec{r}| = |\vec{r}|$

$\Rightarrow (\vec{a} - \vec{r})\cdot(\vec{a} - \vec{r}) = \vec{r}\cdot\vec{r}$

$\Rightarrow a^2 - 2\vec{a}\cdot\vec{r} + r^2 = r^2 \Rightarrow a^2 = 2\vec{a}\cdot\vec{r}$

Similarly, $b^2 = 2\vec{b}\cdot\vec{r}$, $c^2 = 2\vec{c}\cdot\vec{r}$

Since $(\vec{b}\times\vec{c})$, $(\vec{c}\times\vec{a})$ and $(\vec{a}\times\vec{b})$ are non coplanar, then $\vec{r} = x(\vec{b}\times\vec{c}) + y(\vec{c}\times\vec{a}) + z(\vec{a}\times\vec{b})$

$$\Rightarrow \vec{a}\cdot\vec{r} = x\vec{a}\cdot(\vec{b}\times\vec{c}) + y\cdot 0 + z\cdot 0 = x[\vec{a}\,\vec{b}\,\vec{c}] \Rightarrow x = \frac{\vec{a}\cdot\vec{r}}{[\vec{a}\,\vec{b}\,\vec{c}]} = \frac{a^2}{2[\vec{a}\,\vec{b}\,\vec{c}]}$$

Similarly, $y = \dfrac{b^2}{2[\vec{a}\,\vec{b}\,\vec{c}]}$ and $z = \dfrac{c^2}{2[\vec{a}\,\vec{b}\,\vec{c}]}$

Hence $\vec{r} = \dfrac{a^2(\vec{b}\times\vec{c}) + b^2(\vec{c}\times\vec{a}) + c^2(\vec{a}\times\vec{b})}{2[\vec{a}\,\vec{b}\,\vec{c}]}$

3.

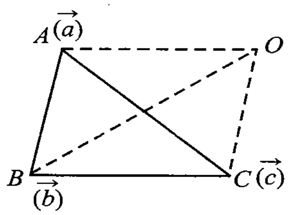


Fig. 2.32

Let O be the origin of reference and A, B, C vertices with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, respectively. A vector normal to plane ABC is $\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}$ and $\overrightarrow{OA} = \vec{a}$.

The angle between a line and a plane is equal to the complement of the angle between the line and the normal to the plane. Thus, if θ denotes the angle between the face and edge, then

$$\sin\theta = \frac{(\vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b})\cdot\vec{a}}{|\vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b}|\cdot|\vec{a}|} = \frac{[\vec{a}\,\vec{b}\,\vec{c}]}{|\vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b}|\cdot|\vec{a}|}$$

Now $[\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix} = k^6 \begin{vmatrix} 1 & \cos 60° & \cos 60° \\ \cos 60° & 1 & \cos 60° \\ \cos 60° & \cos 60° & 1 \end{vmatrix}$, (where k is the length of the side of the tetrahectron)

$$= k^6\left(\frac{3}{4} - \frac{1}{8} - \frac{1}{8}\right) = \frac{1}{2}k^6$$

Also, $(\vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b})$ is twice the area of triangle ABC, which is equilateral with each side k so that this is $\dfrac{\sqrt{3}}{2}k^2$.

Hence $\sin\theta = \dfrac{\dfrac{k^3}{\sqrt{2}}}{\dfrac{\sqrt{3}}{2}k^2\cdot k} = \dfrac{2}{\sqrt{6}} \Rightarrow \cos\theta = \dfrac{1}{\sqrt{3}}$.

4.

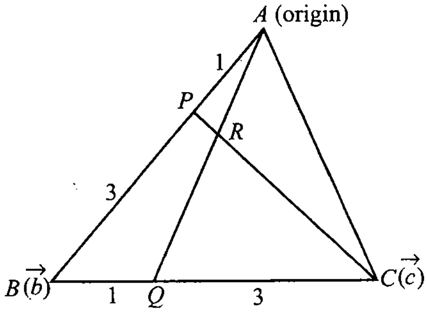


Fig. 2.33

Taking A as origin, let $\vec{b}$ and $\vec{c}$ be the position vectors of B and C, respectively.

The position vector of Q is $\frac{3\vec{b}+\vec{c}}{4}$ and that of P is $\frac{\vec{b}}{4}$.

If $\frac{AR}{QR}=\frac{\lambda}{1}$, then position vector of $R=\lambda\left(\frac{3\vec{b}+\vec{c}}{4}\right)$ (i)

If $\frac{CR}{RP}=\frac{\mu}{1}$, then position vector of $R=\frac{\mu\frac{\vec{b}}{4}+\vec{c}}{\mu+1}$ (ii)

Comparing (i) and (ii), we have

$$\frac{3\lambda}{4}=\frac{\mu}{4(\mu+1)} \text{ and } \frac{\lambda}{4}=\frac{1}{\mu+1}$$

Solving, $\lambda=\frac{4}{13}$ and $\mu=12$

Therefore, position vector R is $\frac{3\vec{b}+\vec{c}}{13}$.

ΔABC and ΔBRC have the same base. Therefore, areas are proportional to AQ and RQ.

$$\frac{\Delta ABC}{\Delta BRC}=\frac{\left|\frac{3\vec{b}+\vec{c}}{4}\right|}{\left|\frac{3\vec{b}+\vec{c}}{4}-\left(\frac{3\vec{b}+\vec{c}}{13}\right)\right|}=\frac{13}{9}$$

Area of $\Delta\, ABC$ is 13/9 units.

5. $$\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta AOC}=\frac{\frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}{\frac{1}{2}|\vec{a}\times\vec{c}|}$$

Now $\quad \vec{a}+2\vec{b}+3\vec{c}=\vec{0}$

Cross multiply with $\vec{b}$, $\vec{a}\times\vec{b}+3\vec{c}\times\vec{b}=\vec{0}$

$\Rightarrow \vec{a}\times\vec{b}=3(\vec{b}\times\vec{c})$

Cross multiply with $\vec{a}$, $2\vec{a}\times\vec{b}+3\vec{a}\times\vec{c}=0$

$\Rightarrow \vec{a}\times\vec{b}=\frac{3}{2}(\vec{c}\times\vec{a})$

$\therefore\ \vec{a}\times\vec{b}=\frac{3}{2}(\vec{c}\times\vec{a})=3(\vec{b}\times\vec{c})$

Let $(\vec{c}\times\vec{a})=\vec{p}$

$\vec{a}\times\vec{b}=\frac{3\vec{p}}{2};\ \vec{b}\times\vec{c}=\frac{\vec{p}}{2}$

$\therefore$ Ratio $=\frac{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}{|\vec{c}\times\vec{a}|}$

$=\frac{\left|\frac{3\vec{p}}{2}+\frac{\vec{p}}{2}+\vec{p}\right|}{|\vec{p}|}$

$=\frac{3|\vec{p}|}{|\vec{p}|}=3$

6. In tetrahedron $OABC$, take O as the initial point and let the position vectors of A, B and C be $\vec{a}$, $\vec{k}$ and $\vec{c}$, respectively; then volume of the tetrahedron is equal to $\frac{1}{6}\vec{a}\cdot(\vec{k}\times\vec{c})$.

Also $\overrightarrow{BC}=\vec{c}-\vec{k}$ so that volume of tetrahedron

$V=\frac{1}{6}\vec{a}\cdot(\vec{k}\times(\vec{k}+\overrightarrow{BC}))=\frac{1}{6}\vec{a}\cdot(\vec{k}\times\overrightarrow{BC})=\frac{1}{6}\vec{k}\cdot(\overrightarrow{BC}\times\vec{a})$

$=\frac{1}{6}\vec{k}\cdot|BC|\,|a|\sin\theta\,\hat{n}$, where $\hat{n}$ is the unit vector along PN, the line perpendicular to both OA and BC.

Also $|BC|=b$.

Here $V=\frac{1}{6}ab\sin\theta\,\vec{k}\cdot\hat{n}=\frac{1}{6}ab\sin\theta$ (projection of OB on PN)

$\frac{1}{6}ab\sin\theta=$ (perpendicular distance between OA and BC) $=\frac{1}{6}ab\sin\theta.d=\frac{1}{6}abd\sin\theta$

7. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors of magnitude $|\vec{a}|$ and equal inclination θ with each other.

The volume of parallelepiped $=\vec{a}\cdot(\vec{b}\times\vec{c})=[\vec{a}\,\vec{b}\,\vec{c}]$

and $[\vec{a}\,\vec{b}\,\vec{c}]^2=\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\\ \vec{a}\cdot\vec{c} & \vec{b}\cdot\vec{c} & \vec{c}\cdot\vec{c}\end{vmatrix}$

$$= |\vec{a}|^6 \begin{vmatrix} 1 & \cos\theta & \cos\theta \\ \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1 \end{vmatrix}$$

$$= |\vec{a}|^6 (2\cos^3\theta - 3\cos^2\theta + 1)$$

$$= |\vec{a}|^6 (1-\cos\theta)^2(1+2\cos\theta)$$

$$\Rightarrow \quad [\vec{a}\,\vec{b}\,\vec{c}] = |\vec{a}|^3 \sqrt{1+2\cos\theta}(1-\cos\theta)$$

8. $\vec{p}$, $\vec{q}$ and $\vec{p}\times\vec{q}$ are perpendicular to each other. We have,

$(\vec{a}\cdot\vec{p})\,\vec{p} + (\vec{a}\cdot\vec{q})\,\vec{q} + (\vec{a}\cdot(\vec{p}\times\vec{q}))\,(\vec{p}\times\vec{q}) = \vec{a}\,|\vec{p}|^2$,

$(\vec{b}\cdot\vec{p})\,\vec{p} + (\vec{b}\cdot\vec{q})\,\vec{q} + (\vec{b}\cdot(\vec{p}\times\vec{q}))\,(\vec{p}\times\vec{q}) = \vec{b}\,|\vec{p}|^2$,

$(\vec{c}\cdot\vec{p})\,\vec{p} + (\vec{c}\cdot\vec{q})\,\vec{q} + (\vec{c}\cdot(\vec{p}\times\vec{q}))\,(\vec{p}\times\vec{q}) = \vec{c}\,|\vec{p}|^2$

Hence, the required distance is $|(\vec{a}+\vec{b}+\vec{c})|\,|\vec{p}|^2$.

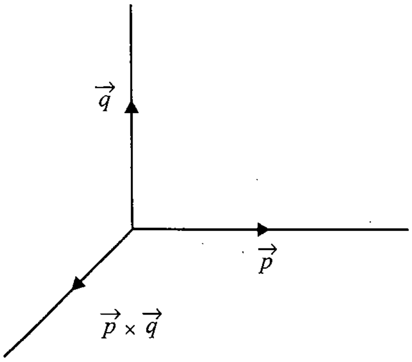


Fig. 2.34

$$= \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2} \times |\vec{p}|^2$$
$$= 14 \times 4^2 = 224$$

9. Here $\vec{A}$, $\vec{B}$ and $\vec{C}$ are the vectors representing the sides of triangle ABC, where $\vec{A} = a\hat{i} + b\hat{j} + c\hat{k}$,

$\vec{B} = d\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{C} = 3\hat{i} + \hat{j} - 2\hat{k}$.

Given that $\vec{A} = \vec{B} + \vec{C}$. Therefore

$$a\hat{i} + b\hat{j} + c\hat{k} = (d+3)\,\hat{i} + 4\hat{j} + 2\hat{k}$$

$$\Rightarrow \quad a = d+3, b = 4, c = 2$$

Now $$\vec{B}\times\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d & 3 & 4 \\ 3 & 1 & -2 \end{vmatrix}$$

$= -10\hat{i} + (2d+12)\hat{j} + (d-9)\hat{k}$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}|\vec{B} \times \vec{C}|$

$= \frac{1}{2}\sqrt{[100 + (2d+12)^2 + (d-9)^2]}$

$= 5\sqrt{6}$ (Given)

$\Rightarrow \sqrt{(5d^2 + 30d + 325)} = 10\sqrt{6}$

$\Rightarrow 5d^2 + 30d - 275 = 0 \Rightarrow d^2 + 6d - 55 = 0$

$\Rightarrow (d+11)(d-5) = 0$

$\Rightarrow d = 5$ or -11

When $d = 5$, $a = 8$, $b = 4$ and $c = 2$, and when $d = -11$, $a = -8$, $b = 4$ and $c = 2$.

10.

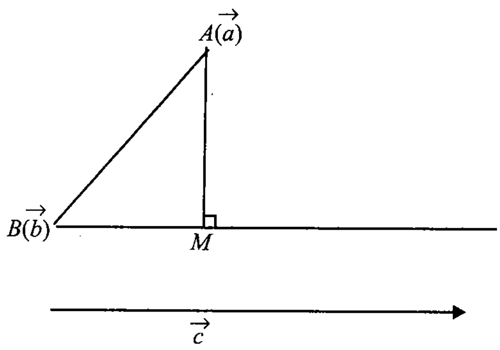


Fig. 2.35

$AM = |\overrightarrow{AB}\sin\theta|$, where θ is the angle between $\overrightarrow{AB}$ and $\vec{c}$

and $\sin\theta = \frac{|\overrightarrow{AB} \times \vec{c}|}{|\overrightarrow{AB}||\vec{c}|}$

$\Rightarrow AM = |\overrightarrow{AB}|\frac{|\overrightarrow{AB} \times \vec{c}|}{|\overrightarrow{AB}||\vec{c}|} = \frac{|(\vec{b}-\vec{a}) \times \vec{c}|}{|\vec{c}|}$

Also $\overrightarrow{BM} = \frac{(\vec{a}-\vec{b})\cdot\vec{c}}{|\vec{c}|}\frac{\vec{c}}{|\vec{c}|}$

And $\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM}$

$\Rightarrow |\overrightarrow{AM}| = \left|\vec{b} - \vec{a} + \frac{(\vec{a}-\vec{b})\cdot\vec{c}}{|\vec{c}|^2}\vec{c}\right|$

11. We know that $[\vec{e}_1\ \vec{e}_2\ \vec{e}_3][\vec{E}_1\ \vec{E}_2\ \vec{E}_3] = \begin{vmatrix} \vec{e}_1\cdot\vec{E}_1 & \vec{e}_1\cdot\vec{E}_2 & \vec{e}_1\cdot\vec{E}_3 \\ \vec{e}_2\cdot\vec{E}_1 & \vec{e}_2\cdot\vec{E}_2 & \vec{e}_2\cdot\vec{E}_2 \\ \vec{e}_3\cdot\vec{E}_1 & \vec{e}_3\cdot\vec{E}_2 & \vec{e}_3\cdot\vec{E}_3 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1$$

Objective Type

1. **c.** If $\vec{x} = \vec{y} \Rightarrow \hat{a}\cdot\vec{x} = \hat{a}\cdot\vec{y}$. This equality must hold for any arbitrary $\hat{a}$

2. **d.** $\vec{a}\times(\vec{a}\times\vec{b}) = \vec{c} \Rightarrow |\vec{a}||\vec{a}\times\vec{b}| = |\vec{c}|$ $(\because \vec{a}\perp(\vec{a}\times\vec{b}))$

$1\,(1\times 5)\sin\theta = 3 \Rightarrow \sin\theta = \frac{3}{5} \Rightarrow \tan\theta = \frac{3}{4}$.

3. **c.** $|\vec{a}+\vec{b}+\vec{c}|^2 = 6$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 6$

$\Rightarrow |\vec{a}| = |\vec{b}| = |\vec{c}|$ and $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cdot\cos\frac{\pi}{3}$

i.e., $\vec{a}\cdot\vec{b} = \frac{1}{2}|\vec{a}|^2$

$\therefore 3|\vec{a}|^2 + 3|\vec{a}|^2 = 6$

$\Rightarrow |\vec{a}|^2 \Rightarrow |\vec{a}| = 1$

4. **b.** Let $\vec{\alpha} = \frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|} + \frac{\vec{c}}{|\vec{c}|}$

Since $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular vectors, if $\vec{\alpha}$ makes angles θ, ϕ ψ with $\vec{a}, \vec{b}$ and $\vec{c}$, respectively, then

$\vec{\alpha}\cdot\vec{a} = \frac{\vec{a}\cdot\vec{a}}{|\vec{a}|}$

$\Rightarrow |\vec{\alpha}|\cdot|\vec{a}|\cos\theta = |\vec{a}|$

$\Rightarrow \cos\theta = \frac{1}{|\vec{\alpha}|}$

Similarly $\cos\phi = \frac{1}{|\vec{\alpha}|}, \cos\psi = \frac{1}{|\vec{\alpha}|}$

$\therefore \theta = \phi = \psi$

5. **c.** $\vec{r} \times \vec{a} = \vec{b} \times \vec{a} \Rightarrow (\vec{r} - \vec{b}) \times \vec{a} = 0$

$\vec{r} \times \vec{b} = \vec{a} \times \vec{b} \Rightarrow (\vec{r} - \vec{a}) \times \vec{b} = 0$

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x-2 & y & z+1 \\ 1 & 1 & 0 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x-1 & y-1 & z \\ 2 & 0 & -1 \end{vmatrix} = 0$$

$\Rightarrow z + 1 = 0, x - y = 2$ and $y - 1 = 0, x - 1 + 2z = 0$

$\Rightarrow x = 3, y = 1, z = -1$

6. **d.** $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$

$\Rightarrow |\vec{a}||\vec{b}||\cos\theta| = |\vec{a}||\vec{b}||\sin\theta|$ (where θ is the angle between $\vec{a}$ and $\vec{b}$)

$\Rightarrow |\cos\theta| = |\sin\theta|$

$\Rightarrow \theta = \dfrac{\pi}{4}$ or $\dfrac{3\pi}{4}$ (as $0 \le \theta \le \pi$)

But $\vec{a} \cdot \vec{b} < 0$, therefore $\theta = \dfrac{3\pi}{4}$

7. **c.** $|\vec{a} + \vec{b} + \vec{c}|^2 = 1$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2|\vec{a}||\vec{b}|\cos\theta_1 + 2|\vec{b}||\vec{c}|\cos\theta_2 + 2|\vec{c}||\vec{a}|\cos\theta_3 = 1$

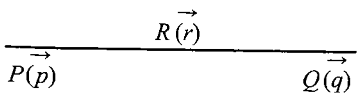
$\Rightarrow \cos\theta_1 + \cos\theta_2 + \cos\theta_3 = -1$

$\Rightarrow$ One of θ_1, θ_2 and θ_3 should be an obtuse angle.

8. **b.** $|\vec{a} \times \vec{b} - \vec{a} \times \vec{c}|^2 = |\vec{a} \times (\vec{b} - \vec{c})|^2 = |\vec{a}|^2 |\vec{b} - \vec{c}|^2 - (\vec{a} \cdot (\vec{b} - \vec{c}))^2 = |\vec{b} - \vec{c}|^2$

$= |\vec{b}|^2 + |\vec{c}|^2 - 2|\vec{b}||\vec{c}| \cos\dfrac{\pi}{3} = 1$

9. **c.** $R(\vec{r})$ moves on PQ.



10. **b.** $|\overrightarrow{AC} \times \overrightarrow{BD}| = 2|\overrightarrow{AB} \times \overrightarrow{AD}|$

$$= 2\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & -5 \\ 1 & 2 & 3 \end{vmatrix}$$

$= |2[\hat{i}(12 + 10) - \hat{j}(6 + 5) + \hat{k}(4 - 4)]|$

$= |2[22\hat{i} - 11\hat{j}]|$

$= 22|[2\hat{i} - \hat{j}]|$

$= 22 \times \sqrt{5}$

11. **c.** $(\hat{a} + \hat{b} + \hat{c})^2 \geq 0$

$3 + 2(\hat{a} \cdot \hat{b} + \vec{b} \cdot \hat{c} + \vec{c} \cdot \vec{a}) \geq 0$

$3 + 6 \cos \theta \geq 0$

$\cos \theta \geq -\frac{1}{2}$

$\Rightarrow \theta = \frac{2\pi}{3}$

12. **c.** $\vec{a} \times \vec{b}$ is a vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$. Similarly, $\vec{c} \times \vec{d}$ is a vector perpendicular to the plane containing $\vec{c}$ and $\vec{d}$.

Thus, the two planes will be parallel if their normals, i.e., $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$, are parallel.

$\Rightarrow (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{0}$

13. **d.** Let $\vec{r} \neq \vec{0}$. Then $\vec{r} \cdot \vec{a} = \vec{r} \cdot \vec{b} = \vec{r} \cdot \vec{c} = 0$

$\Rightarrow \vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, which is a contradiction.

Therefore, $\vec{r} = \vec{0}$

14. **c.** $\vec{a} \times (\hat{i} + 2\hat{j} + \hat{k}) = \hat{i} - \hat{k} = (\hat{j} \times (\hat{i} + 2\hat{j} + \hat{k}))$

$\Rightarrow (\vec{a} - \hat{j}) \times (\hat{i} + 2\hat{j} + \hat{k}) = \vec{0}$

$\Rightarrow \vec{a} - \hat{j} = \lambda(\hat{i} + 2\hat{j} + \hat{k})$

$\Rightarrow \vec{a} = \lambda\hat{i} + (2\lambda + 1)\hat{j} + \lambda\hat{k}, \lambda \in R$

15. **a.** $(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + \vec{b}) = 0$

$\Rightarrow 6|\vec{a}|^2 - 5|\vec{b}|^2 = 7\vec{a} \cdot \vec{b}$

Also, $(\vec{a} + 4\vec{b}) \cdot (\vec{b} - \vec{a}) = 0$

$\Rightarrow -|\vec{a}|^2 + 4|\vec{b}|^2 = 3\vec{a} \cdot \vec{b}$

$\Rightarrow \frac{6}{7}|\vec{a}|^2 - \frac{5}{7}|\vec{b}|^2 = -\frac{1}{3}|\vec{a}|^2 + \frac{4}{3}|\vec{b}|^2$

$\Rightarrow 25|\vec{a}|^2 = 43|\vec{b}|^2$

$\Rightarrow 3\vec{a} \cdot \vec{b} = -|\vec{a}|^2 + 4|\vec{b}|^2 = \frac{57}{25}|\vec{b}|^2$

$\Rightarrow 3|\vec{a}||\vec{b}|\cos\theta = \frac{57}{25}|\vec{b}|^2$

$\Rightarrow 3\sqrt{\frac{43}{25}}|\vec{b}|^2 \cos\theta = \frac{57}{25}|\vec{b}|^2$

$\Rightarrow \cos\theta = \frac{19}{5\sqrt{43}}$

16. **a.** Let l, m and n be the direction cosines of the required vector.
Then, $l = m$ (given). Therefore

Required vector $\vec{r} = l\hat{i} + m\hat{j} + n\hat{k} = l\hat{i} + l\hat{j} + n\hat{k}$

Now, $l^2 + m^2 + n^2 = 1 \Rightarrow 2l^2 + n^2 = 1$ (i)

Since, $\hat{r}$ is perpendicular to $-\hat{i} + 2\hat{j} + 2\hat{k}$,

$\vec{r}\cdot(-\hat{i} + 2\hat{j} + 2\hat{k}) = 0 \Rightarrow -l + 2l + 2n = 0 \Rightarrow l + 2n = 0$ (ii)

From (i) and (ii), we get: $n = \mp\frac{1}{3}, l = \pm\frac{2}{3}$

Hence, required vector $\vec{r} = \frac{1}{3}(\pm 2\hat{i} \pm 2\hat{j} \mp \hat{k}) = \pm\frac{1}{3}(2\hat{i} + 2\hat{j} - \hat{k})$

17. **d.** The angle between $\vec{a}$ and $\vec{b}$ is obtuse. Therefore,

$\vec{a}\cdot\vec{b} < 0$

$\Rightarrow 14x^2 - 8x + x < 0$

$\Rightarrow 7x(2x-1) < 0$

$\Rightarrow 0 < x < 1/2$ (i)

The angle between $\vec{b}$ and the *z-axis* is acute and less than $\pi/6$. Therefore,

$\frac{\vec{b}\cdot\vec{k}}{|\vec{b}||\vec{k}|} > \cos\pi/6$ $(\because \theta < \pi/6 \Rightarrow \cos\theta > \cos\pi/6)$

$\Rightarrow \frac{x}{\sqrt{x^2+53}} > \frac{\sqrt{3}}{2}$

$\Rightarrow 4x^2 > 3x^2 + 159$

$\Rightarrow x^2 > 159$

$\Rightarrow x > \sqrt{159}$ or $x < -\sqrt{159}$ (ii)

Clearly, (i) and (ii) cannot hold together.

18. **c.**

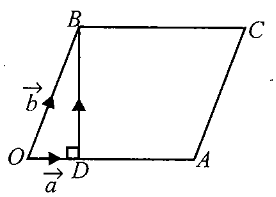


Fig. 2.36

Let $\overrightarrow{OD} = t\vec{a}$

$\therefore\ \overrightarrow{DB} = \vec{b} - t\vec{a}$

$\therefore\ (\vec{b} - t\vec{a})\cdot\vec{a} = 0\ \ (\because\ \overrightarrow{DB} \perp \overrightarrow{OA})$

$$\Rightarrow t = \frac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}$$

$$\therefore\ \overrightarrow{DB} = \vec{b} - \frac{(\vec{b}\cdot\vec{a})\,\vec{a}}{|\vec{a}|^2}$$

19. d. $(3\vec{a} + \vec{b})\cdot(\vec{a} - 4\vec{b})$

$= 3\,|\vec{a}|^2 - 11\vec{a}\cdot\vec{b} - 4\,|\vec{b}|^2$

$= 3 \times 36 - 11 \times 6 \times 8 \cos\pi - 4 \times 64 > 0$

Therefore, the angle between $\vec{a}$ and $\vec{b}$ is acute.

The longer diagonal is given by

$\vec{\alpha} = (3\vec{a} + \vec{b}) + (\vec{a} - 4\vec{b}) = 4\vec{a} - 3\vec{b}$

Now, $|\vec{\alpha}|^2 = |4\vec{a} - 3\vec{b}|^2 = 16|\vec{a}|^2 + 9|\vec{b}|^2 - 24\vec{a}\cdot\vec{b}$

$= 16 \times 36 + 9 \times 64 - 24 \times 6 \times 8 \cos\pi$

$= 16 \times 144$

$\Rightarrow |4\vec{a} - 3\vec{b}| = 48$

20. b. $\vec{c} = m\vec{a} + n\vec{b} + p(\vec{a} \times \vec{b})$

Taking dot product with $\vec{a}$ and $\vec{b}$, we have

$m = n = \cos\theta$

$\Rightarrow |\vec{c}| = |\cos\theta\,\vec{a} + \cos\theta\,\vec{b} + p\,(\vec{a} \times \vec{b})| = 1$

Squaring both sides, we get

$\cos^2\theta + \cos^2\theta + p^2 = 1$

$$\Rightarrow \cos\theta = \pm\frac{\sqrt{1 - p^2}}{\sqrt{2}}$$

Now $-\dfrac{1}{\sqrt{2}} \le \cos\theta \le \dfrac{1}{\sqrt{2}}$ (for real value of θ)

$\therefore\ \dfrac{\pi}{4} \le \cos\theta \le \dfrac{3\pi}{4}$

21. a. $\vec{b} - 2\vec{c} = \lambda\vec{a}$

$\Rightarrow \vec{b} = 2\vec{c} + \lambda\vec{a}$

$\Rightarrow |\vec{b}|^2 = |2\vec{c} + \lambda\vec{a}|^2$

$\Rightarrow 16 = 4|\vec{c}|^2 + \lambda^2|\vec{a}|^2 + 4\lambda\,\vec{a}\cdot\vec{c}$

$\Rightarrow 16 = 4 + \lambda^2 + 4\lambda\frac{1}{4}$

$\Rightarrow \lambda^2 + \lambda - 12 = 0$

$\Rightarrow \lambda = 3, -4$

22. **a.** A vector perpendicular to the plane of O, P and Q is $\overrightarrow{OP} \times \overrightarrow{OQ}$.

Now, $\overrightarrow{OP} \times \overrightarrow{OQ} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 1 & \lambda \\ 2 & -1 & \lambda \end{vmatrix} = 2\lambda\hat{i} - 2\lambda\hat{j} - 6\hat{k}$

Therefore, $\hat{i} - \hat{j} + 6\hat{k}$ is parallel to $2\lambda\hat{i} - 2\lambda\hat{j} - 6\hat{k}$

Hence $\frac{1}{2\lambda} = \frac{-1}{-2\lambda} = \frac{6}{-6}$

$\lambda = -\frac{1}{2}$

23. **a.** A vector coplanar with $\vec{a}$ and $\vec{b}$ and perpendicular to $\vec{c}$ is $\lambda((\vec{a}\times\vec{b})\times\vec{c})$.

But $\lambda((\vec{a}\times\vec{b})\times\vec{c}) = \lambda[(\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}]$

$= \lambda[4\vec{b} - 4\vec{a}]$

$= 4\lambda[\hat{j} - \hat{k}]$

Now $4|\lambda|\sqrt{2} = \sqrt{2}$ (Given) $\Rightarrow \lambda = \pm\frac{1}{4}$

Hence the required vector is $\hat{j} - \hat{k}$ or $-\hat{j} + \hat{k}$

24. **a.** $\vec{a} - \vec{p} + \vec{b} - \vec{p} + \vec{c} - \vec{p} = 0$

$\Rightarrow \vec{p} = \frac{\vec{a} + \vec{b} + \vec{c}}{3}$

$\Rightarrow P$ is centroid

25. **b.**

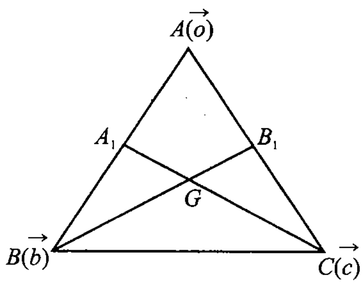


Fig. 2.37

Let P.V. of A, B and C be $\vec{0}, \vec{b}$ and $\vec{c}$, respectively. Therefore,

$$\vec{G} = \frac{\vec{b}+\vec{c}}{3}$$

$$\vec{A_1} = \frac{\vec{b}}{2}, \vec{B_1} = \frac{\vec{c}}{2}$$

$$\Delta_{AB_1G} = \frac{1}{2}|\overrightarrow{AG} \times \overrightarrow{AB_1}| = \frac{1}{2}\left|\frac{\vec{b}+\vec{c}}{3} \times \left(\frac{\vec{c}}{2}\right)\right|$$

$$= \frac{1}{12}|\vec{b} \times \vec{c}|$$

$$\Delta_{AA_1G} = \frac{1}{2}|\overrightarrow{AG} \times \overrightarrow{AA_1}| = \frac{1}{2}\left|\frac{\vec{b}+\vec{c}}{3} \times \left(\frac{\vec{b}}{2}\right)\right| = \frac{1}{12}|\vec{b} \times \vec{c}|$$

$$\Rightarrow \Delta_{GA_1AB_1} = \frac{1}{6}|\vec{b} \times \vec{c}| = \frac{1}{3}\cdot\frac{1}{2}|\vec{b} \times \vec{c}| = \frac{1}{3}\Delta_{ABC}$$

$$\Rightarrow \frac{\Delta}{\Delta_1} = 3$$

26. **a.** Points $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are coplanar. Therefore,

$\sin\alpha + 2\sin 2\beta + 3\sin 3\gamma = 1$

Now $|\sin\alpha + 2\sin 2\beta + 3\sin 3\gamma| \le \sqrt{1+4+9}.\sqrt{\sin^2\alpha + \sin^2 2\beta + \sin^2 3\gamma}$

$$\Rightarrow \sin^2\alpha + \sin^2 2\beta + \sin^2 3\gamma \ge \frac{1}{14}$$

27. **c.** $1 + 9(\vec{a}\cdot\vec{b})^2 - 6(\vec{a}\cdot\vec{b}) + 4|\vec{a}|^2 + |\vec{b}|^2 + 9|\vec{a}\times\vec{b}|^2 + 4\vec{a}\cdot\vec{b} = 47$

$\Rightarrow 1 + 4 + 4 + 36 - 4\cos\theta = 47$

$$\Rightarrow \cos\theta = -\frac{1}{2}$$

$\Rightarrow$ Angle between $\vec{a}$ and $\vec{b}$ is $\dfrac{2\pi}{3}$.

28. **c.** $k = |2(\vec{a}\times\vec{b})| + |3(\vec{a}\cdot\vec{b})|$

$= 12\sin\theta + 18\cos\theta$

$\Rightarrow$ maximum value of k is $\sqrt{12^2+18^2} = 6\sqrt{13}$

29. **b.** $|\vec{a} + \vec{b} + 3\vec{c}|^2 = 16$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 9|\vec{c}|^2 + 2\cos\theta_1 + 6\cos\theta_2 + 6\cos\theta_3 = 16,\ \theta_3 \in [\pi/6, 2\pi/3]$

$\Rightarrow 2\cos\theta_1 + 6\cos\theta_2 = 5 - 6\cos\theta_3$

$\Rightarrow (\cos\theta_1 + 3\cos\theta_2)_{max} = 4$

30. **c.** $|\vec{a} \times \vec{r}| = |\vec{c}|$

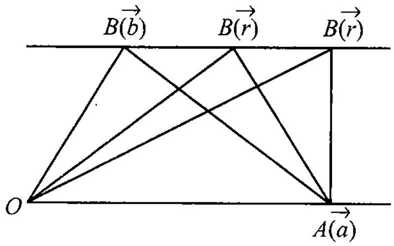


Fig. 2.38

Triangles on the same base and between the same parallel will have the same area.

31. **c.** Given $\vec{v} \cdot \vec{u} = \vec{w} \cdot \vec{u}$

and $\vec{v} \perp \vec{w} \Rightarrow \vec{v} \cdot \vec{w} = 0$

Now, $|\vec{u} - \vec{v} + \vec{w}|^2$

$= |\vec{u}|^2 + |\vec{v}|^2 + |\vec{w}|^2 - 2\vec{u} \cdot \vec{v} - 2\vec{w} \cdot \vec{v} + 2\vec{u} \cdot \vec{w}$

$= 1 + 4 + 9$

so $|\vec{u} - \vec{v} + \vec{w}| = \sqrt{14}$

32. **b.** We have

$\vec{p} \cdot \vec{q} = 0$

$\Rightarrow (5\vec{a} - 3\vec{b}) \cdot (-\vec{a} - 2\vec{b}) = 0$

$\Rightarrow 6|\vec{b}|^2 - 5|\vec{a}|^2 - 7\vec{a} \cdot \vec{b} = 0$ (i)

Also $\vec{r} \cdot \vec{s} = 0$

$\Rightarrow (-4\vec{a} - \vec{b})\ (-\vec{a} + \vec{b}) = 0$

$\Rightarrow 4|\vec{a}|^2 - |\vec{b}|^2 - 3\vec{a} \cdot \vec{b} = 0$ (ii)

Now $\vec{x} = \frac{1}{3}(\vec{p} + \vec{r} + \vec{s}) = \frac{1}{3}(5\vec{a} - 3\vec{b} - 4\vec{a} - \vec{b} - \vec{a} + \vec{b}) = -\vec{b}$

and $\vec{y} = \frac{1}{5}(\vec{r} + \vec{s}) = \frac{1}{5}(-5\vec{a}) = -\vec{a}$

Angle between $\vec{x}$ and $\vec{y}$, i.e., $\cos\theta = \dfrac{\vec{x} \cdot \vec{y}}{|\vec{x}||\vec{y}|} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$ (iii)

From (i) and (ii), $|\vec{a}| = \sqrt{\frac{25}{19}}\sqrt{\vec{a} \cdot \vec{b}}$ and $|\vec{b}| = \sqrt{\frac{43}{19}}\sqrt{\vec{a} \cdot \vec{b}}$. Therefore

$|\vec{a}||\vec{b}| = \dfrac{\sqrt{25 \times 43}}{19} \cdot \vec{a} \cdot \vec{b}$

$\theta = \cos^{-1}\left(\dfrac{19}{5\sqrt{43}}\right)$

33. **a.** $\vec{\alpha} \parallel (\vec{\beta} \times \vec{\gamma}) \Rightarrow \vec{\alpha} \perp \vec{\beta}$ and $\vec{\alpha} \perp \vec{\gamma}$

Now, $(\vec{\alpha} \times \vec{\beta}) \cdot (\vec{\alpha} \times \vec{\gamma}) = |\vec{\alpha}|^2 (\vec{\beta} \cdot \vec{\gamma}) - (\vec{\alpha} \cdot \vec{\beta})(\vec{\alpha} \cdot \vec{\gamma}) = |\vec{\alpha}|^2 \cdot (\vec{\beta} \cdot \vec{\gamma})$

34. **b.** Since, $\overrightarrow{OA} = \hat{i} + \hat{j} + \hat{k}$

$\overrightarrow{OB} = \hat{i} + 5\hat{j} - \hat{k}$

$\overrightarrow{OC} = 2\hat{i} + 3\hat{j} + 5\hat{k}$

$a = BC = |\overrightarrow{BC}| = |\overrightarrow{OC} - \overrightarrow{OB}| = |\hat{i} - 2\hat{j} + 6\hat{k}| = \sqrt{41}$

$b = CA = |\overrightarrow{CA}| = |\overrightarrow{OA} - \overrightarrow{OC}| = |-\hat{i} - 2\hat{j} - 4\hat{k}| = \sqrt{21}$

and $c = AB = |\overrightarrow{AB}| = |\overrightarrow{OB} - \overrightarrow{OA}| = |0\hat{i} + 4\hat{j} - 2\hat{k}| = \sqrt{20}$

Since $a > b > c$, A is the greatest angle. Therefore,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{21 + 20 - 41}{2 \cdot \sqrt{21} \cdot \sqrt{20}} = 0$$

$\therefore \angle A = 90°$

35. **b.** $\vec{a} + \vec{b} = \lambda \vec{c}$ (i)

and $\vec{b} + \vec{c} = \mu \vec{a}$ (ii)

$\therefore (\lambda \vec{c} - \vec{a}) + \vec{c} = \mu \vec{a}$ (putting $\vec{b} = \lambda \vec{c} - \vec{a}$)

$\Rightarrow (\lambda + 1)\vec{c} = (\mu + 1)\vec{a}$

$\Rightarrow \lambda = \mu = -1$

$\Rightarrow \vec{a} + \vec{b} + \vec{c} = 0$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$

$\Rightarrow \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = -3$

36. **d.** $\vec{0} = (\vec{a} + \vec{b}) \cdot (2\vec{a} + 3\vec{b}) \times (3\vec{a} - 2\vec{b})$

$= (\vec{a} + \vec{b}) \cdot (-4\vec{a} \times \vec{b} - 9\vec{a} \times \vec{b})$

$= -13\ (\vec{a} + \vec{b}) \cdot (\vec{a} \times \vec{b})$

which is true for all values of $\vec{a}$ and $\vec{b}$.

37. **c.** We have

$$\overrightarrow{AB}\cdot\overrightarrow{AC}+\overrightarrow{BC}\cdot\overrightarrow{BA}+\overrightarrow{CA}\cdot\overrightarrow{CB}=(AB)(AC)\cos\theta+(BC)(BA)\sin\theta+0$$
$$=AB\,(AC\cos\theta+BC\sin\theta)$$
$$=AB\left(\frac{(AC)^2}{AB}+\frac{(BC)^2}{AB}\right)$$
$$=AC^2+BC^2=AB^2=p^2$$

38. **c.** $\vec{a}_1=(\vec{a}\cdot\hat{b})\hat{b}=\dfrac{(\vec{a}\cdot\vec{b})\vec{b}}{|\vec{b}|^2}$

$$\Rightarrow \vec{a}_2=\vec{a}-\vec{a}_1=\vec{a}-\frac{(\vec{a}\cdot\vec{b})\vec{b}}{|\vec{b}|^2}$$

Thus, $\vec{a}_1\times\vec{a}_2=\dfrac{(\vec{a}\cdot\vec{b})\vec{b}}{|\vec{b}|^2}\times\left(\vec{a}-\dfrac{(\vec{a}\cdot\vec{b})\vec{b}}{|\vec{b}|^2}\right)=\dfrac{(\vec{a}\cdot\vec{b})(\vec{b}\times\vec{a})}{|\vec{b}|^2}$.

39. **b.** Let the required vector be $\vec{r}$. Then $\vec{r}=x_1\vec{b}+x_2\vec{c}$ and $\vec{r}\cdot\vec{a}=\sqrt{\dfrac{2}{3}}\ (|\vec{a}|)=2$

Now, $\vec{r}\cdot\vec{a}=x_1\,\vec{a}\cdot\vec{b}+x_2\,\vec{a}\cdot\vec{c}\Rightarrow 2=x_1(2-2-1)+x_2(2-1-2)\Rightarrow x_1+x_2=-2$

$$\Rightarrow \vec{r}=x_1(\hat{i}+2\hat{j}-\hat{k})+x_2(\hat{i}+\hat{j}-2\hat{k})=\hat{i}(x_1+x_2)+\hat{j}(2x_1+x_2)-\hat{k}(2x_2+x_1)$$
$$=-2\hat{i}+\hat{j}(x_1-2)-\hat{k}(-4-x_1),\ \text{where } x_1\in R$$

40. **a.** Let P.V. of P, A, B and C be $\vec{p}$, $\vec{a}$, $\vec{b}$ and $\vec{c}$, respectively, and $O(\vec{0})$ be the circumcentre of equilateral triangle ABC. Then

$$|\vec{p}|=|\vec{b}|=|\vec{a}|=|\vec{c}|=\frac{l}{\sqrt{3}}$$

Now $|\overrightarrow{PA}|^2=|\vec{a}-\vec{p}|^2=|\vec{a}|^2+|\vec{p}|^2-2\vec{p}\cdot\vec{a}$

Similarly, $|\overrightarrow{PB}|^2=|\vec{b}|^2+|\vec{p}|^2-2\vec{p}\cdot\vec{b}$

and $|\overrightarrow{PC}|^2=|\vec{c}|^2+|\vec{p}|^2-2\vec{p}\cdot\vec{c}$

$$\Rightarrow \Sigma|\overrightarrow{PA}|^2=6\cdot\frac{l^2}{3}-2\vec{p}\cdot(\vec{a}+\vec{b}+\vec{c})=2l^2\quad \text{as } (\vec{a}+\vec{b}+\vec{c}/3=\vec{0})$$

41. **d.** For minimum value $|\vec{r}+b\vec{s}|=0$.

Let $\vec{r}$ and $\vec{s}$ are anti parallel so $b\vec{s}=-\vec{r}$

so $|b\vec{s}|^2+|\vec{r}+b\vec{s}|^2=|-\vec{r}|^2+|\vec{r}-\vec{r}|^2=|\vec{r}|^2$

42. **c.** Let the required vector $\vec{r}$ be such that

$$\vec{r}=x_1\vec{a}+x_2\vec{b}+x_3\vec{a}\times\vec{b}$$

We must have $\vec{r}\cdot\vec{a} = \vec{r}\cdot\vec{b} = \vec{r}\cdot(\vec{a}\times\vec{b})$ (as $\vec{r}, \vec{a}, \vec{b}$ and $\vec{a}\times\vec{b}$ are unit vectors and $\vec{r}$ is equally inclined to $\vec{a}, \vec{b}$ and $\vec{a}\times\vec{b}$)

Now $\vec{r}\cdot\vec{a} = x_1$, $\vec{r}\cdot\vec{b} = x_2$, $\vec{r}\cdot(\vec{a}\times\vec{b}) = x_3$

$\Rightarrow \vec{r} = \lambda(\vec{a}+\vec{b}+(\vec{a}\times\vec{b}))$

Also, $\vec{r}\cdot\vec{r} = 1$

$\Rightarrow \lambda^2(\vec{a}+\vec{b}+\vec{a}\times\vec{b})\cdot(\vec{a}+\vec{b}+(\vec{a}\times\vec{b})) = 1$

$\Rightarrow \lambda^2(|\vec{a}|^2+|\vec{b}|^2+|\vec{a}\times\vec{b}|^2) = 1$

$\Rightarrow \lambda^2 = \dfrac{1}{3}$

$\Rightarrow \lambda = \pm\dfrac{1}{\sqrt{3}}$

$\Rightarrow \vec{r} = \pm\dfrac{1}{\sqrt{3}}(\vec{a}+\vec{b}+\vec{a}\times\vec{b})$

43. **d.** $\vec{a}+\vec{b} = \mu\vec{p}$ $\quad \vec{b}\cdot\vec{q} = 0,\ |\vec{b}|^2 = 1$

$\because\ \vec{a}+\vec{b} = \mu\vec{p}$

$\Rightarrow (\vec{a}+\vec{b})\times\vec{a} = \mu\vec{p}\times\vec{a},\ \vec{b}\times\vec{a} = \mu\vec{p}\times\vec{a} \Rightarrow \vec{q}\times(\vec{b}\times\vec{a}) = \mu\vec{q}\times(\vec{p}\times\vec{a})$

$\Rightarrow (\vec{q}\cdot\vec{a})\vec{b} - (\vec{q}\cdot\vec{b})\vec{a} = \mu\vec{q}\times(\vec{p}\times\vec{a}) \Rightarrow (\vec{q}\cdot\vec{a})\vec{b} = \mu\vec{q}\times(\vec{p}\times\vec{a})$

$\because\ \vec{a}+\vec{b} = \mu\vec{p}$

$\Rightarrow \vec{q}\cdot(\vec{a}+\vec{b}) = \mu\vec{q}\cdot\vec{p}$

$\Rightarrow \vec{q}\cdot\vec{a}+\vec{q}\cdot\vec{b} = \mu\vec{p}\cdot\vec{q}$

$\Rightarrow \mu = \dfrac{\vec{q}\cdot\vec{a}}{\vec{p}\cdot\vec{q}}$

$\Rightarrow (\vec{q}\cdot\vec{a})\vec{b} = \dfrac{\vec{q}\cdot\vec{a}}{\vec{p}\cdot\vec{q}}[(\vec{q}\cdot\vec{a})\cdot\vec{p} - (\vec{q}\cdot\vec{p})\vec{a}]$

$\Rightarrow |(\vec{q}\cdot\vec{a})\vec{p} - (\vec{q}\cdot\vec{p})\vec{a}| = |(\vec{p}\cdot\vec{q})\vec{b}| = |(\vec{p}\cdot\vec{q})|\cdot|\vec{b}|$

$\Rightarrow |(\vec{q}\cdot\vec{a})\vec{p} - (\vec{q}\cdot\vec{p})\vec{a}| = |\vec{p}.\vec{q}|$

44. **c.** $\vec{d}\cdot\hat{a} = \vec{d}\cdot\hat{b} = \vec{d}\cdot\hat{c}$

$\Rightarrow \lambda(\hat{a}\cdot\hat{b}+\hat{a}\cdot\hat{c}) = \lambda(1+\hat{b}\cdot\hat{c}) = \lambda(1+\hat{b}\cdot\hat{c}) \Rightarrow 1+\hat{b}\cdot\hat{c} = \hat{a}\cdot\hat{b}+\hat{a}\cdot\hat{c}$

$\Rightarrow 1-\hat{a}\cdot\hat{b}+\hat{b}\cdot\hat{c}-\hat{a}\cdot\hat{c} = 0 \Rightarrow 1-\hat{a}\cdot\hat{b}+(\hat{b}-\hat{a})\cdot\hat{c} = 0 \Rightarrow \hat{a}\cdot(\hat{a}-\hat{b})+(\hat{b}-\hat{a})\cdot\hat{c} = 0$

$\Rightarrow (\hat{a}-\hat{c})\cdot(\hat{a}-\hat{b}) = 0 \Rightarrow \hat{a}-\hat{c}$ is perpendicular to $(\hat{a}-\hat{b}) \Rightarrow$ The triangle is right angled.

45. **c.** The given relation can be rewritten as the vector expression

$$(\sqrt{a^2-4}\,\hat{i}+a\hat{j}+\sqrt{a^2+4}\,\hat{k})\cdot(\tan A\hat{i}+\tan B\hat{j}+\tan C\hat{k})=6a$$

$$\Rightarrow \sqrt{a^2-4+a^2+a^2+4}\sqrt{\tan^2 A+\tan^2 B+\tan^2 C}\cdot(\cos\theta)=6a \quad (\because \vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta)$$

$$\sqrt{3}\,a\sqrt{\tan^2 A+\tan^2 B+\tan^2 C}\cdot(\cos\theta)=6a$$

$$\tan^2 A+\tan^2 B+\tan^2 C=12\sec^2\theta\ge 12 \quad (\because \sec^2\theta\ge 1)$$

The least value of $\tan^2 A+\tan^2 B+\tan^2 C$ is 12.

46. **d.** $\Delta=\frac{1}{2}|(\hat{j}+\lambda\hat{k})\times(\hat{i}+\lambda\hat{k})|=\frac{1}{2}|-\hat{k}+\lambda\hat{i}+\lambda\hat{j}|=\frac{1}{2}\sqrt{2\lambda^2+1}$

$$\Rightarrow \frac{9}{4}\le\frac{1}{4}(2\lambda^2+1)\le\frac{33}{4}$$

$$\Rightarrow 4\le\lambda^2\le 16$$

$$\Rightarrow 2\le|\lambda|\le 4$$

47. **c.** Let the projection be x, then $\vec{a}=\frac{x(\hat{i}+\hat{j})}{\sqrt{2}}+\frac{x(-\hat{i}+\hat{j})}{\sqrt{2}}+x\hat{k}$

$$\therefore \vec{a}=\frac{2x\hat{j}}{\sqrt{2}}+x\hat{k} \quad\Rightarrow\quad \hat{a}=\frac{\sqrt{2}}{\sqrt{3}}\hat{j}+\frac{\hat{k}}{\sqrt{3}}$$

48. **b.** Let $\vec{r}$ be the new position. Then $\vec{r}=\lambda\hat{k}+\mu(\hat{i}+\hat{j})$

Also $\vec{r}\cdot\hat{k}=-\frac{1}{\sqrt{2}} \Rightarrow \lambda=-\frac{1}{\sqrt{2}}$

Also, $\lambda^2+2\mu^2=1\Rightarrow 2\mu^2=\frac{1}{2}\Rightarrow\mu=\pm\frac{1}{2}$

$$\therefore \vec{r}=\pm\frac{1}{2}(\hat{i}+\hat{j})-\frac{\hat{k}}{\sqrt{2}}$$

49. **c.**

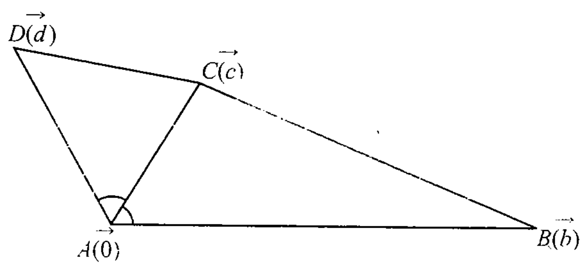


Fig. 2.39

Let $|\overrightarrow{AC}| = \lambda > 0$

Then from $15\,|\overrightarrow{AC}| = 3\,|\overrightarrow{AB}| = 5\,|\overrightarrow{AD}|$

$|\overrightarrow{AB}| = 5\lambda$

Let θ be the angle between $\overrightarrow{BA}$ and $\overrightarrow{CD}$.

$$\Rightarrow \cos\theta = \frac{\overrightarrow{BA}\cdot\overrightarrow{CD}}{|\overrightarrow{BA}||\overrightarrow{CD}|} = \frac{-\vec{b}\cdot(\vec{d}-\vec{c})}{|\vec{b}||\vec{d}-\vec{c}|} \qquad \text{(i)}$$

Now $-\vec{b}\cdot(\vec{d}-\vec{c}) = \vec{b}\cdot\vec{c} - \vec{b}\cdot\vec{d}$

$$= |\vec{b}||\vec{c}|\cos\frac{\pi}{3} - |\vec{b}||\vec{d}|\cos\frac{2\pi}{3}$$

$$= (5\lambda)(\lambda)\frac{1}{2} + (5\lambda)(3\lambda)\frac{1}{2}$$

$$= \frac{5\lambda^2 + 15\lambda^2}{2}$$

$$= 10\lambda^2$$

Denominator of (i) $= |\vec{b}||\vec{d}-\vec{c}|$

Now $|\vec{d}-\vec{c}|^2 = |\vec{d}|^2 + |\vec{c}|^2 - 2\,\vec{c}\cdot\vec{d}$

$$= 9\lambda^2 + \lambda^2 - 2(\lambda)(3\lambda)(1/2)$$

$$= 10\lambda^2 - 3\lambda^2$$

$$= 7\lambda^2$$

Denominator of (i) $= (5\lambda)(\sqrt{7}\,\lambda) = 5\sqrt{7}\,\lambda^2$

$$\therefore \cos\theta = \frac{10\lambda^2}{5\sqrt{7}\,\lambda^2} = \frac{2}{\sqrt{7}}$$

50. **a.** Let A be the origin. $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{AD} = \vec{b}$

so, $\overrightarrow{AE} = \vec{b} + \frac{3}{2}\vec{a}$, $\overrightarrow{AG} = \vec{a} + 3\vec{b}$.

$$\text{So the required ratio} = \frac{\frac{1}{2}\left|(\vec{a}+3\vec{b})\times\left(\vec{b}+\frac{3}{2}\vec{a}\right)\right|}{\frac{1}{2}|\vec{a}\times\vec{b}|}$$

$$= \frac{7}{2}$$

51. **b.** Let $\vec{a} = \lambda\,\vec{b} + \mu\,\vec{c}$

$\vec{a}$ is equally inclined to $\vec{b}$ and $\vec{d}$ where $\vec{d} = \hat{j} + 2\hat{k}$.

$$\Rightarrow \frac{\vec{a}\cdot\vec{b}}{ab} = \frac{\vec{a}\cdot\vec{d}}{ad}$$

$$\Rightarrow \frac{(\lambda\vec{b}+\mu\vec{c})\cdot\vec{b}}{b} = \frac{(\lambda\vec{b}+\mu\vec{c})\cdot\vec{d}}{d}$$

$$\Rightarrow \frac{[\lambda(2\hat{i}+\hat{j})+\mu(\hat{i}-\hat{j}+\hat{k})]\cdot(2\hat{i}+\hat{j})}{\sqrt{5}} = \frac{[\lambda(2\hat{i}+\hat{j})+\mu(\hat{i}-\hat{j}+\hat{k})]\cdot(\hat{j}+2\hat{k})}{\sqrt{5}}$$

$$\Rightarrow \lambda(4+1)+\mu(2-1)=\lambda(1)+\mu(-1+2)$$

$$\Rightarrow 4\lambda=0, \text{ i.e., } \lambda=0$$

$$\therefore \hat{a} = \frac{\hat{i}-\hat{j}+\hat{k}}{\sqrt{3}}$$

52. **a.** Area of $\Delta BCD = \frac{1}{2}|\overrightarrow{BC}\times\overrightarrow{BD}| = \frac{1}{2}|(b\hat{i}-c\hat{j})\times(b\hat{i}-d\hat{k})|$

$$= \frac{1}{2}|bd\,\hat{j}+bc\,\hat{k}+dc\,\hat{i}|$$

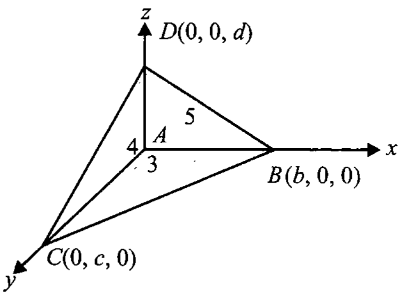


Fig. 2.40

$$= \frac{1}{2}\sqrt{b^2c^2+c^2d^2+d^2b^2} \quad \text{(i)}$$

Now $6 = bc$; $8 = cd$; $10 = bd$

$b^2c^2 + c^2d^2 + d^2b^2 = 200$

Substituting the value in (i)

$$A = \frac{1}{2}\sqrt{200} = 5\sqrt{2}$$

53. **d.** $\vec{f}\left(\frac{5}{4}\right) = \left[\frac{5}{4}\right]\hat{i} + \left(\frac{5}{4}-\left[\frac{5}{4}\right]\right)\hat{j} + \left[\frac{5}{4}+1\right]\hat{k}$

$$= \hat{i} + \left(\frac{5}{4}-1\right)\hat{j} + 2\hat{k}$$

$$= \hat{i} + \frac{1}{4}\hat{j} + 2\hat{k}$$

When $0<t<1$, $\vec{f}(t) = 0\vec{i} + \{t-0\}\vec{j} + \vec{k} = t\vec{j} + \vec{k}$

$\vec{f}\left(\frac{5}{4}\right)\cdot\vec{f}(t) = 2+\frac{t}{4}$

So $\cos\theta = \dfrac{2+\frac{t}{4}}{\left|\vec{i}+\frac{1}{4}\vec{j}+2\vec{k}\right| |t\vec{j}+\vec{k}|} = \dfrac{2+\frac{t}{4}}{\sqrt{1+\frac{1}{16}+4}\sqrt{1+t^2}}$

$= \dfrac{8+t}{9\sqrt{1+t^2}}$

54. **a.** $(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{u}$, where $\vec{u} = \vec{a}\times\vec{c}$

$\Rightarrow \vec{a}\cdot(\vec{b}\times\vec{u}) = \vec{a}\cdot[\vec{b}\times(\vec{a}\times\vec{c})]$

$= \vec{a}\cdot[(\vec{b}\cdot\vec{c})\,\vec{a} - (\vec{a}\cdot\vec{b})\,\vec{c}]$

$= \vec{a}\cdot(\vec{b}\cdot\vec{c})\,\vec{a}$ $(\because\ \vec{a}\cdot\vec{b}=0)$

$= |\vec{a}|^2\,(\vec{b}\cdot\vec{c})$

55. **d.** $(\hat{i}+\hat{j})\times(\hat{j}+\hat{k}) = \hat{i}-\hat{j}+\hat{k}$ so that unit vector perpendicular to the plane of $\hat{i}+\hat{j}$ and $\hat{j}+\hat{k}$ is $\frac{1}{\sqrt{3}}(\hat{i}-\hat{j}+\hat{k})$.

Similarly, the other two unit vectors are $\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}-\hat{k})$ and $\frac{1}{\sqrt{3}}(-\hat{i}+\hat{j}+\hat{k})$.

The required volume $= \dfrac{3}{\sqrt{3}}\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix} = 4\sqrt{3}$

56. **c.** $\vec{d}\cdot\vec{c} = \vec{d}\cdot\vec{b} = \vec{d}\cdot\vec{c} = [\vec{a}\,\vec{b}\,\vec{c}]$

Then $|(\vec{d}\cdot\vec{c})(\vec{a}\times\vec{b}) + (\vec{d}\cdot\vec{a})(\vec{b}\times\vec{c}) + (\vec{d}\cdot\vec{b})(\vec{c}\times\vec{a})| = 0$

$\Rightarrow [\vec{a}\,\vec{b}\,\vec{c}]\,|\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}| = 0$

$\Rightarrow [\vec{a}\,\vec{b}\,\vec{c}] = 0$ $(\because\ \vec{d}$ is non-zero)

$\Rightarrow \vec{a}, \vec{b}, \vec{c}$ are coplanar.

57. **a.** $(\vec{a}\times(\vec{a}\times(\vec{a}\times(\vec{a}\times\vec{b})))) = (\vec{a}\times(\vec{a}\times((\vec{a}\cdot\vec{b})\vec{a} - (\vec{a}\cdot\vec{b})\vec{b})))$

$= (\vec{a}\times(\vec{a}\times(-4\vec{b})))$

$$= -4(\vec{a} \times (\vec{a} \times \vec{b}))$$
$$= -4((\vec{a} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{a})\vec{b})$$
$$= -4(-4\vec{b}) = 16\vec{b} = 48\hat{b}$$

58. **d.** Let $\vec{a} = 6\hat{i} + 6\hat{k}, \vec{b} = 4\hat{j} + 2\hat{k}, \vec{c} = 4\hat{j} - 8\hat{k}$

then $\vec{a} \times \vec{b} = -24\hat{i} - 12\hat{j} + 24\hat{k}$
$= 12(-2\hat{i} - \hat{j} + 2\hat{k})$

$\therefore$ Area of the base of the parallelepiped $= \frac{1}{2}|\vec{a} \times \vec{b}|$
$= \frac{1}{2}(12 \times 3)$
$= 18$

Height of the parallelepiped = length of projection of $\vec{c}$ on $\vec{a} \times \vec{b}$

$$= \frac{|\vec{c} \cdot \vec{a} \times \vec{b}|}{|\vec{a} \times \vec{b}|}$$
$$= \frac{|12(-4-16)|}{36}$$
$$= \frac{20}{3}$$

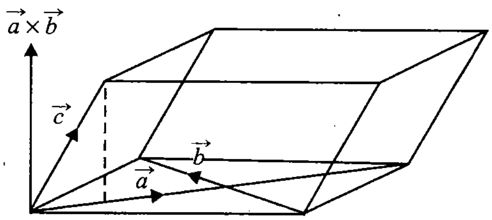


Fig. 2.41

$\therefore$ Volume of the parallelepiped $= 18 \times \frac{20}{3} = 120$

59. **c.** $3 = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$

$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 18$

Volume of the required parallelepiped

$= [\vec{a} + \vec{b}\ \ \vec{b} + \vec{c}\ \ \vec{c} + \vec{a}]$

$= 2[\vec{a}\ \vec{b}\ \vec{c}] = 36$

60. **b.** Here $[\vec{a}\ \vec{b}\ \vec{c}] = \pm 1$

$$[\vec{a} + \vec{b} + \vec{c}\ \ \vec{a} + \vec{b}\ \ \vec{b} + \vec{c}] = (\vec{a} + \vec{b} + \vec{c}) \times (\vec{a} + \vec{b}) \cdot (\vec{b} + \vec{c})$$
$$= \vec{c} \times (\vec{a} + \vec{b}) \cdot (\vec{b} + \vec{c})$$
$$= (\vec{c} \times \vec{a} + \vec{c} \times \vec{b}) \cdot (\vec{b} + \vec{c})$$
$$= \vec{c} \times \vec{a} \cdot \vec{b} = [\vec{a}\ \vec{b}\ \vec{c}] = \pm 1$$

61. **a.** Let $\vec{c} = \lambda(\vec{a} \times \vec{b})$.

Hence $\lambda(\vec{a} \times \vec{b}) \cdot (\hat{i} + 2\hat{j} - 7\hat{k}) = 10$

$$\Rightarrow \lambda \begin{vmatrix} 2 & -3 & 1 \\ 1 & -2 & 3 \\ 1 & 2 & -7 \end{vmatrix} = 10$$

$\Rightarrow \lambda = -1$

$\Rightarrow \vec{c} = -(\vec{a} \times \vec{b})$

62. **d.** $\vec{a} \perp \vec{b} \Rightarrow x - y + 2 = 0$

$\vec{a} \cdot \vec{c} = 4 \Rightarrow x + 2y = 4$

Solving we get $x = 0;\ y = 2$

$\Rightarrow \vec{a} = 2\hat{j} + 2\hat{k}$

$$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 0 & 2 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 0 \end{vmatrix} = 8 = |\vec{a}|^2$$

63. **c.** $(\vec{a} \times \vec{b} \cdot \vec{c})^2 = |\vec{a}|^2 |\vec{b}|^2 |\vec{c}|^2 \sin^2\theta \cos^2\phi$ (θ is the angle between $\vec{a}$ and $\vec{b}$, $\phi = 0$)

$= \frac{1}{4}(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$

64. **c.** $\vec{r} \cdot \vec{a} = 0, |\vec{r} \times \vec{b}| = |\vec{r}||\vec{b}|$ and $|\vec{r} \times \vec{c}| = |\vec{r}||\vec{c}|$

$\Rightarrow \vec{r} \perp \vec{a}, \vec{b}, \vec{c}$

$\therefore [\vec{a}\ \vec{b}\ \vec{c}] = 0$

65. **b.** $\vec{c} = \lambda(\vec{a} \times \vec{b})$

$\Rightarrow \vec{c} \cdot \vec{c} = \lambda(\vec{a} \times \vec{b}) \cdot \vec{c}$

$\Rightarrow \frac{1}{3} = \lambda$

Also $|\vec{c}|^2 = \lambda^2 |\vec{a} \times \vec{b}|^2$

$\Rightarrow \frac{1}{3} = \frac{1}{9}(a^2 b^2 \sin^2\theta) = \frac{1}{9} \times 2 \times 3 \sin^2\theta$

$\Rightarrow \sin^2\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{4}$

66. **c.** $4\vec{a} + 5\vec{b} + 9\vec{c} = 0 \Rightarrow$ Vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

$\Rightarrow \vec{b} \times \vec{c}$ and $\vec{c} \times \vec{a}$ are collinear $\Rightarrow (\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a}) = \vec{0}$.

67. **a.** $[\vec{a}\times\vec{b}\ \ \vec{a}\times\vec{c}\ \ \vec{d}]$

$=(\vec{a}\times\vec{b})\cdot((\vec{a}\times\vec{c})\times\vec{d})$

$=(\vec{a}\times\vec{b})\cdot((\vec{a}\cdot\vec{d})\vec{c}-(\vec{c}\cdot\vec{d})\vec{a})$

$=(\vec{a}\cdot\vec{d})[\vec{a}\,\vec{b}\,\vec{c}]$

68. **a.** Let $\vec{r}=x_1\hat{a}+x_2\hat{b}+x_3(\hat{a}\times\hat{b})$

$\Rightarrow \vec{r}\cdot\hat{a}=x_1+x_2\hat{a}\cdot\hat{b}+x_3\hat{a}\cdot(\hat{a}\times\hat{b})=x_1$

Also, $\vec{r}\cdot\hat{b}=x_1\hat{a}\cdot\hat{b}+x_2+x_3\hat{b}\cdot(\hat{a}+\hat{b})=x_2$

and $\vec{r}\cdot(\hat{a}\times\hat{b})=x_1\hat{a}\cdot(\hat{a}\times\hat{b})+x_2\hat{b}\cdot(\hat{a}\times\hat{b})+x_3(\hat{a}\times\hat{b})\cdot(\hat{a}\times\hat{b})=x_3$

$\Rightarrow \vec{r}=(\vec{r}\cdot\hat{a})\hat{a}+(\vec{r}\cdot\hat{b})\hat{b}+(\vec{r}\cdot(\hat{a}\times\hat{b}))(\hat{a}\times\hat{b})$

69. **a.** $[\vec{a}+(\vec{a}\times\vec{b})\ \ \vec{b}+(\vec{a}\times\vec{b})\ \ \vec{a}\times\vec{b}]$

$=(\vec{a}+(\vec{a}\times\vec{b}))\cdot((\vec{b}+(\vec{a}\times\vec{b}))\times(\vec{a}\times\vec{b}))$

$=(\vec{a}+(\vec{a}\times\vec{b}))\cdot(\vec{b}\times(\vec{a}\times\vec{b}))$

$=(\vec{a}+(\vec{a}\times\vec{b}))\cdot(\vec{a}-(\vec{a}\cdot\vec{b})\vec{b})$

$=\vec{a}\cdot\vec{a}=1$ (as $\vec{a}\cdot\vec{b}=0,\ \vec{a}\cdot(\vec{a}\times\vec{b})=0$)

70. **d.** $|\vec{a}|=1, |\vec{b}|=4, \vec{a}\cdot\vec{b}=2$

$\vec{c}=(2\vec{a}\times\vec{b})-3\vec{b}$

$\Rightarrow \vec{c}+3\vec{b}=2\vec{a}\times\vec{b}$

$\because\ \vec{a}\cdot\vec{b}=2$

$\Rightarrow |\vec{a}|\cdot|\vec{b}|\cos\theta=2$

$\Rightarrow \cos\theta=\dfrac{2}{|\vec{a}|\cdot|\vec{b}|}=\dfrac{2}{4}$

$\Rightarrow \cos\theta=\dfrac{1}{2}$

$\because\ \theta=\dfrac{\pi}{3}$

$\Rightarrow |\vec{c}+3\vec{b}|^2=|2\vec{a}\times\vec{b}|^2$

$\Rightarrow |\vec{c}|^2+9|\vec{b}|^2+2\vec{c}\cdot3\vec{b}=4|\vec{a}|^2|\vec{b}|^2\sin^2\theta$

$\Rightarrow |\vec{c}|^2+144+6\vec{b}\cdot\vec{c}=48$

$\Rightarrow |\vec{c}|^2+96+6(\vec{b}\cdot\vec{c})=0$ (i)

$\because\quad \vec{c}=2\vec{a}\times\vec{b}-3\vec{b}$

$\Rightarrow \quad \vec{b}\cdot\vec{c} = 0 - 3\times 16$

$\therefore \quad \vec{b}\cdot\vec{c} = -48$

Putting value of $\vec{b}\cdot\vec{c}$ in Eq. (i)

$|\vec{c}|^2 + 96 - 6\times 48 = 0$

$\Rightarrow \quad |\vec{c}|^2 = 48\times 4$

$\Rightarrow \quad |\vec{c}|^2 = 192$

Again, putting the value of $|\vec{c}|$ in Eq. (i),

$192 + 96 + 6|\vec{b}|\cdot|\vec{c}|\cos\alpha = 0$

$\Rightarrow \quad 6\times 4\times 8\sqrt{3}\cos\alpha = -288$

$\Rightarrow \quad \cos\alpha = -\dfrac{288}{6\times 4\times 8\sqrt{3}} = -\dfrac{3}{2\sqrt{3}} \Rightarrow \cos\alpha = -\dfrac{\sqrt{3}}{2}$

$\therefore \quad \alpha = \dfrac{5\pi}{6}$

71. **d.** $((\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c}))\times(\vec{b}\times\vec{c})$

$= (\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c}) + (\vec{a}\times\vec{c})\times(\vec{b}\times\vec{c})$

$= ((\vec{a}\times\vec{b})\cdot\vec{c})\vec{b} - ((\vec{a}\times\vec{b})\cdot\vec{b})\vec{c} + ((\vec{a}\times\vec{c})\cdot\vec{c})\vec{b} - ((\vec{a}\times\vec{c})\cdot\vec{b})\vec{c}$

$= [\vec{a}\,\vec{b}\,\vec{c}](\vec{b} + \vec{c})$

$\Rightarrow ((\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c}))\times(\vec{b}\times\vec{c}))\cdot(\vec{b} - \vec{c})$

$= [\vec{a}\,\vec{b}\,\vec{c}](\vec{b} + \vec{c}).(\vec{b} - \vec{c})$

$= [\vec{a}\,\vec{b}\,\vec{c}](|\vec{b}|^2 - |c|^2) = 0$

72. **a.** $\vec{a}\times\vec{b} = \vec{c}$

$\Rightarrow \vec{a}\times(\vec{a}\times\vec{b}) = \vec{a}\times\vec{c}$

$\Rightarrow (\vec{a}\cdot\vec{b})\vec{a} - |\vec{a}|^2\vec{b} = \vec{a}\times\vec{c}$

$\Rightarrow \vec{b} = \dfrac{\beta\vec{a} - \vec{a}\times\vec{c}}{|\vec{a}|^2} \qquad (\because \vec{a}\cdot\vec{b} = \beta)$

73. **b.** Taking dot product of $a(\vec{\alpha}\times\vec{\beta}) + b(\vec{\beta}\times\vec{\gamma}) + c(\vec{\gamma}\times\vec{\alpha}) = 0$ with $\vec{\gamma}, \vec{\alpha}$ and $\vec{\beta}$, respectively, we have

$a[\vec{\alpha}\vec{\beta}\vec{\gamma}] = 0$

$b[\vec{\alpha}\vec{\beta}\vec{\gamma}] = 0$

$c[\vec{\alpha}\vec{\beta}\vec{\gamma}] = 0$

$\because$ At least one of a, b and $c \neq 0$

$\therefore\ [\vec{\alpha}\vec{\beta}\vec{\gamma}]=0$

Hence $\vec{\alpha}$, $\vec{\beta}$ and $\vec{\gamma}$ are coplanar.

74. **c.** $(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})=\vec{b}$

$\Rightarrow\ [\vec{a}\,\vec{b}\,\vec{c}]\vec{b}=\vec{b}$

$\Rightarrow [\vec{a}\,\vec{b}\,\vec{c}]=1$

$\therefore\ \vec{a}, \vec{b}$ and $\vec{c}$ cannot be coplanar.

75. **c.** Any vector $\vec{r}$ can be represented in terms of three non-coplanar vectors $\vec{a}, \vec{b}$ and $\vec{c}$ as

$$\vec{r}=x(\vec{a}\times\vec{b})+y(\vec{b}\times\vec{c})+z(\vec{c}\times\vec{a}) \quad \text{(i)}$$

Taking dot product with $\vec{a}, \vec{b}$ and $\vec{c}$, respectively, we have,

$$x=\frac{\vec{r}\cdot\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]},\ y=\frac{\vec{r}\cdot\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]} \text{ and } z=\frac{\vec{r}\cdot\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

From (i)

$$[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}=\frac{1}{2}(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})$$

$\therefore$ Area of ΔABC

$=\frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$

$=|[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}|$

76. **a.** Differentiate the curve

$6x+8(xy_1+y)+4yy_1=0$

m_T at $(1,0)$ is $6+8(y_1(0))=0$

$y_1(0)=-\frac{3}{4}$

$m_N=\frac{4}{3}$

Unit vector $=\pm\dfrac{(3\hat{i}+4\hat{j})}{5}$

Again normal vector of magnitude $10=\pm(6\hat{i}+8\hat{j})$

77 **a.** $\{\vec{a}\times(\vec{b}+\vec{a}\times\vec{b})\}\cdot\vec{b}$

$=\{\vec{a}\times\vec{b}+\vec{a}\times(\vec{a}\times\vec{b})\}\cdot\vec{b}$

$=[\vec{a}\,\vec{b}\,\vec{b}]+\{(\vec{a}\cdot\vec{b})\vec{a}-(\vec{a}\cdot\vec{a})\vec{b}\}\cdot\vec{b}$

$=0+(\vec{a}\cdot\vec{b})^2-(\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b})$

$=\cos^2\frac{\pi}{3}-1=-\frac{3}{4}$

78. a. $\vec{r} \times \vec{a} = \lambda \vec{a} + \mu \vec{b} + \gamma \vec{a} \times \vec{b}$

$\therefore [\vec{r}\,\vec{a}\,\vec{a}] = \lambda \vec{a} \cdot \vec{a} + \mu \vec{b} \cdot \vec{a} + \gamma [\vec{a}\,\vec{b}\,\vec{a}]$

$0 = \lambda |\vec{a}|^2 + 0 + 0$

$\lambda = 0$

Also $[\vec{r}\,\vec{a}\,\vec{b}] = \lambda \vec{a} \cdot \vec{b} + \mu \vec{b} \cdot \vec{b} + \gamma [\vec{a}\,\vec{b}\,\vec{b}] = \mu$

Also $(\vec{r} \times \vec{a}) \times \vec{b} = \gamma (\vec{a} \times \vec{b}) \times \vec{b}$

$\Rightarrow (\vec{r} \cdot \vec{b}) \vec{a} - (\vec{a} \cdot \vec{b}) \vec{r} = \gamma \{(\vec{a} \cdot \vec{b}) \vec{b} - (\vec{b} \cdot \vec{b}) \vec{a}\}$

$\Rightarrow (\vec{r} \cdot \vec{b}) \vec{a} = -\gamma \vec{a}, \quad \gamma = -(\vec{r} \cdot \vec{b})$

79. c. The given equation reduces to $[\vec{a}\;\vec{b}\;\vec{c}]^2 x^2 + 2[\vec{a}\;\vec{b}\;\vec{c}]x + 1 = 0 \Rightarrow D = 0$

80. b. $\vec{x} + \vec{c} \times \vec{y} = \vec{a}$ (i)

$\vec{y} + \vec{c} \times \vec{x} = \vec{b}$ (ii)

Taking cross with $\vec{c}$

$\vec{c} \times \vec{y} + \vec{c} \times (\vec{c} \times \vec{x}) = \vec{c} \times \vec{b}$

$\Rightarrow (\vec{a} - \vec{x}) + (\vec{c} \cdot \vec{x})\vec{c} - (\vec{c} \cdot \vec{c})\vec{x} = \vec{c} \times \vec{b}$

Also $\vec{x} + \vec{c} \times \vec{y} = \vec{a}$

$\Rightarrow \quad \vec{c} \cdot \vec{x} + \vec{c} \cdot (\vec{c} \times \vec{y}) = \vec{c} \cdot \vec{a}$

$\Rightarrow \quad \vec{c} \cdot \vec{x} + 0 = \vec{c} \cdot \vec{a}$

$\therefore \quad \vec{c} \cdot \vec{x} = \vec{c} \cdot \vec{a}$

$\Rightarrow \vec{a} - \vec{x} + (\vec{c} \cdot \vec{a})\vec{c} - (\vec{c} \cdot \vec{c})\vec{x} = \vec{c} \times \vec{b}$

$\Rightarrow \vec{x}(1 + (\vec{c} \cdot \vec{c})) = \vec{b} \times \vec{c} + \vec{a} + (\vec{c} \cdot \vec{a}) \cdot \vec{c}$

$$\therefore \; \vec{x} = \frac{\vec{b} \times \vec{c} + \vec{a} + (\vec{c} \cdot \vec{a})\vec{c}}{1 + \vec{c} \cdot \vec{c}}$$

Similarly on taking cross product of Eq. (i), we find

$$\vec{y} = \frac{\vec{a} \times \vec{c} + \vec{b} + (\vec{c} \cdot \vec{b})\vec{c}}{1 + \vec{c} \cdot \vec{c}}$$

81. c. $\vec{r} \times \vec{a} = \vec{b}$

$\Rightarrow \vec{d} \times (\vec{r} \times \vec{a}) = \vec{d} \times \vec{b}$

$\Rightarrow (\vec{a} \cdot \vec{d}) \vec{r} - (\vec{d} \cdot \vec{r}) \vec{a} = \vec{d} \times \vec{b}$ (i)

$\vec{r} \times \vec{c} = \vec{d}$

$\Rightarrow \vec{b} \times (\vec{r} \times \vec{c}) = \vec{b} \times \vec{d}$

$\Rightarrow (\vec{b}\cdot\vec{c})\vec{r} - (\vec{b}\cdot\vec{r})\vec{c} = \vec{b}\times\vec{d}$ (ii)

Adding (i) and (ii) we get,

$(\vec{a}\cdot\vec{d} + \vec{b}\cdot\vec{c})\vec{r} - (\vec{d}\cdot\vec{r})\vec{a} - (\vec{b}\cdot\vec{r})\vec{c} = \vec{0}$

Now $\vec{r}\cdot\vec{d} = 0$ and $\vec{b}\cdot\vec{r} = 0$ as $\vec{d}$ and $\vec{r}$ as well as $\vec{b}$ and $\vec{r}$ are mutually perpendicular.

Hence, $(\vec{b}\cdot\vec{c} + \vec{a}\cdot\vec{d})\vec{r} = \vec{0}$

82. **b.** Let $\vec{a}\times\vec{b} = x\hat{i} + y\hat{j} + z\hat{k}$. Therefore,

$[\vec{a}\,\vec{b}\,\hat{i}] = (\vec{a}\times\vec{b})\cdot\hat{i} = x$

$[\vec{a}\,\vec{b}\,\hat{j}] = (\vec{a}\times\vec{b})\cdot\hat{j} = y$

$[\vec{a}\,\vec{b}\,\vec{k}] = (\vec{a}\times\vec{b})\cdot\hat{k} = z$

Hence, $[\vec{a}\,\vec{b}\,\hat{i}]\hat{i} + [\vec{a}\,\vec{b}\,\hat{j}]\hat{j} = [\vec{a}\,\vec{b}\,\hat{k}]\hat{k} = x\hat{i} + y\hat{j} + z\hat{k} = \vec{a}\times\vec{b}$

83. **a.** $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = 5(\hat{i} + 2\hat{j} + 2\hat{k}) - 6(\hat{i} + \hat{j} + 2\hat{k})$

$\Rightarrow (1+\alpha)\hat{i} + \beta(1+\alpha)\hat{j} + \gamma(1+\alpha)(1+\beta)\hat{k} = -\hat{i} + 4\hat{j} - 2\hat{k}$

$\Rightarrow 1+\alpha = -1, \beta = -4$ and $\gamma(-1)(-3) = -2$

$\Rightarrow \gamma = -\frac{2}{3}$

84. **b.** If $\vec{a}(x)$ and $\vec{b}(x)$ are $\perp$, then $\vec{a}\cdot\vec{b} = 0$

$\Rightarrow \sin x\cos 2x + \cos x\sin 2x = 0$

$\sin(3x) = 0 = \sin 0$

$3x = n\pi \quad \Rightarrow x = \frac{n\pi}{3}$

Therefore, the two vectors are $\perp$ for infinite values of 'x'.

85. **b.** $(\vec{a}\times\hat{i})\cdot(\vec{b}\times\hat{i}) = \begin{vmatrix} \vec{a}\cdot\vec{b} & \vec{a}\cdot\hat{i} \\ \vec{b}\cdot\hat{i} & \hat{i}\cdot\hat{i} \end{vmatrix} = (\vec{a}\cdot\vec{b}) - (\vec{a}\cdot\hat{i})(\vec{b}\cdot\hat{i})$

Similarly, $(\vec{a}\times\hat{j})\cdot(\vec{b}\times\hat{j}) = (\vec{a}\cdot\vec{b}) - (\vec{a}\cdot\hat{j})(\vec{b}\cdot\hat{j})$

and $(\vec{a}\times\hat{k})\cdot(\vec{b}\times\hat{k}) = \vec{a}\cdot\vec{b} - (\vec{a}\cdot\hat{k})(\vec{b}\cdot\hat{k})$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. Therefore,

$(\vec{a}\cdot\hat{i}) = a_1$, $\vec{a}\cdot\hat{j} = a_2$, $\vec{a}\cdot\hat{k} = a_3$, $\vec{b}\cdot\hat{i} = b_1$, $\vec{b}\cdot\hat{j} = b_2$, $(\vec{b}\cdot\hat{k}) = b_3$

$\Rightarrow (\vec{a}\times\hat{i})\cdot(\vec{b}\times\hat{i}) + (\vec{a}\times\hat{j})\cdot(\vec{b}\times\hat{j}) + (\vec{a}\times\hat{k})\cdot(\vec{b}\times\hat{k})$

$= 3\vec{a}\cdot\vec{b} - (a_1b_1 + a_2b_2 + a_3b_3)$

$= 3\vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{b} = 2\vec{a}\cdot\vec{b}$

86. **b.** $(\vec{a}\times\vec{b})\times(\vec{r}\times\vec{c})=((\vec{a}\times\vec{b})\cdot\vec{c})\vec{r}-((\vec{a}\times\vec{b})\cdot\vec{r})\cdot\vec{c}=[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}-[\vec{a}\,\vec{b}\,\vec{r}]\vec{c}$

Similarly, $(\vec{b}\times\vec{c})\times(\vec{r}\times\vec{a})=[\vec{b}\,\vec{c}\,\vec{a}]\vec{r}-[\vec{b}\,\vec{c}\,\vec{r}]\vec{a}$

and, $(\vec{c}\times\vec{a})\times(\vec{r}\times\vec{b})=[\vec{c}\,\vec{a}\,\vec{b}]\vec{r}-[\vec{c}\,\vec{a}\,\vec{r}]\vec{b}$

$\Rightarrow (\vec{a}\times\vec{b})\times(\vec{r}\times\vec{c})+(\vec{b}\times\vec{c})\times(\vec{r}\times\vec{a})+(\vec{c}\times\vec{a})\times(\vec{r}\times\vec{b})$

$= 3[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}-([\vec{b}\,\vec{c}\,\vec{r}]\vec{a}+[\vec{c}\,\vec{a}\,\vec{r}]\vec{b}+[\vec{a}\,\vec{b}\,\vec{r}]\vec{c})$

$= 3[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}-[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}=2[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}$

87. **a.** We have,

$$\vec{a}\cdot\vec{p}=\vec{a}\cdot\frac{(\vec{b}\times\vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]}=\frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]}=\frac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}=1$$

$$\vec{a}.\vec{q}=\vec{a}\cdot\frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}=\frac{[\vec{a}\,\vec{c}\,\vec{a}]}{[\vec{a}\,\vec{b}\,\vec{c}]}=0$$

Similarly, $\vec{a}\cdot\vec{r}=0,\ \vec{b}\cdot\vec{p}=0, \vec{b}\cdot\vec{q}=1, \vec{b}\cdot\vec{r}=0, \vec{c}\cdot\vec{p}=0,\ \vec{c}\cdot\vec{q}=0$ and $\vec{c}\cdot\vec{r}=1$

$\therefore\ (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{p}+\vec{q}+\vec{r}) =\vec{a}\cdot\vec{p}+\vec{a}\cdot\vec{q}+\vec{a}\cdot\vec{r}+\vec{b}\cdot\vec{p}+\vec{b}\cdot\vec{q}+\vec{b}\cdot\vec{r}+\vec{c}\cdot\vec{p}+\vec{c}\cdot\vec{q}+\vec{c}\cdot\vec{r}$

$=1+1+1=3$

88. **b.** A vector perpendicular to the plane of $A(\vec{a})$, $B(\vec{b})$ and $C(\vec{c})$ is

$(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})=\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}.$

Now for any point $R(\vec{r})$ in the plane of A, B and C is

$(\vec{r}-\vec{a})\cdot(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})=0.$

$\vec{r}\cdot(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})-\vec{a}\cdot(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})=0$

$\vec{r}\cdot(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})=0+\vec{a}\cdot\vec{b}\times\vec{c}+\vec{0}$

$\vec{r}\cdot(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})=[\vec{a}\,\vec{b}\,\vec{c}]$

89. **c.** Given that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar.

$\Rightarrow [\vec{a}\,\vec{b}\,\vec{c}]\neq 0$ (i)

Again $\vec{a}\times(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{c})=0$

$\Rightarrow [(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}]\cdot(\vec{a}\times\vec{c})=0$

$\Rightarrow (\vec{a}\cdot\vec{c})[\vec{b}\,\vec{a}\,\vec{c}]=0$

$\Rightarrow (\vec{a}\cdot\vec{c})=0$

$\Rightarrow \vec{a}$ and $\vec{c}$ are perpendicular. (ii)

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} \Rightarrow [\vec{a}\times(\vec{b}\times\vec{c})]\times\vec{c} = \vec{0}$

90. **c.** Consider a tetrahedron with vertices $O(0, 0, 0)$, $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Its volume $V = \frac{1}{6}[\vec{a}\,\vec{b}\,\vec{c}]$

Now centroids of the faces OAB, OAC, OBC and ABC are $G_1(a/3, b/3, 0)$, $G_2(a/3, 0, c/3)$, $G_3(0, b/3, c/3)$ and $G_4(a/3, b/3, c/3)$, respectively.

$G_4G_1 = \vec{c}/3,\ \overrightarrow{G_4G_2} = \vec{b}/3,\ \overrightarrow{G_4G_3} = \vec{a}/3$.

Volume of tetrahedron by centroids $V' = \frac{1}{6}\begin{bmatrix}\frac{\vec{a}}{3} & \frac{\vec{b}}{3} & \frac{\vec{c}}{3}\end{bmatrix} = \frac{1}{27}V$

$\Rightarrow K = 27$

91. **c.** $[(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})\quad(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\quad(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})]$

$= [[\vec{a}\,\vec{b}\,\vec{c}]\vec{b}\quad[\vec{a}\,\vec{b}\,\vec{c}]\,\vec{c}\quad[\vec{a}\,\vec{b}\,\vec{c}]\cdot\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^3\,[\vec{b}\,\vec{c}\,\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^4$

92. **d.** $\vec{r} = x_1(\vec{a}\times\vec{b}) + x_2(\vec{b}\times\vec{c}) + x_3(\vec{c}\times\vec{a})$

$\Rightarrow \vec{r}\cdot\vec{a} = x_2[\vec{a}\,\vec{b}\,\vec{c}],\ \vec{r}\cdot\vec{b} = x_3[\vec{b}\,\vec{c}\,\vec{a}]$

and $\vec{r}\cdot\vec{c} = x_1[\vec{c}\,\vec{a}\,\vec{b}] = x_1[\vec{a}\,\vec{b}\,\vec{c}]$

$\Rightarrow x_1 + x_2 + x_3 = 4\vec{r}.(\vec{a} + \vec{b} + \vec{c})$

93. **a.** Let $\vec{v} = x\vec{a} + y\vec{b} + z\vec{a}\times\vec{b}$

Given: $\vec{a}\cdot\vec{b} = 0$, $\vec{v}\cdot\vec{a} = 0$, $\vec{v}\cdot\vec{b} = 1$, $[\vec{v}\ \vec{a}\ \vec{b}] = 1$

$\Rightarrow \vec{v}\cdot\vec{a} = x\,\vec{a}\cdot\vec{a} = x|\vec{a}|^2 \quad (\because \vec{a}\cdot\vec{b} = 0, \vec{a}\cdot\vec{a}\times\vec{b} = 0)$

$\Rightarrow x = 0$

Again, $\vec{v}\cdot\vec{b} = y|\vec{b}|^2 \Rightarrow 1 = yb^2$

$\therefore y = \frac{1}{b^2}$ (ii)

Again, $\vec{v}\cdot(\vec{a}\times\vec{b}) = z(\vec{a}\times\vec{b})^2$

$\Rightarrow 1 = z(\vec{a}\times\vec{b})^2 \Rightarrow z = \frac{1}{|\vec{a}\times\vec{b}|^2}$

Hence, $\vec{v} = \frac{1}{|\vec{b}|^2}\vec{b} + \frac{1}{|\vec{a}\times\vec{b}|^2}\vec{a}\times\vec{b}$

94. **d.** Volume of the parallelepiped formed by $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ is 4.

Therefore, the volume of the parallelepiped formed by $\vec{a}$, $\vec{b}$ and $\vec{c}$ is $\frac{1}{4}$.

$\vec{b} \times \vec{c} = [\vec{a}\,\vec{b}\,\vec{c}]\,\vec{a}' = \frac{1}{4}\vec{a}'$

$|\vec{b} \times \vec{c}| = \frac{\sqrt{2}}{4} = \frac{1}{2\sqrt{2}}$

Length of altitude $= \frac{1}{4} \times 2\sqrt{2} = \frac{1}{\sqrt{2}}$

95. **d.** $\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]} = \frac{\hat{i} + \hat{j} - \hat{k}}{2}$

Multiple Correct Answers Type

1. **a., b.** We have, $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2(\vec{a}\cdot\vec{b})$

$\Rightarrow |\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos 2\theta$

$\Rightarrow |\vec{a} - \vec{b}|^2 = 2 - 2\cos 2\theta \quad (\because |\vec{a}| = |\vec{b}| = 1)$

$\Rightarrow |\vec{a} - \vec{b}|^2 = 4\sin^2\theta$

$\Rightarrow |\vec{a} - \vec{b}| = 2|\sin\theta|$

Now, $|\vec{a} - \vec{b}| < 1$

$\Rightarrow 2|\sin\theta| < 1$

$\Rightarrow |\sin\theta| < \frac{1}{2}$

$\Rightarrow \theta \in [0, \pi/6)$ or $\theta \in (5\pi/6, \pi]$

2. **a., c.** $\vec{a} \times (\vec{b} \times \vec{c}) + (\vec{a}\cdot\vec{b})\vec{b} = (4 - 2x - \sin y)\vec{b} + (x^2 - 1)\vec{c}$

$\Rightarrow (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} + (\vec{a}\cdot\vec{b})\vec{b} = (4 - 2x - \sin y)\vec{b} + (x^2 - 1)\vec{c}$

Now, $(\vec{c}\cdot\vec{c})\vec{a} = \vec{c}$. Therefore,

$(\vec{c}\cdot\vec{c})(\vec{a}\cdot\vec{c}) = (\vec{c}\cdot\vec{c}) \Rightarrow \vec{a}\cdot\vec{c} = 1$

$\Rightarrow 1 + \vec{a}\cdot\vec{b} = 4 - 2x - \sin y,\ x^2 - 1 = -(\vec{a}\cdot\vec{b})$

$\Rightarrow 1 = 4 - 2x - \sin y + x^2 - 1$

$\Rightarrow \sin y = x^2 - 2x + 2 = (x - 1)^2 + 1$

But $\sin y \le 1 \Rightarrow x = 1, \sin y = 1$

$\Rightarrow y = (4n + 1)\frac{\pi}{2}, \quad n \in I$

3. **a., b., c., d.** Since $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors inclined at an angle θ.

$|\vec{a}| = |\vec{b}| = 1$ and $\cos\theta = \vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c}$

Now, $\vec{c} = \alpha\vec{a} + \beta\vec{b} + \gamma(\vec{a}\times\vec{b})$ (i)

$\Rightarrow \vec{a}\cdot\vec{c} = \alpha(\vec{a}\cdot\vec{a}) + \beta(\vec{a}\cdot\vec{b}) + \gamma\{\vec{a}\cdot(\vec{a}\times\vec{b})\}$

$\Rightarrow \cos\theta = \alpha|\vec{a}|^2 \quad (\because \vec{a}\cdot\vec{b} = 0,\ \vec{a}\cdot(\vec{a}\times\vec{b}) = 0)$

$\Rightarrow \cos\theta = \alpha$

Similarly, by taking dot product on both sides of (i) by $\vec{b}$, we get $\beta = \cos\theta$

$\therefore \alpha = \beta$

Again, $\vec{c} = \alpha\vec{a} + \beta\vec{b} + \gamma(\vec{a}\times\vec{b})$

$\Rightarrow |\vec{c}|^2 = |\alpha\vec{a} + \beta\vec{b} + \gamma(\vec{a}\times\vec{b})|^2$

$= \alpha^2|\vec{a}|^2 + \beta^2|\vec{b}|^2 + \gamma^2|\vec{a}\times\vec{b}|^2 + 2\alpha\beta(\vec{a}\cdot\vec{b}) + 2\alpha\gamma\{\vec{a}\cdot(\vec{a}\times\vec{b})\} + 2\beta\gamma(\vec{b}\cdot\{\vec{a}\times\vec{b}\})$

$\Rightarrow 1 = \alpha^2 + \beta^2 + \gamma^2|\vec{a}\times\vec{b}|^2$

$\Rightarrow 1 = 2\alpha^2 + \gamma^2\{|\vec{a}|^2|\vec{b}|^2\sin^2\pi/2\}$

$\Rightarrow 1 = 2\alpha^2 + \gamma^2 \Rightarrow \alpha^2 = \dfrac{1-\gamma^2}{2}$

But $\alpha = \beta = \cos\theta$.

$1 = 2\alpha^2 + \gamma^2 \Rightarrow \gamma^2 = 1 - 2\cos^2\theta = -\cos 2\theta$

$\therefore \beta^2 = \dfrac{1-\gamma^2}{2} = \dfrac{1+\cos 2\theta}{2}$

4. **a., b., c.** We have,

AM = projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|}$

$\therefore \overrightarrow{AM} = \left(\dfrac{\vec{a}.\vec{b}}{|\vec{a}|^2}\right)\vec{a}$

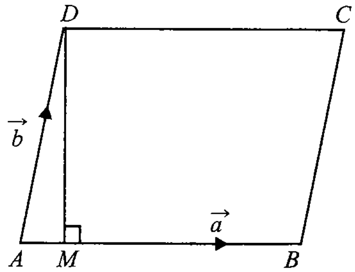


Fig. 2.42

Now, in ΔADM

$\overrightarrow{AD} = \overrightarrow{AM} + \overrightarrow{MD} \Rightarrow \overrightarrow{DM} = \overrightarrow{AM} - \overrightarrow{AD}$

$\Rightarrow \overrightarrow{DM} = \dfrac{(\vec{a}\cdot\vec{b})\vec{a}}{|\vec{a}|^2} - \vec{b}$

Also, $\overrightarrow{DM} = \dfrac{1}{|\vec{a}|^2}[(\vec{a}\cdot\vec{b})\vec{a} - |\vec{a}|^2\vec{b}]$

$\Rightarrow \overrightarrow{MD} = \frac{1}{|\vec{a}|^2}[|\vec{a}|^2 \vec{b} - (\vec{a}\cdot\vec{b})\vec{a}]$

Now, $\frac{\vec{a}\times(\vec{a}\times\vec{b})}{|\vec{a}|^2} = \frac{1}{|\vec{a}|^2}[(\vec{a}\cdot\vec{b})\vec{a} - (\vec{a}\cdot\vec{a})\vec{b}] = \overrightarrow{DM}$

5. **a., c.** $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$ and $(\vec{a}\times\vec{b})\times\vec{c} = -(\vec{c}\cdot\vec{b})\vec{a} + (\vec{a}\cdot\vec{c})\vec{b}$

We have been given $(\vec{a}\times(\vec{b}\times\vec{c}))\cdot((\vec{a}\times\vec{b})\times\vec{c}) = 0$. Therefore,

$((\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c})\cdot((\vec{a}\cdot\vec{c})\vec{b} - (\vec{c}\cdot\vec{b})\vec{a}) = 0$

$\Rightarrow (\vec{a}\cdot\vec{c})^2|\vec{b}|^2 - (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{b}) - (\vec{a}\cdot\vec{b})(\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{c}) + (\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})(\vec{c}\cdot\vec{a}) = 0$

$\Rightarrow (\vec{a}\cdot\vec{c})^2|\vec{b}|^2 = (\vec{a}\cdot\vec{c})(\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})$

$\Rightarrow (\vec{a}\cdot\vec{c})((\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{b}) - (\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})) = 0$

$\vec{a}\cdot\vec{c} = 0$ or $(\vec{a}\cdot\vec{c})|\vec{b}|^2 = (\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})$

6. **a., c.** We have $[\vec{p}\ \vec{q}\ \vec{r}] = \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}$. Therefore,

$[\vec{p}\ \vec{q}\ \vec{r}] > 0$

a. $x > 0, x[\vec{a}\ \vec{b}\ \vec{c}] + \frac{[\vec{p}\ \vec{q}\ \vec{r}]}{x} \geq 2$ (using A.M. $\geq$ G.M.)

b. Similarly, use A.M. $\geq$ G.M.

7. **a., b., c., d.** $a_1 + a_2\cos 2x + a_3\sin^2 x = 0\ \forall\ x \in R$

$\Rightarrow (a_1 + a_2) + \sin^2 x(a_3 - 2a_2) = 0$

$\Rightarrow a_1 + a_2 = 0$ and $a_3 - 2a_2 = 0$

$\frac{a_1}{-1} = \frac{a_2}{1} = \frac{a_3}{2} = \lambda(\neq 0)$

$\Rightarrow a_1 = -\lambda, a_2 = \lambda, a_3 = 2\lambda$

8. **a., b., c., d.** $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\ \hat{n}$

$\Rightarrow |\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

$\Rightarrow \sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$ (i)

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow \cos\theta = \frac{|\vec{a}\cdot\vec{b}|}{|\vec{a}||\vec{b}|}$ (ii)

From (i) and (ii),

$\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow |\vec{a}\times\vec{b}|^2 + (\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$

If $\theta = \pi/4$, then $\sin\theta = \cos\theta = 1/\sqrt{2}$. Therefore,

$$|\vec{a}\times\vec{b}| = \frac{|\vec{a}||\vec{b}|}{\sqrt{2}} \text{ and } \vec{a}\cdot\vec{b} = \frac{|\vec{a}||\vec{b}|}{\sqrt{2}}$$

$$|\vec{a}\times\vec{b}| = \vec{a}\cdot\vec{b}$$

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n} = \frac{|\vec{a}||\vec{b}|}{\sqrt{2}}\hat{n}$$

$$\vec{a}\times\vec{b} = (\vec{a}\cdot\vec{b})\hat{n}$$

9. **a., b., c., d.** Since $\vec{a}, \vec{b}$ and $\vec{a}\times\vec{b}$ are non-coplanar,

$$\vec{r} = x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b})$$

$$\therefore\ \vec{r}\times\vec{b} = \vec{a} \Rightarrow x\vec{a}\times\vec{b} + z\{(\vec{a}\cdot\vec{b})\vec{b} - (\vec{b}\cdot\vec{b})\vec{a}\} = \vec{a}$$

$$\Rightarrow -(1+z|\vec{b}|^2)\vec{a} + x\vec{a}\times\vec{b} = 0 \quad (\text{since } \vec{a}\cdot\vec{b} = 0)$$

$$\therefore\ x = 0 \text{ and } z = -\frac{1}{|\vec{b}|^2}$$

Thus, $\vec{r} = y\vec{b} - \dfrac{\vec{a}\times\vec{b}}{|\vec{b}|^2}$, where y is the parameter.

10. **b., d.** Since $\vec{a} = (1, 3, \sin 2\alpha)$ makes an obtuse angle with the z-axis, its z-component is negative.

$\Rightarrow -1 \le \sin 2\alpha < 0$ (i)

But $\vec{b}\cdot\vec{c} = 0$ ($\because$ orthogonal)

$\tan^2\alpha - \tan\alpha - 6 = 0$

$\therefore$ $(\tan\alpha - 3)(\tan\alpha + 2) = 0$

$\Rightarrow \tan\alpha = 3, -2$

Now, $\tan\alpha = 3$. Therefore,

$$\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha} = \frac{6}{1+9} = \frac{3}{5} \text{ (not possible as } \sin 2\alpha < 0)$$

Now, if $\tan\alpha = -2$,

$$\Rightarrow \sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha} = \frac{-4}{1+4} = \frac{-4}{5}$$

$\Rightarrow \tan 2\alpha > 0$

$\Rightarrow 2\alpha$ is the third quadrant. Also, $\sqrt{\sin\alpha/2}$ is meaningful. If $0 < \sin\alpha/2 < 1$, then $\alpha = (4n+1)\pi - \tan^{-1}2$ and $\alpha = (4n+2)\pi - \tan^{-1}2$.

11. **b., d.** $\vec{a}\times(\vec{r}\times\vec{a})=\vec{a}\times\vec{b}$

$3\vec{r}-(\vec{a}\cdot\vec{r})\,\vec{a} = \vec{a}\times\vec{b}$

Also $|\vec{r}\times\vec{a}| = |\vec{b}|$

$\Rightarrow \sin^2\theta = \dfrac{2}{3}$

$\Rightarrow (1-\cos^2\theta) = \dfrac{2}{3}$

$\Rightarrow \dfrac{1}{3} = \cos^2\theta$

$\Rightarrow \vec{a}.\vec{r} = \pm 1$

$\Rightarrow 3\vec{r}\pm\vec{a} = \vec{a}\times\vec{b}$

$\Rightarrow \vec{r} = \dfrac{1}{3}(\vec{a}\times\vec{b}\pm\vec{a})$

12. **b., d.** $(\vec{a}-\vec{b})\times[(\vec{b}+\vec{a})\times(2\vec{a}+\vec{b})] = \vec{b}+\vec{a}$

$\Rightarrow \{(\vec{a}-\vec{b})\cdot(2\vec{a}+\vec{b})\}(\vec{b}+\vec{a}) - \{(\vec{a}-\vec{b})\cdot(\vec{b}+\vec{a})\}(2\vec{a}+\vec{b}) = \vec{b}+\vec{a}$

$\Rightarrow (2-\vec{a}.\vec{b}-1)(\vec{b}+\vec{a}) = \vec{b}+\vec{a}$

$\Rightarrow$ either $\vec{b}+\vec{a} = \vec{0}$ or $1-\vec{a}\cdot\vec{b} = 1$

$\Rightarrow$ either $\vec{b} = -\vec{a}$ or $\vec{a}\cdot\vec{b} = 0$

$\Rightarrow$ either $\theta = \pi$ or $\theta = \pi/2$

13. **a., d.** Given $\vec{c} = \lambda_1\vec{a}+\lambda_2\vec{b}+\lambda_3(\vec{a}\times\vec{b})$ (i)

and $\vec{a}\cdot\vec{b} = 0$, $|\vec{a}| = 1$, $|\vec{b}| = 1$

From (i), $\vec{a}\cdot\vec{c} = \lambda_1$, $\vec{c}\cdot\vec{b} = \lambda_2$

and $\vec{c}\cdot(\vec{a}\times\vec{b}) = |\vec{a}\times\vec{b}|^2\,\lambda_3$

$= (1.1\sin 90°)^2\,\lambda_3 = \lambda_3$

Hence $\lambda_1+\lambda_2+\lambda_3 = (\vec{a}+\vec{b}+\vec{a}\times\vec{b})\cdot\vec{c}$

14. **b., c., d.** Obviously, $\dfrac{\vec{a}}{|\vec{a}|}+\dfrac{\vec{b}}{|\vec{b}|}$ is a vector in the plane of $\vec{a}$ and $\vec{b}$ and hence perpendicular to $\vec{a}\times\vec{b}$. It is also equally inclined to $\vec{a}$ and $\vec{b}$ as it is along the angle bisector.

15. a., d. $|\vec{a}+\vec{b}|=|\vec{a}-2\vec{b}|$

$\Rightarrow \vec{a}\cdot\vec{b}=\dfrac{|\vec{b}|^2}{2}$

Also $\vec{a}\cdot\vec{b}+\dfrac{1}{|\vec{b}|^2+2}$

$=\dfrac{|\vec{b}|^2+2}{2}+\dfrac{1}{|\vec{b}|^2+2}-1$

$\geq\sqrt{2}-1$ (using A.M. $\geq$ G.M.)

16. b., d. $\vec{V_1}=\vec{V_2}$

$\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\times\vec{c}$

$\Rightarrow (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}$

$\Rightarrow (\vec{a}\cdot\vec{b})\vec{c}=(\vec{b}\cdot\vec{c})\vec{a}$

$\Rightarrow$ either $\vec{c}$ and $\vec{a}$ are collinear or $\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{c}\Rightarrow\vec{b}=\lambda(\vec{a}\times\vec{c})$

17. b., c. We have $\vec{A}+\vec{B}=\vec{a}$

$\Rightarrow \vec{A}\cdot\vec{a}+\vec{B}\cdot\vec{a}=\vec{a}\cdot\vec{a}$

$\Rightarrow 1+\vec{B}\cdot\vec{a}=a^2$ (given $\vec{A}\cdot\vec{a}=1$)

$\Rightarrow \vec{B}\cdot\vec{a}=a^2-1$ (i)

Also $\vec{A}\times\vec{B}=\vec{b}$

$\Rightarrow \vec{a}\times(\vec{A}\times\vec{B})=\vec{a}\times\vec{b}$

$\Rightarrow (\vec{a}\cdot\vec{B})\vec{A}-(\vec{a}\cdot\vec{A})\vec{B}=\vec{a}\times\vec{b}$

$\Rightarrow (a^2-1)\vec{A}-\vec{B}=\vec{a}\times\vec{b}$ (using (i) and $\vec{a}\cdot\vec{A}=1$) (ii)

and $\vec{A}+\vec{B}=\vec{a}$ (iii)

From (ii) and (iii)

$\vec{A}=\dfrac{(\vec{a}\times\vec{b})+\vec{a}}{a^2}$

$$\vec{B} = \vec{a} - \left\{\frac{(\vec{a} \times \vec{b}) + \vec{a}}{a^2}\right\}$$

or $\vec{B} = \dfrac{(\vec{b} \times \vec{a}) + \vec{a}(a^2 - 1)}{a^2}$

Thus $\vec{A} = \dfrac{(\vec{a} \times \vec{b}) + \vec{a}}{a^2}$ and $\vec{B} = \dfrac{(\vec{b} \times \vec{a}) + \vec{a}(a^2 - 1)}{a^2}$

18. **c., d.** Since $[\vec{a}\,\vec{b}\,\vec{c}\,] = 0$, $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar vectors.

Further, since $\vec{d}$ is equally inclined to $\vec{a}, \vec{b}$ and $\vec{c}$,

$$\vec{d}\cdot\vec{a} = \vec{d}\cdot\vec{b} = \vec{d}\cdot\vec{c} = 0$$

$$\vec{d}\cdot\vec{x} = \vec{d}\cdot\vec{y} = \vec{d}\cdot\vec{z} = 0$$

$$\vec{d}\cdot\vec{r} = 0$$

19. **b., d.** Let $\vec{\alpha} = \hat{i} - \hat{j} - \hat{k}$, $\vec{\beta} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{\gamma} = -\hat{i} + \hat{j} + \hat{k}$.

Let required vector $\vec{a} = x\hat{i} + y\hat{j} + z\hat{j}$.

$\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are coplanar

$$\Rightarrow \begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 0 \Rightarrow y = z$$

Also, $\vec{a}$ and $\vec{\alpha}$ are perpendicular

$\Rightarrow$ $x - y - z = 0$

$\Rightarrow$ $x = zy$

$\Rightarrow$ Options b and d are correct.

20. **b., d.**

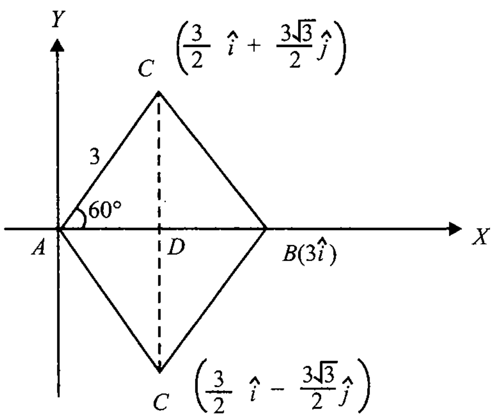


Fig. 2.43

21. **a., b., c.** Consider $\vec{V}_1 \cdot \vec{V}_2 = 0 \Rightarrow A = 90°$

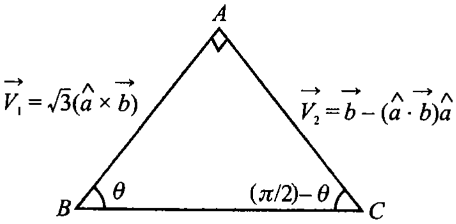


Fig. 2.44

Using the sine law, $\left|\dfrac{\vec{b} - (\hat{a} \cdot \vec{b})\hat{a}}{\sin\theta}\right| = \dfrac{\sqrt{3}\,|\hat{a} \times \vec{b}|}{\cos\theta}$

$$\Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \frac{|\vec{b} - (\hat{a} \cdot \vec{b})\hat{a}|}{|\hat{a} \times \vec{b}|}$$

$$= \frac{1}{\sqrt{3}} \frac{|(\hat{a} \times \vec{b}) \times \hat{a}|}{|\hat{a} \times \vec{b}|}$$

$$= \frac{1}{\sqrt{3}} \frac{|\hat{a} \times \vec{b}||\hat{a}|\sin 90°}{|\hat{a} \times \vec{b}|} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta = \frac{\pi}{6}$$

22. **a., b.** Given, $\frac{1}{6}\hat{i} - \frac{1}{3}\hat{j} + \frac{1}{3}\hat{k} = (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

$$= [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$$

$$= [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} \qquad \text{(i)}$$

[$\because$ $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar]

$$[\vec{a}\,\vec{b}\,\vec{d}] = (\vec{a} \times \vec{b}) \cdot \vec{d}$$

$$= |\vec{a} \times \vec{b}||\vec{d}|\cos\theta \quad (\because \vec{d} \perp \vec{a}, \vec{d} \perp \vec{b}, \therefore \vec{d} \parallel \vec{a} \times \vec{b})$$

$$= ab \sin 30° \cdot 1 \cdot (\pm 1) \quad (\because \theta = 0 \text{ or } \pi)$$

$$= 1 \cdot 1 \cdot \frac{1}{2} .1(\pm 1) = \pm\frac{1}{2}$$

From (i),

$$\vec{c} = \pm\left(\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}\right) = \pm\frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$$

23. **a., b., c.** We know that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, then $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$

Given $\vec{a}+2\vec{b}+3\vec{c}=\vec{0} \quad \Rightarrow 2\vec{a}\times\vec{b}=6\vec{b}\times\vec{c}=3\vec{c}\times\vec{a}$

Hence $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=2(\vec{a}\times\vec{b})$ or $6(\vec{b}\times\vec{c})$ or $3(\vec{c}\times\vec{a})$

24. **a., b.** $\vec{u}=\vec{a}-(\vec{a}\cdot\vec{b})\vec{b}$

$$=\vec{a}(\vec{b}\cdot\vec{b})-(\vec{a}\cdot\vec{b})\vec{b}$$

$$=\vec{b}\times(\vec{a}\times\vec{b})$$

$$\Rightarrow |\vec{u}|=|\vec{b}\times(\vec{a}\times\vec{b})|$$

$$=|\vec{b}||\vec{a}\times\vec{b}|\sin 90^\circ$$

$$=|\vec{b}||\vec{a}\times\vec{b}|$$

$$=|\vec{v}|$$

Also $\vec{u}\cdot\vec{b}=\vec{b}\cdot\vec{b}\times(\vec{a}\times\vec{b})$

$$=[\vec{b}\ \vec{b}\ \vec{a}\times\vec{b}]$$

$$=0$$

$$\Rightarrow |\vec{v}|=|\vec{u}|+|\vec{u}\cdot\vec{b}|$$

25. **a., c.** $\vec{a}\times\vec{b}=\vec{c},\ \vec{b}\times\vec{c}=\vec{a}$

Taking cross with $\vec{b}$ in the first equation, we get $\vec{b}\times(\vec{a}\times\vec{b})=\vec{b}\times\vec{c}=\vec{a}$

$\Rightarrow |\vec{b}|^2\,\vec{a}-(\vec{a}\cdot\vec{b})\vec{b}=\vec{a} \Rightarrow |\vec{b}|=1$ and $\vec{a}\cdot\vec{b}=0$

Also $|\vec{a}\times\vec{b}|=|\vec{c}| \Rightarrow |\vec{a}||\vec{b}|\sin\frac{\pi}{2}=|\vec{c}| \Rightarrow |\vec{a}|=|\vec{c}|$

26. **b., d.** $\vec{d}\cdot\vec{a}=[\vec{a}\ \vec{b}\ \vec{c}]\cos y=-\vec{d}\cdot(\vec{b}+\vec{c})$

$$\Rightarrow \cos y=-\frac{\vec{d}\cdot(\vec{b}+\vec{c})}{[\vec{a}\cdot\vec{b}\cdot\vec{c}]}$$

Similarly, $\sin x = -\frac{\vec{d}\cdot(\vec{a}+\vec{b})}{[\vec{a}\ \vec{b}\ \vec{c}]}$ and $\frac{\vec{d}\cdot(\vec{a}+\vec{c})}{[\vec{a}\ \vec{b}\ \vec{c}]} = -2$

$\therefore\ \sin x + \cos y + 2 = 0$
$\Rightarrow\ \sin x + \cos y = -2$
$\Rightarrow \sin x = -1, \cos y = -1$
Since we want the minimum value of $x^2 + y^2, x = -\pi/2,\ y = \pi$
$\therefore$ The minimum value of $x^2 + y^2$ is $5\pi^2/4$

27. **b., c.** $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{1}{2}\vec{b}$

$\Rightarrow (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = \frac{1}{2}\vec{b}$

$\Rightarrow \vec{a}\cdot\vec{c} = \frac{1}{2}$ and $\vec{a}\cdot\vec{b} = 0$
$\Rightarrow 1\cdot 1\cos\alpha = \frac{1}{2}$ and $\vec{a} \perp \vec{b}$

$\Rightarrow \alpha = \frac{\pi}{3}$ and $\vec{a} \perp \vec{b}$

28. **a., b., c.** $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\overrightarrow{BC} = \frac{2\vec{u}}{|\vec{u}|} - \frac{\vec{u}}{|\vec{u}|} + \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{u}}{|\vec{u}|} + \frac{\vec{v}}{|\vec{v}|}$

$\overrightarrow{AB}\cdot\overrightarrow{BC} = \left(\frac{\vec{u}}{|\vec{u}|} - \frac{\vec{v}}{|\vec{v}|}\right)\left(\frac{\vec{u}}{|\vec{u}|} + \frac{\vec{v}}{|\vec{v}|}\right) = (\hat{u} - \hat{v})\cdot(\hat{u} + \hat{v}) = 1 - 1 = 0$

$\Rightarrow \angle B = 90°$
$\Rightarrow 1 + \cos 2A + \cos 2B + \cos 2C = 0$

29. **a., b., c.** Let $\vec{A} = \vec{a} \times \vec{b},\ \vec{B} = \vec{c} \times \vec{d}$ and $\vec{C} = \vec{e} \times \vec{f}$

We know that $\vec{A}\cdot(\vec{B} \times \vec{C}) = \vec{B}\cdot(\vec{C} \times \vec{A}) = \vec{C}\cdot(\vec{A} \times \vec{B})$

$= (\vec{a} \times \vec{b})\cdot[(\vec{c} \times \vec{d}) \times (\vec{e} \times \vec{f})]$

$= (\vec{a} \times \vec{b})\cdot[\{(\vec{c} \times \vec{d})\cdot\vec{f}\}\vec{e} - \{(\vec{c} \times \vec{d})\cdot\vec{e}\}\vec{f}\}$

$= [\vec{c}\ \vec{d}\ \vec{f}][\vec{a}\ \vec{b}\ \vec{e}] - [\vec{c}\ \vec{d}\ \vec{e}][\vec{a}\ \vec{b}\ \vec{f}]$

Similarly, other parts can be obtained.

30. **a., c.** Here $(l\vec{a} + m\vec{b}) \times \vec{b} = \vec{c} \times \vec{b} \Rightarrow l\vec{a} \times \vec{b} = \vec{c} \times \vec{b}$

$\Rightarrow l(\vec{a}\times\vec{b})^2 = (\vec{c}\times\vec{b})\cdot(\vec{a}\times\vec{b}) \Rightarrow l = \dfrac{(\vec{c}\times\vec{b})\cdot(\vec{a}\times\vec{b})}{(\vec{a}\times\vec{b})^2}$

Similarly, $m = \dfrac{(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{a})}{(\vec{b}\times\vec{a})^2}$

31. **b., c., d.** $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})\cdot(\vec{a}\times\vec{d}) = 0$

$\Rightarrow ([\vec{a}\,\vec{c}\,\vec{d}]\vec{b} - [\vec{b}\,\vec{c}\,\vec{d}]\vec{a})\cdot(\vec{a}\times\vec{d}) = 0$

$\Rightarrow [\vec{a}\,\vec{c}\,\vec{d}][\vec{b}\,\vec{a}\,\vec{d}] = 0$

$\Rightarrow$ Either $\vec{c}$ or $\vec{b}$ must lie in the plane of $\vec{a}$ and $\vec{d}$.

32. **a., b.** Let $\overrightarrow{EB} = p$, $\overrightarrow{AB}$ and $\overrightarrow{CE} = q\overrightarrow{CD}$.

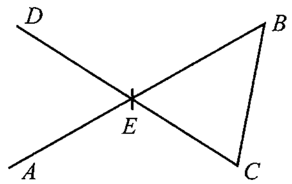


Fig. 2.45

Then $0 < p$ and $q \le 1$

Since $\overrightarrow{EB} + \overrightarrow{BC} + \overrightarrow{CE} = \vec{0}$

$pm(2\hat{i} - 6\hat{j} + 2\hat{k}) + (\hat{i} - 2\hat{j}) + qn(-6\hat{i} + 15\hat{j} - 3\hat{k}) = \vec{0}$

$\Rightarrow (2pm + 1 - 6qn)\hat{i} + (-6pm - 2 + 15qn)\hat{j} + (2pm - 6qn)\hat{k} = \vec{0}$

$\Rightarrow 2pm - 6qn + 1 = \vec{0}, -6pm - 2 + 15qn = \vec{0}, 2pm - 6qn = \vec{0}$

Solving these, we get

$p = 1/(2m)$ and $q = 1/(3n)$

$\therefore 0 < 1/(2m) \le 1$ and $0 < 1/(3n) \le 1$

$\Rightarrow m \ge 1/2$ and $n \ge 1/3$

33. **a., b., d.** $\left.\begin{aligned} \vec{V}_1 &= l\vec{a} + m\vec{b} + n\vec{c} \\ \vec{V}_2 &= n\vec{a} + l\vec{b} + m\vec{c} \\ \vec{V}_3 &= m\vec{a} + n\vec{b} + l\vec{c} \end{aligned}\right\}$ when $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar.

Therefore,

$$[\vec{V}_1\,\vec{V}_2\,\vec{V}_3] = \begin{vmatrix} l & m & n \\ n & l & m \\ m & n & l \end{vmatrix} = 0$$

$$\Rightarrow (l+m+n)\,[(l-m)^2+(m-n)^2+(n-l)^2]=0$$

$$\Rightarrow l+m+n=0 \qquad \text{(i)}$$

Obviously, $lx^2+mx+n=0$ is satisfied by $x=1$ due to (i).

$l^3+m^3+n^3=3lmn$

$\Rightarrow (l+m+n)(l^2+m^2+n^2-lm-mn-ln)=0$, which is true

34. **a., b., c.** It is given that $\vec{\alpha}$, $\vec{b}$ and $\vec{\gamma}$ are coplanar vectors. Therefore,

$[\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}]=0$

$$\Rightarrow \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$$\Rightarrow 3abc-a^3-b^3-c^3=0$$

$$\Rightarrow a^3+b^3+c^3-3abc=0$$

$$\Rightarrow (a+b+c)(a^2+b^2+c^2-ab-bc-ca)=0$$

$$\Rightarrow a+b+c=0 \quad [\because a^2+b^2+c^2-ab-bc-ca \neq 0]$$

$$\Rightarrow \vec{v}\cdot\vec{\alpha} = \vec{v}\cdot\vec{\beta} = \vec{v}\cdot\vec{\gamma} = 0$$

$\Rightarrow \vec{v}$ is perpendicular to $\vec{\alpha}$, $\vec{\beta}$ and $\vec{\gamma}$

35. **b., d.** For $\vec{A}$, $\vec{B}$ and $\vec{C}$ to form a left-handed system

$[\vec{A}\,\vec{B}\,\vec{C}]<0$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 1 & 1 & 5 \end{vmatrix} = 11\hat{i}-6\hat{j}-\hat{k} \qquad \text{(i)}$$

(i) is satisfied by options **(b)** and **(d).**

Reasoning Type

1. **b.** A vector along the bisector is $\frac{\vec{a}}{|\vec{a}|}+\frac{\vec{b}}{|\vec{b}|}=\frac{-5\hat{i}+7\hat{j}+2\hat{k}}{9}$

Hence $\vec{c}=-5\hat{i}+7\hat{j}+2\hat{k}$ is along the bisector. It is obvious that $\vec{c}$ makes an equal angle with $\vec{a}$ and $\vec{b}$. However, Statement 2 does not explain Statement 1, as a vector equally inclined to given two vectors is not necessarily coplanar.

2. **c.** Component of vector $\vec{b}=4\hat{i}+2\hat{j}+3\hat{k}$ in the direction of $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ is $\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\frac{\vec{a}}{|\vec{a}|}$ or $3\hat{i}+3\hat{j}+3\hat{k}$.

Then component in the direction perpendicular to the direction of $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ is

$\vec{b}-3\hat{i}+3\hat{j}+3\hat{k}=\hat{i}-\hat{j}$

3. **d.** $\overrightarrow{AD}=2\hat{j}-\hat{k}$, $\overrightarrow{BD}=-2\hat{i}-\hat{j}-3\hat{k}$ and $\overrightarrow{CD}=2\hat{i}-\hat{j}$

Volume of tetrahedron is $\frac{1}{6}[\overrightarrow{AD}\ \overrightarrow{BD}\ \overrightarrow{CD}]=\frac{1}{6}\left\|\begin{matrix}0 & 2 & -1\\ -2 & -1 & -3\\ 2 & -1 & 0\end{matrix}\right\|=\frac{8}{3}.$

Also, the area of the triangle ABC is $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{1}{2}\left\|\begin{matrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 2\\ -2 & 3 & -1\end{matrix}\right\|$

$$=\frac{1}{2}|-9\hat{i}-2\hat{j}+12\hat{k}|$$

$$=\frac{\sqrt{229}}{2}$$

Then $\frac{8}{3}=\frac{1}{3}\times$ (distance of D from base ABC) $\times$ (area of triangle ABC)

Distance of D from base $ABC=16/\sqrt{229}$

4. **b.** $\vec{r}\cdot\vec{a}=\vec{r}\cdot\vec{b}=\vec{r}\cdot\vec{c}=0$ only if $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

$\Rightarrow[\vec{a}\,\vec{b}\,\vec{c}]=0$

Hence, Statement 2 is true.

Also, $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}]=0$ even if $[\vec{a}\ \vec{b}\ \vec{c}]\neq 0$.

Hence, Statement 2 is not the correct explanation for Statement 1.

5. **a.** Let the three given unit vectors be $\hat{a}$, $\hat{b}$ and $\hat{c}$. Since they are mutually perpendicular, $\hat{a}\cdot(\hat{b}\times\hat{c})=1$. Therefore,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 1$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 1$$

Hence, $a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$, $a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ and $a_3\hat{i} + b_3\hat{j} + c_3\hat{k}$ may be mutually perpendicular.

6. **d.** $\vec{A}\times((\vec{A}\cdot\vec{B})\vec{A}-(\vec{A}\cdot\vec{A})\vec{B})\cdot\vec{C}$

$$= \Big(\underbrace{\vec{A}\times(\vec{A}\cdot\vec{B})\vec{A}}_{\text{zero}}-(\vec{A}\cdot\vec{A})\vec{A}\times\vec{B}\Big)\cdot\vec{C} = -|\vec{A}|^2\,[\vec{A}\;\vec{B}\;\vec{C}]$$

Now, $|\vec{A}|^2 = 4+9+36 = 49$

$$[\vec{A}\vec{B}\vec{C}] = \begin{vmatrix} 2 & 3 & 6 \\ 1 & 1 & -2 \\ 1 & 2 & 1 \end{vmatrix} = 2(1+4) - 1(3-12) + 1(-6-6)$$

$$= 10 + 9 - 12 = 7$$

$\therefore |-|\vec{A}|^2[\vec{A}\vec{B}\vec{C}]| = 49\times 7 = 343$

7. **b.** Let $\vec{d} = \lambda_1\vec{a} + \lambda_2\vec{b} + \lambda_3\vec{c}$

$\Rightarrow [\vec{d}\,\vec{a}\,\vec{b}] = \lambda_3[\vec{c}\,\vec{a}\,\vec{b}] \Rightarrow \lambda_3 = 1$

$[\vec{c}\,\vec{a}\,\vec{b}] = 1$ (because $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three mutually perpendicular unit vectors)

Similarly, $\lambda_1 = \lambda_2 = 1$

$\Rightarrow \vec{d} = \vec{a} + \vec{b} + \vec{c}$

Hence Statement 1 and Statement 2 are correct, but Statement 2 does not explain Statement 1 as it does not give the value of dot products.

8. **a.** Statement 2 is true (see properties of dot product)

Also, $(\hat{i}\times\vec{a})\cdot\vec{b} = \hat{i}\cdot(\vec{a}\times\vec{b})$

$\Rightarrow \vec{a}\times\vec{b} = (\hat{i}\cdot(\vec{a}\times\vec{b}))\hat{i} + (\hat{j}\cdot(\vec{a}\times\vec{b}))\hat{j} + (\hat{k}\cdot(\vec{a}\times\vec{b}))\hat{k}$

Linked Comprehension Type

For Problems 1–3

1. b., 2. c., 3. d.

Sol.

Taking dot product of $\vec{u}+\vec{v}+\vec{w}=\vec{a}$ with $\vec{u}$, we have

$$1+\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}=\vec{a}\cdot\vec{u}=\frac{3}{2}\Rightarrow \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}=\frac{1}{2} \quad \text{(i)}$$

Similarly, taking dot product with $\vec{v}$, we have

$$\vec{u}\cdot\vec{v}+\vec{w}\cdot\vec{v}=\frac{3}{4} \quad \text{(ii)}$$

Also, $\vec{a}\cdot\vec{u}+\vec{a}\cdot\vec{v}+\vec{a}\cdot\vec{w}=\vec{a}\cdot\vec{a}=4$

$$\Rightarrow \vec{a}\cdot\vec{w}=4-\left(\frac{3}{2}+\frac{7}{4}\right)=\frac{3}{4}$$

Again, taking dot product with $\vec{w}$, we have

$$\vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w}=\frac{3}{4}-1=-\frac{1}{4} \quad \text{(iii)}$$

Adding (i), (ii) and (iii), we have

$$2(\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w})=1$$

$$\Rightarrow \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w}=\frac{1}{2} \quad \text{(iv)}$$

Subtracting (i), (ii) and (iii) from (iv), we have

$$\vec{v}\cdot\vec{w}=0,\ \vec{u}\cdot\vec{w}=-\frac{1}{4} \text{ and } \vec{u}\cdot\vec{v}=\frac{3}{4}$$

Now, the equations $\vec{u}\times(\vec{v}\times\vec{w})=\vec{b}$ and $(\vec{u}\times\vec{v})\times\vec{w}=\vec{c}$ can be written as $(\vec{u}\cdot\vec{w})\vec{v}-(\vec{u}\cdot\vec{v})\vec{w}=\vec{b}$

and $(\vec{u}\cdot\vec{w})\vec{v}-(\vec{v}\cdot\vec{w})\vec{u}=\vec{c}\Rightarrow -\frac{1}{4}\vec{v}-\frac{3}{4}\vec{w}=\vec{b},\ -\frac{1}{4}\vec{v}=\vec{c}$, i.e., $\vec{v}=-4\vec{c}$

$$\Rightarrow \vec{c}-\frac{3}{4}\vec{w}=\vec{b}\Rightarrow \vec{w}=\frac{4}{3}(\vec{c}-\vec{b}) \text{ and } \vec{u}=\vec{a}-\vec{v}-\vec{w}=\vec{a}+4\vec{c}-\frac{4}{3}\vec{c}+\frac{4}{3}\vec{b}=\vec{a}+\frac{4}{3}\vec{b}+\frac{8}{3}\vec{c}$$

For Problems 4–6

4. d., 5. c., 6. b.

Sol.

Given that $|\vec{x}|=|\vec{y}|=|\vec{z}|=\sqrt{2}$ and they are inclined at an angle of 60° with each other.

$$\therefore\ \vec{x}\cdot\vec{y}=\vec{y}\cdot\vec{z}=\vec{z}\cdot\vec{x}=\sqrt{2}\cdot\sqrt{2}\cos 60^\circ=1$$

$$\vec{x}\times(\vec{y}\times\vec{z})=\vec{a}\Rightarrow(\vec{x}\cdot\vec{z})\vec{y}-(\vec{x}\cdot\vec{y})\vec{z}=\vec{a}\Rightarrow\vec{y}-\vec{z}=\vec{a} \quad \text{(i)}$$

Similarly, $\vec{y}\times(\vec{z}\times\vec{x})=\vec{b}\Rightarrow\vec{z}-\vec{x}=\vec{b}$ (ii)

$\vec{y}=\vec{a}+\vec{z},\ \vec{x}=\vec{z}-\vec{b}$ (from (i) and (ii)) (iii)

Now, $\vec{x}\times\vec{y}=\vec{c}$

$\Rightarrow(\vec{z}-\vec{b})\times(\vec{z}+\vec{a})=\vec{c}$

$\Rightarrow\vec{z}\times\vec{a}-\vec{b}\times\vec{z}-\vec{b}\times\vec{a}=\vec{c}$

$\Rightarrow\vec{z}\times(\vec{a}+\vec{b})=\vec{c}+(\vec{b}\times\vec{a})$ (iv)

$\Rightarrow(\vec{a}+\vec{b})\times\{\vec{z}\times(\vec{a}+\vec{b})\}=(\vec{a}+\vec{b})\times\vec{c}+(\vec{a}+\vec{b})\times(\vec{b}\times\vec{a})$

$\Rightarrow(\vec{a}+\vec{b})^2\vec{z}-\{(\vec{a}+\vec{b})\cdot\vec{z}\}(\vec{a}+\vec{b})=(\vec{a}+\vec{b})\times\vec{c}+|\vec{a}|^2\vec{b}-|\vec{b}|^2\vec{a}+(\vec{a}\cdot\vec{b})(\vec{b}-\vec{a})$ (v)

Now, (i) $\Rightarrow|\vec{a}|^2=|\vec{y}-\vec{z}|^2=2+2-2=2$

Similarly, (ii) $\Rightarrow|\vec{b}|^2=2$

Also (i) and (ii) $\Rightarrow\vec{a}+\vec{b}=\vec{y}-\vec{x}\Rightarrow|\vec{a}+\vec{b}|^2=2$ (vi)

Also $(\vec{a}+\vec{b})\cdot\vec{z}=(\vec{y}-\vec{x})\cdot\vec{z}=\vec{y}\cdot\vec{z}-\vec{x}\cdot\vec{z}=1-1=0$

and $\vec{a}\cdot\vec{b}=(\vec{y}-\vec{z})\cdot(\vec{z}-\vec{x})$

$=\vec{y}\cdot\vec{z}-\vec{x}\cdot\vec{y}-|z|^2+\vec{x}\cdot\vec{z}=-1$

Thus from (v), we have $2\vec{z}=(\vec{a}+\vec{b})\times\vec{c}+2(\vec{b}-\vec{a})-(\vec{b}-\vec{a})$ or $\vec{z}=(1/2)[(\vec{a}+\vec{b})\times\vec{c}+\vec{b}-\vec{a}]$

$\therefore\ \vec{y}=\vec{a}+\vec{z}=(1/2)[(\vec{a}+\vec{b})\times\vec{c}+\vec{b}+\vec{a}]$ and $\vec{x}=\vec{z}-\vec{b}=(1/2)[(\vec{a}+\vec{b})\times\vec{c}-(\vec{a}+\vec{b})]$

For Problems 7–9

7. b., 8. a., 9. c.

Sol.

Given

$\vec{x}\times\vec{y}=\vec{a}$ (i)

$\vec{y}\times\vec{z}=\vec{b}$ (ii)

$\vec{x}\cdot\vec{b}=\gamma$ (iii)

$\vec{x}\cdot\vec{y}=1$ (iv)

$\vec{y}\cdot\vec{z}=1$ (v)

From (ii), $\vec{x}\cdot(\vec{y}\times\vec{z})=\vec{x}\cdot\vec{b}=\gamma\Rightarrow[\vec{x}\ \vec{y}\ \vec{z}]=\gamma$

From (i) and (ii), $(\vec{x}\times\vec{y})\times(\vec{y}\times\vec{z})=\vec{a}\times\vec{b}$

$\therefore\ [\vec{x}\ \vec{y}\ \vec{z}]\vec{y}-[\vec{y}\ \vec{y}\ \vec{z}]\vec{x}=\vec{a}\times\vec{b}\Rightarrow\vec{y}=\dfrac{\vec{a}\times\vec{b}}{\gamma}$ (vi)

Also from (i), we get $(\vec{x}\times\vec{y})\times\vec{y}=\vec{a}\times\vec{y}$

$\Rightarrow(\vec{x}\cdot\vec{y})\vec{y}-(\vec{y}\cdot\vec{y})\vec{x}=\vec{a}\times\vec{y}\Rightarrow\vec{x}=(1/|\vec{y}|^2)(\vec{y}-\vec{a}\times\vec{y})=\dfrac{\gamma^2}{|\vec{a}\times\vec{b}|^2}\left[\dfrac{\vec{a}\times\vec{b}}{\gamma}-\dfrac{\vec{a}\times(\vec{a}\times\vec{b})}{\gamma}\right]$

$$\Rightarrow \vec{x} = \frac{\gamma}{|\vec{a} \times \vec{b}|^2}[\vec{a} \times \vec{b} - \vec{a} \times (\vec{a} \times \vec{b})]$$

Also from (ii), $(\vec{y} \times \vec{z}) \times \vec{y} = \vec{b} \times \vec{y} \Rightarrow |\vec{y}|^2\, \vec{z} - (\vec{z} \cdot \vec{y})\, \vec{y} = \vec{b} \times \vec{y}$

$$\Rightarrow \vec{z} = \frac{1}{|\vec{y}|^2}[\vec{y} + \vec{b} \times \vec{y}] = \frac{\gamma}{|\vec{a} \times \vec{b}|^2}[\vec{a} \times \vec{b} + \vec{b} \times (\vec{a} \times \vec{b})]$$

For Problems 10–12

10. b., 11. b., 12. d.

Sol.

$\vec{P} \times \vec{B} = \vec{A} - \vec{P}$ and $|\vec{A}| = |\vec{B}| = 1$ and $\vec{A} \cdot \vec{B} = 0$ is given

Now $\vec{P} \times \vec{B} = \vec{A} - \vec{P}$ (i)

$(\vec{P} \times \vec{B}) \times \vec{B} = (\vec{A} - \vec{P}) \times \vec{B}$ (taking cross product with $\vec{B}$ on both sides)

$\Rightarrow (\vec{P} \cdot \vec{B})\, \vec{B} - (\vec{B} \cdot \vec{B}) \vec{P} = \vec{A} \times \vec{B} - \vec{P} \times \vec{B}$

$\Rightarrow (\vec{P} \cdot \vec{B})\, \vec{B} - \vec{P} = \vec{A} \times \vec{B} - \vec{A} + \vec{P}$

$\Rightarrow 2\vec{P} = \vec{A} - \vec{A} \times \vec{B} - (\vec{P} \cdot \vec{B})\vec{B}$

$$\Rightarrow \vec{P} = \frac{\vec{A} - \vec{A} \times \vec{B} - (\vec{P} \cdot \vec{B})\, \vec{B}}{2} \quad \text{(ii)}$$

Taking dot product with $\vec{B}$ on both sides of (i), we get

$\vec{P} \cdot \vec{B} = \vec{A} \cdot \vec{B} - \vec{P} \cdot \vec{B}$

$\Rightarrow \vec{P} \cdot \vec{B} = 0$ (iii)

$$\Rightarrow \vec{P} = \frac{\vec{A} + \vec{B} \times \vec{A}}{2}$$

Now

$(\vec{P} \times \vec{B}) \times \vec{B} = (\vec{P} \cdot \vec{B})\, \vec{B} - (\vec{B} \cdot \vec{B})\, \vec{P} = -\vec{P}$

$\vec{P}, \vec{A}, \vec{P} \times \vec{B}\ (= \vec{A} - \vec{P})$ are dependent

Also $\vec{P} \cdot \vec{B} = 0$

and $|\vec{P}|^2 = \left|\frac{\vec{A} - \vec{A} \times \vec{B}}{2}\right|^2$

$$= \frac{|\vec{A}|^2 + |\vec{A} \times \vec{B}|^2}{4}$$

$$= \frac{1+1}{4} = \frac{1}{2} \Rightarrow |\vec{P}| = \frac{1}{\sqrt{2}}$$

For Problems 13–15

13. b., 14. a., 15. c.

Sol.

13. b. $\vec{a}_1 = \left[(2\hat{i}+3\hat{j}-6\hat{k})\cdot\frac{(2\hat{i}-3\hat{j}+6\hat{k})}{7}\right]\frac{2\hat{i}-3\hat{j}+6\hat{k}}{7} = \frac{-41}{49}(2\hat{i}-3\hat{j}+6\hat{k})$

$$\vec{a}_2 = \frac{-41}{49}\left((2\hat{i}-3\hat{j}+6\hat{k})\cdot\frac{(-2\hat{i}+3\hat{j}+6\hat{k})}{7}\right)\frac{(-2\hat{i}+3\hat{j}+6\hat{k})}{7}$$

$$= \frac{-41}{(49)^2}(-4-9+36)(-2\hat{i}+3\hat{j}+6\hat{k}) = \frac{943}{49^2}(2\hat{i}-3\hat{j}-6\hat{k})$$

14. a. $\vec{a}_1\cdot\vec{b} = \frac{-41}{49}(2\hat{i}-3\hat{j}+6\hat{k})\cdot(2\hat{i}-3\hat{j}+6\hat{k}) = -41$

15. c. $\vec{a}$, $\vec{a}_1$ and $\vec{b}$ are coplanar because $\vec{a}_1$ and $\vec{b}$ are collinear.

For Problems 16–18

16. b., 17. c., 18. a.

Sol.

Point G is $\left(\frac{4}{3},\frac{1}{3},\frac{8}{3}\right)$. Therefore,

$$\left|\overrightarrow{AG}\right|^2 = \left(\frac{5}{3}\right)^2 + \frac{1}{9} + \left(\frac{5}{3}\right)^2 = \frac{51}{9}$$

$$\Rightarrow \left|\overrightarrow{AG}\right| = \frac{\sqrt{51}}{3}$$

$$\overrightarrow{AB} = -4\hat{i}+4\hat{j}+0\hat{k}$$

$$\overrightarrow{AC} = 2\hat{i}+2\hat{j}+2\hat{k}$$

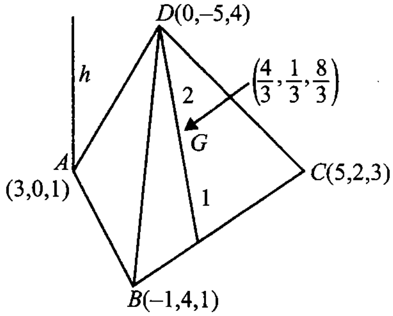


Fig. 2.46

$$\therefore\ \overrightarrow{AB} \times \overrightarrow{AC} = -8\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 8(\hat{i} + \hat{j} - 2\hat{k})$$

Area of $\Delta ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = 4\sqrt{6}$

$\overrightarrow{AD} = -3\hat{i} - 5\hat{j} + 3\hat{k}$

The length of the perpendicular from the vertex D on the opposite face

$= |$ projection of $\overrightarrow{AD}$ on $\overrightarrow{AB} \times \overrightarrow{AC}|$

$$= \left|\frac{(-3\hat{i} - 5\hat{j} + 3\hat{k})(\hat{i} + \hat{j} - 2\hat{k})}{\sqrt{6}}\right|$$

$$= \left|\frac{-3-5-6}{\sqrt{6}}\right| = \frac{14}{\sqrt{6}}$$

For Problems 19–21

19. c., 20. b., 21. d.

Sol.

19. **c.** Let point D be (a_1, a_2, a_3)

$a_1 + 1 = 3 \Rightarrow a_1 = 2$

$a_2 + 0 = 1 \Rightarrow a_2 = 1$

$a_3 - 1 = 7 \Rightarrow a_3 = 8$

$\therefore D(2, 1, 8)$

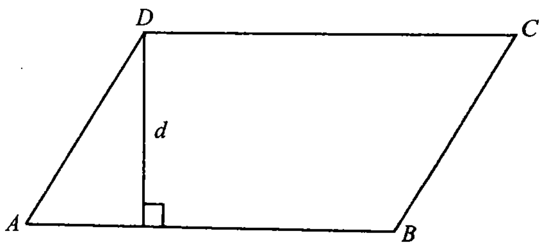


Fig. 2.47

$$\vec{d} = \left|\frac{(\overrightarrow{AB}) \times (\overrightarrow{AD})}{|\overrightarrow{AB}|}\right|$$

$\overrightarrow{AB} = -\hat{i} + \hat{j} - 5\hat{k}$

$\overrightarrow{AD} = 0\hat{i} + 2\hat{j} + 4\hat{k}$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -5 \\ 0 & 2 & 4 \end{vmatrix}$$

$$= 14\hat{i} + 4\hat{j} - 2\hat{k}$$

$$= 2(7\hat{i} + 2\hat{j} - \hat{k})$$

$\Rightarrow d = 2\sqrt{2}$

20. **b.**

$\vec{n} = 7\hat{i} + 2\hat{j} - \hat{k}$ is normal to the plane $P \equiv (8, 2, -12)$.

$|\overrightarrow{AP}| = 6\hat{i} + 3\hat{j} - 16\hat{k}$

$\therefore$ distance $d = \left|\dfrac{\overrightarrow{AP} \cdot \vec{n}}{|\vec{n}|}\right|$

$= \left|\dfrac{42 + 6 + 16}{\sqrt{49 + 4 + 1}}\right|$

$= \dfrac{64}{\sqrt{54}}$

$= \dfrac{64}{3\sqrt{6}} = \dfrac{64\sqrt{6}}{18} = \dfrac{32\sqrt{6}}{9}$

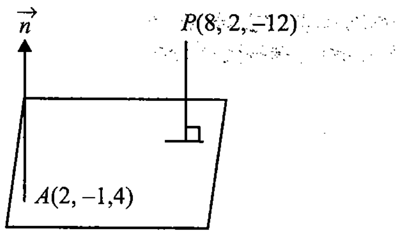


Fig. 2.48

21. **d.** Vector normal to the plane

$\overrightarrow{AD} \times \overrightarrow{AB} = +2(7\hat{i} + 2\hat{j} - \hat{k})$

Projection on $xy = 2$

Projection on $yz = 14$

Projection on $zx = 4$

For Problems 22–24

22. d., 23. c., 24. c.

Sol.

Let $\vec{r} = x\hat{i} + y\hat{j}$

$x^2 + y^2 + 8x - 10y + 40 = 0$, which is a circle

centre $C(-4, 5)$, radius $r = 1$

$p_1 = \max\{(x + 2)^2 + (y - 3)^2\}$

$p_2 = \min\{(x + 2)^2 + (y - 3)^2\}$

Let P be $(-2, 3)$. Then

$CP = 2\sqrt{2}, r = 1$

$p_2 = (2\sqrt{2} - 1)^2$

$p_1 = (2\sqrt{2} + 1)^2$

$p_1 + p_2 = 18$

Slope $= AB = \left(\dfrac{dy}{dx}\right)_{(2, 2)} = -2$

Equation of AB, $2x + y = 6$

$\overrightarrow{OA} = 2\hat{i} + 2\hat{j}$, $\overrightarrow{OB} = 3\hat{i}$

$\overrightarrow{AB} = \hat{i} - 2\hat{j}$

$\overrightarrow{AB} \cdot \overrightarrow{OB} = (\hat{i} - 2\hat{j})(3\hat{i}) = 3$

Matrix-Match Type

1. a → p, q, r, s; b → p, q; c → p, r; d → r

a. Given equations are consistent if

$$(\hat{i}+\hat{j})+\lambda(\hat{i}+2\hat{j}-\hat{k})=(\hat{i}+2\hat{j})+\mu(-\hat{i}+\hat{j}+a\hat{k})$$

$\Rightarrow 1+\lambda=1-\mu, 1+2\lambda=2+\mu, -\lambda=a\mu$

$\Rightarrow \lambda=1/3$ and $\mu=-1/3$

$\Rightarrow a=1$

b. $\vec{a}=\lambda\hat{i}-3\hat{j}-\hat{k}$

$\vec{b}=2\lambda\hat{i}+\lambda\hat{j}-\hat{k}$

Angle between $\vec{a}$ and $\vec{b}$ is acute. Therefore,

$\vec{a}\cdot\vec{b}>0$

$\Rightarrow 2\lambda^2-3\lambda+1>0$

$\Rightarrow (2\lambda-1)(\lambda-1)>0$

$\Rightarrow \lambda\in\left(-\infty,\frac{1}{2}\right)\cup(1,\infty)$

Also $\vec{b}$ makes an obtuse angle with the axes. Therefore,

$\vec{b}\cdot\hat{i}<0\Rightarrow\lambda<0$

$\vec{b}\cdot\hat{j}<0\Rightarrow\lambda<0$ (ii)

Combining these two, we get $\lambda=-4,-2$

c. If vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}+(1+a)\hat{k}$ and $3\hat{i}+a\hat{j}+5\hat{k}$ are coplanar, then

$$\begin{vmatrix}2 & -1 & 1\\ 1 & 2 & 1+a\\ 3 & a & 5\end{vmatrix}=0$$

$\Rightarrow a^2+2a-8=0$

$\Rightarrow (a+4)(a-2)=0$

$\Rightarrow a=-4, 2$

d. $\vec{A}=2\hat{i}+\lambda\hat{j}+3\hat{k}$

$B=2\hat{i}+\lambda\hat{j}+\hat{k}$

$C=3\hat{i}+\hat{j}+0.\hat{k}$

$\therefore \vec{A}+\lambda\vec{B}=2(1+\lambda)\hat{i}+(\lambda+\lambda^2)\hat{j}+(3+\lambda)\hat{k}$

Now $(\vec{A}+\lambda\vec{B})\perp\vec{C}$. Therefore,

$$(\vec{A}+\lambda\vec{B})\cdot\vec{C}=0$$
$$\Rightarrow 6(1+\lambda)+(\lambda+\lambda^2)+0=0$$
$$\Rightarrow \lambda^2+7\lambda+6=0$$
$$\Rightarrow (\lambda+6)(\lambda+1)=0$$
$$\Rightarrow \lambda=-6,=-1$$
$$\Rightarrow |2\lambda|=12,2$$

2. a → r; b → p; c → s; d → q

a. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular, then

$$[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}]=[\vec{a}\ \vec{b}\ \vec{c}]^2=(|\vec{a}||\vec{b}||\vec{c}|)^2=16$$

b. Given $\vec{a}$ and $\vec{b}$ are two unit vectors, i.e., $|\vec{a}|=|\vec{b}|=1$ and angle between them is $\pi/3$.

$$\sin\theta=\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}\Rightarrow \sin\frac{\pi}{3}=|\vec{a}\times\vec{b}|$$

$$\frac{\sqrt{3}}{2}=|\vec{a}\times\vec{b}|$$

Now

$$[\vec{a}\ \ \vec{b}+\vec{a}\times\vec{b}\ \ \vec{b}]=[\vec{a}\ \vec{b}\ \vec{b}]+[\vec{a}\ \ \vec{a}\times\vec{b}\ \ \vec{b}]$$
$$=0+[\vec{a}\ \ \vec{a}\times\vec{b}\ \ \vec{b}]$$
$$=(\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{a})$$
$$=-(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})$$
$$=-|\vec{a}\times\vec{b}|^2$$
$$=-\frac{3}{4}$$

c. If $\vec{b}$ and $\vec{c}$ are orthogonal, $\vec{b}\cdot\vec{c}=0$.

Also, it is given that $\vec{b}\times\vec{c}=\vec{a}$. Now

$$[\vec{a}+\vec{b}+\vec{c}\ \ \vec{a}+\vec{b}\ \ \vec{b}+\vec{c}]=[\vec{a}\ \ \vec{a}+\vec{b}\ \ \vec{b}+\vec{c}]+[\vec{b}+\vec{c}\ \ \vec{a}+\vec{b}\ \ \vec{b}+\vec{c}]$$
$$=[\vec{a}\ \vec{b}\ \vec{c}]$$
$$=\vec{a}\cdot(\vec{b}\times\vec{c})$$
$$=\vec{a}\cdot\vec{a}=|\vec{a}|^2=1 \quad \text{(because } \vec{a} \text{ is a unit vector)}$$

d. $[\vec{x}\ \vec{y}\ \vec{a}]=0$

Therefore, $\vec{x}$, $\vec{y}$ and $\vec{a}$ are coplanar. (i)

$[\vec{x}\ \vec{y}\ \vec{b}]=0$

Therefore, $\vec{x}$, $\vec{y}$ and $\vec{b}$ are coplanar. (ii)

Also, $[\vec{a}\ \vec{b}\ \vec{c}]=0$

Therefore, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar (iii)

From (i), (ii) and (iii),

$\vec{x}$, $\vec{y}$ and $\vec{c}$ are coplanar. Therefore,

$[\vec{x}\ \vec{y}\ \vec{c}]=0$

3. **a → q; b → s; c → p; d → r**

a. $|\vec{a}+\vec{b}+\vec{c}| = \sqrt{6} \Rightarrow \vec{a}^2+\vec{b}^2+\vec{c}^2+2\vec{a}\cdot\vec{b}+2\vec{b}\cdot\vec{c}+2\vec{c}\cdot\vec{a}=6$

$\therefore |\vec{a}| = 1$

b. $\vec{a}$ is perpendicular to $\vec{b}+\vec{c} \Rightarrow \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=0$ (i)

$\vec{b}$ is perpendicular to $\vec{a}+\vec{c} \Rightarrow \vec{b}\cdot\vec{c}+\vec{b}\cdot\vec{c}=0$ (ii)

$\vec{c}$ is perpendicular to $\vec{a}+\vec{b} \Rightarrow \vec{c}\cdot\vec{a}+\vec{a}\cdot\vec{c}\cdot\vec{b}=0$ (iii)

From (i), (ii) and (iii), we get

$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$

$\therefore |\vec{a}+\vec{b}+\vec{c}|=7$

c. $(\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d})=21$

d. We know that $[\vec{a}\times\vec{b}\quad \vec{b}\times\vec{c}\quad \vec{c}\times\vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$

and $[\vec{a}\ \vec{b}\ \vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$

$=\begin{vmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{vmatrix}$

$=32$

$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}]=4\sqrt{2}$

4. **a → s; b → r; c → q; d → p**

a. $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 1 & 2\\ -1 & -2 & -1\end{vmatrix}=3\hat{i}-3\hat{j}+3\hat{k}$

Hence, the area of the triangle is $\frac{3\sqrt{3}}{2}$.

b. The area of the parallelogram is $3\sqrt{3}$.

c. The area of a parallelogram whose diagonals are $2\vec{a}$ and $4\vec{b}$ is $\frac{1}{2}|2\vec{a}\times 4\vec{b}|=12\sqrt{3}$.

d. The volume of the parallelepiped $=|(\vec{a}\times\vec{b})\cdot\vec{c}|=\sqrt{9+36+9}=3\sqrt{6}$

5. **a → p, r; b → q; c → s; d → p**

a. Vectors $-3\hat{i}+3\hat{j}+4\hat{k}$ and $\hat{i}+\hat{j}$ are coplanar with $\vec{a}$ and $\vec{b}$.

b. $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 2 & 2\\ -2 & 1 & 2\end{vmatrix}$

$=2\hat{i}-2\hat{j}+3\hat{k}$

c. If $\vec{c}$ is equally inclined to $\vec{a}$ and $\vec{b}$, then we must have $\vec{a}\cdot\vec{c}=\vec{b}\cdot\vec{c}$, which is true for $\vec{c}=\hat{i}-\hat{j}+5\hat{k}$.

d. Vector is forming a triangle with $\vec{a}$ and $\vec{b}$. Then $\vec{c}=\vec{a}+\vec{b}=-3\hat{i}+3\hat{j}+4\hat{k}$

6. **a → q; b → s; c → p; d → r**

a. $|\vec{a}+\vec{b}|=|\vec{a}+2\vec{b}|$

$a^2+b^2+2\vec{a}\cdot\vec{b}=a^2+4b^2+4\vec{a}\cdot\vec{b}$

$\Rightarrow 2\vec{a}\cdot\vec{b}=-3b^2<0$

Hence, angle between $\vec{a}$ and $\vec{b}$ is obtuse.

b. $|\vec{a}+\vec{b}|=|\vec{a}-2\vec{b}|$

$\Rightarrow a^2+b^2+2\vec{a}\cdot\vec{b}=a^2+4b^2-4\vec{a}\cdot\vec{b}$

$\Rightarrow 6\vec{a}\cdot\vec{b}=3b^2$

Hence, angle between $\vec{a}$ and $\vec{b}$ is acute.

c. $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$

$\Rightarrow \vec{a}\cdot\vec{b}$

$\Rightarrow \vec{a}$ is perpendicular to $\vec{b}$.

d. $\vec{c}\times(\vec{a}\times\vec{b})$ lies in the plane of vectors $\vec{a}$ and $\vec{b}$.

A vector perpendicular to this plane is parallel to $\vec{a}\times\vec{b}$

Hence angle is 0°.

7. **a → r; b → s; c → q; d → p**

$[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] = 36$

$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 6$

$\Rightarrow$ Volume of tetrahedron formed by vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is $\frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}] = 1$.

$[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}] = 12$

$\vec{a}-\vec{b}$, $\vec{b}-\vec{c}$ and $\vec{c}-\vec{a}$ are coplanar $\Rightarrow [\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}] = 0$

Integer Answer Type

1. **(5)** Let angle between $\vec{a}$ and $\vec{b}$ be θ.

We have $|\vec{a}| = |\vec{b}| = 1$

Now $|\vec{a}+\vec{b}| = 2\cos\frac{\theta}{2}$ and $|\vec{a}-\vec{b}| = 2\sin\frac{\theta}{2}$

Consider $F(\theta) = \frac{3}{2}\left(2\cos\frac{\theta}{2}\right) + 2\left(2\sin\frac{\theta}{2}\right)$

$\therefore F(\theta) = 3\cos\frac{\theta}{2} + 4\sin\frac{\theta}{2}$, $\theta \in [0, \pi]$

2. **(1)** Since angle between $\vec{u}$ and $\hat{i}$ is 60°,

$\vec{u}\cdot i = |\vec{u}||\hat{i}|\cos 60° = \frac{|\vec{u}|}{2}$

Given that $|\vec{u}-\hat{i}|, |\vec{u}|, |\vec{u}-2\hat{i}|$ are in G.P., so $|\vec{u}-\hat{i}|^2 = |\vec{u}||\vec{u}-2\hat{i}|$

Squaring both sides, $[|\vec{u}|^2 + |\hat{i}|^2 - 2\vec{u}\cdot\hat{i}]^2 = |\vec{u}|^2[|\vec{u}|^2 + 4|\hat{i}|^2 - 4\vec{u}\cdot\hat{i}]$

$[|\vec{u}|^2 + 1 - \frac{2|\vec{u}|}{2}]^2 = |\vec{u}|^2[|\vec{u}|^2 + 4 - 4\frac{|\vec{u}|}{2}] \Rightarrow |\vec{u}|^2 + 2|\vec{u}| - 1 = 0 \Rightarrow |\vec{u}| = -\frac{2\pm 2\sqrt{2}}{2} \Rightarrow |\vec{u}| = \sqrt{2}-1$

3. **(2)** $\overline{AB} = 2\hat{i} + \hat{j} + \hat{k}$, $\overline{AC} = (t+1)\hat{i} + 0\hat{j} - \hat{k}$

$$\overline{AB} \times \overline{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ t+1 & 0 & -1 \end{vmatrix} = -\hat{i} + (t+3)\hat{j} - (t+1)\hat{k}$$

$$|\overline{AB} \times \overline{AC}| = \sqrt{1 + (t+3)^2 + (t+1)^2} = \sqrt{2t^2 + 8t + 11}$$

$$\text{Area of } \Delta ABC = \frac{1}{2}|\overline{AB} \times \overline{AC}| \Rightarrow \Delta = \frac{1}{2}\sqrt{2t^2 + 8t + 1}$$

$$\text{Let } f(t) = \Delta^2 = \frac{1}{4}(2t^2 + 8t + 1)$$

$f'(t) = 0 \Rightarrow t = -2$

At $t = -2$, $f''(t) > 0$

So Δ is minimum at $t = -2$

4. **(7)** $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

L.H.S. $= [3\vec{a} + \vec{b} \quad 3\vec{b} + \vec{c} \quad 3\vec{c} + \vec{a}]$

$= [3\vec{a}\ 3\vec{b}\ 3\vec{c}] + [\vec{b}\ \vec{c}\ \vec{a}]$

$= 3^3[\vec{a}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{b}\ \vec{c}]$

$= 28\,[\vec{a}\ \vec{b}\ \vec{c}]$

5. **(4)** $\vec{a} = \alpha\hat{i} + 2\hat{j} - 3\hat{k}$, $\vec{b} = \hat{i} + 2\alpha\hat{j} - 2\hat{k}$, $\vec{c} = 2\hat{i} - \alpha\hat{j} + \hat{k}$

$\{(\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c})\} \times (\vec{c} \times \vec{a}) = \vec{0}$

$\Rightarrow \quad \{[\vec{a}\ \vec{b}\ \vec{c}]\vec{b} - [\vec{a}\ \vec{b}\ \vec{b}]\vec{c}\} \times (\vec{c} \times \vec{a}) = \vec{0}$

$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}]\vec{b} \times (\vec{c} \times \vec{a}) = \vec{0}$

$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] \quad ((\vec{a} \cdot \vec{b})\vec{c} - (\vec{b} \cdot \vec{c})\vec{a}) = \vec{0}$

$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = 0 \quad (\because \quad \vec{a}$ and $\vec{c}$ are not collinear)

$$\Rightarrow \begin{vmatrix} \alpha & 2 & -3 \\ 1 & 2\alpha & -2 \\ 2 & -\alpha & 1 \end{vmatrix}$$

$\Rightarrow \quad \alpha(2\alpha - 2\alpha) - 2(1+4) - 3(-\alpha - 4\alpha) = 0$

$\Rightarrow \quad 10 - 15\alpha = 0$

$\therefore \alpha = 2/3$

6. **(9)** Since $\vec{x}$ and $\vec{y}$ are non-collinear vectors, therefore $\vec{x}$, $\vec{y}$ and $\vec{x} \times \vec{y}$ are non-coplanar vectors.

$[(a-2)\alpha^2 + (b-3)\alpha + c] + [(a-2)\beta^2 + (b-3)\,\beta\beta + c]\,\vec{y} + [(a-2)\gamma^2 + (b-3)\,\gamma + c]\,(\vec{x} \times \vec{y}) = 0$

Coefficient of each vector $\vec{x}$, $\vec{y}$ and $\vec{x} \times \vec{y}$ is zero.

$(a-2)\alpha^2 + (b-3)\alpha + c = 0$

$(a-2)\beta^2 + (b-3)\beta + c = 0$

$(a-2)\,\gamma^2 + (b-3)\,\beta + c = 0$

The above three equations will satisfy if the coefficients of α, β and γ are zero because α, β and γ are three distinct real numbers

$a - 2 = 0 \Rightarrow a = 2,$

$b - 3 = 0 \Rightarrow b = 3$ and $c = 0$

$\therefore a^2 + b^2 + c^2 = 2^2 + 3^2 + 0^2 = 4 + 9 = 13$

7. **(1)** Given, $\vec{u} \times \vec{v} + \vec{u} = \vec{w}$ and $\vec{w} \times \vec{u} = \vec{v}$

$\Rightarrow (\vec{u} \times \vec{v} + \vec{u}) \times \vec{u} = \vec{v} \Rightarrow (\vec{u} \times \vec{v}) \times \vec{u} = \vec{v} \Rightarrow \vec{v} - (\vec{u} \cdot \vec{v}) = \vec{v} \Rightarrow (\vec{u} \cdot \vec{v})\vec{u} = 0 \Rightarrow (\vec{u}\;\vec{v}) = 0$

Now, $[\vec{u}\,\vec{v}\,\vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$

$= \vec{u} \cdot (\vec{v} \times (\vec{u} \times \vec{v} + \vec{u})) = \vec{u} \cdot (\vec{v} \times (\vec{u} \times \vec{v}) + \vec{v} \times \vec{u}) = \vec{u}(v^2\,\vec{u} - (\vec{u} \cdot \vec{v})\vec{v} + \vec{v} \times \vec{u}) = \vec{v}^2\,\vec{u}^2 = 1$

8. **(7)** Let the vertices are A, B, C, D and O is the origin.

$\therefore \quad \vec{OA} = \hat{i} - 6\hat{j} + 10\hat{k},\ \vec{OB} = \hat{i} - 3\hat{j} + 7\hat{k},\ \vec{OC} = -5\hat{i} - \hat{j} + \lambda\hat{k},\ \vec{OD} = 7\hat{i} - 4\hat{j} + 7\hat{k}$

$\therefore \quad \vec{AB} = \vec{OB} - \vec{OA} = -2\hat{i} + 3\hat{j} - 3\hat{k}$

$\vec{AC} = \vec{OC} - \vec{OA} = 4\hat{i} + 5\hat{j} + (\lambda - 10)\hat{k}$

$\vec{AD} = \vec{OD} - \vec{OA} = 6\hat{i} + 2\hat{j} - 3\hat{k}$

Volume of tetrahedron $= \frac{1}{6}[\vec{AB}\ \vec{AC}\ \vec{AD}]$

$$= \frac{1}{6}\begin{vmatrix} -2 & 3 & -3 \\ 4 & 5 & \lambda-10 \\ 6 & 2 & -3 \end{vmatrix}$$

$$= \frac{1}{6}\{-2(-15-2\lambda+20)-3(-12-6\lambda+60)-3(8-30)\}$$

$$= \frac{1}{6}\{4\lambda-10-144+18\lambda+66\}$$

$$= \frac{1}{6}(22\lambda-88) = 11 \quad \text{(given)}$$

$\Rightarrow\ 2\lambda-8=6$

$\therefore\ \lambda=7$

9. **(6)** Let $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{u} = \hat{i} - 2\hat{j} + 3\hat{k};\ \vec{v} = 2\hat{i} + \hat{j} + 4\hat{k};\ \vec{w} = \hat{i} + 3\hat{j} + 3\hat{k}$

$(\vec{u}\cdot\vec{R}-15)\hat{i} + (\vec{v}\cdot\vec{R}-30)\hat{j} + (\vec{w}\cdot\vec{R}-25)\hat{k} = \vec{0}$ (given)

So $\vec{u}\cdot\vec{R} = 15 \Rightarrow x-2y+3z=15$ (i)

$\vec{v}\cdot\vec{R} = 30 \Rightarrow 2x+y+4z=30$ (ii)

$\vec{w}\cdot\vec{R} = 25 \Rightarrow x+3y+3z=25$ (iii)

Solving, we get

$x=4$

$y=2$

$z=5$

10. **(6)** $2\vec{V} + \vec{V}\times(\hat{i}+2\hat{j}) = (2\hat{i}+\hat{k})$ (i)

$\Rightarrow 2\vec{V}\cdot(\hat{i}+2\hat{j}) + 0 = (2\hat{i}+\hat{k})\cdot(\hat{i}+2\hat{j})$

$\Rightarrow 2\vec{V}\cdot(\hat{i}+2\hat{j}) = 2$

$\Rightarrow |\vec{V}\cdot(\hat{i}+2\hat{j})|^2 = 1$

$\Rightarrow |\vec{V}|^2\cdot|\hat{i}+2\hat{j}|^2\cos^2\theta = 1$ (θ is the angle between $\vec{V}$ and $\hat{i}+2\hat{j}$)

$\Rightarrow |\vec{V}|^2\, 5(1-\sin^2\theta) = 1$

$\Rightarrow |\vec{V}|^2\, 5\sin^2\theta = 5|\vec{V}|^2 - 1$ (ii)

From Eq. (i)

$\Rightarrow |2\vec{V} + \vec{V} \times (\hat{i} + 2\hat{j})|^2 = |2\hat{i} + \hat{k}|^2$

$\Rightarrow 4|\vec{V}|^2 + |\vec{V} \times (\hat{i} + 2\hat{j})|^2 = 5$

$\Rightarrow 4|\vec{V}|^2 + |\vec{V}|^2 \cdot |\hat{i} + 2\hat{j}|^2 \sin^2\theta = 5$

$\Rightarrow 4|\vec{V}|^2 + 5|\vec{V}|^2 \sin^2\theta = 5$

$\Rightarrow 4|\vec{V}|^2 + 5|\vec{V}|^2 - 1 = 5$

$\Rightarrow 9|\vec{V}|^2 = 6$

$\Rightarrow 3|\vec{V}| = \sqrt{6}$

$\Rightarrow 3|\vec{V}| = \sqrt{6} = \sqrt{m}$

$\therefore m = 6$

11. **(1)** $\vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$

$\vec{a} \cdot \vec{c} = 0 \Rightarrow \vec{a} \perp \vec{c}$

$\Rightarrow \vec{a} \perp \vec{b} - \vec{c}$

$|\vec{a} \times \vec{b} - \vec{a} \times \vec{c}| = |\vec{a} \times (\vec{b} - \vec{c})| = |\vec{a}||\vec{b} - \vec{c}| = |\vec{b} - \vec{c}|$

Now $|\vec{b} - \vec{c}|^2 = |\vec{b}|^2 + |\vec{c}|^2 - 2|\vec{b}||\vec{c}|\cos\frac{\pi}{3} = 2 - 2x \times \frac{1}{2} = 1$

$|\vec{b} - \vec{c}| = 1$

12. **(6)** Here $\vec{OA} = \vec{a}, \vec{OB} = 10\vec{a} + 2\vec{b}, \vec{OC} = \vec{b}$

q = Area of parallelogram with OA and OC as adjacent sides.

$\therefore \quad q = |\vec{a} \times \vec{b}|$ (i)

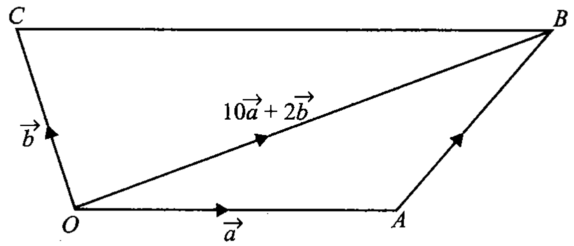


Fig. 2.49

p = Area of quadrilateral $OABC$

= Area of ΔOAB + are of ΔOBC

$= \frac{1}{2}|\vec{a} \times (10\vec{a} + 2\vec{b})| + \frac{1}{2}|(10\vec{a} + 2\vec{b}) \times \vec{b}|$

$= |\vec{a} \times \vec{b}| + 5|\vec{a} \times \vec{b}|$

$\therefore \quad p = 6|\vec{a} \times \vec{b}|$

$\Rightarrow \quad p = 6q$ [From Eq. (i)]

$\therefore \quad k = 6$

13. **(9)** Here $\vec{F} = 3\hat{i} - \hat{j} - 2\hat{k}$

$\vec{AB}$ = P.V. of B – P.V. of A

$\therefore \quad \vec{AB} = (-\hat{i} - \hat{j} - 2\hat{k}) - (-3\hat{i} - 4\hat{j} + \hat{k})$

$= 2\hat{i} + 3\hat{j} - 3\hat{k}$

Let $\vec{s} = \vec{AB}$ be the displacement vector

Now work done $= \vec{F} \cdot \vec{s}$

$= (3\hat{i} - \hat{j} - 2\hat{k}) \cdot (2\hat{i} + 3\hat{j} - 3\hat{k})$

$= 6 - 3 + 6 = 9$

Archives

Subjective Type

1. Let with respect to O, position vectors of points A, B, C, D, E and F be $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}$ and $\vec{f}$. Let perpendiculars from A to EF and from B to DF meet each other at H. Let position vectors of H be $\vec{r}$. We join CH. In order to prove the statement given in the question, it is sufficient to prove that CH is perpendicular to DE.

Now, as $OD \perp BC \Rightarrow \vec{d} \cdot (\vec{b} - \vec{c}) = 0$

$\Rightarrow \vec{d} . \vec{b} = \vec{d} \cdot \vec{c}$ (i)

as $OE \perp AC \Rightarrow \vec{e} \cdot (\vec{c} - \vec{a}) = 0 \Rightarrow \vec{e} \cdot \vec{c} = \vec{e} \cdot \vec{a}$ (ii)

as $OF \perp AB \Rightarrow \vec{f} \cdot (\vec{a} - \vec{b}) = 0 \Rightarrow \vec{f} \cdot \vec{a} = \vec{f} \cdot \vec{b}$ (iii)

Also $AH \perp EF \Rightarrow (\vec{r} - \vec{a}) \cdot (\vec{e} - \vec{f}) = 0$

$\Rightarrow \vec{r} \cdot \vec{e} - \vec{r} \cdot \vec{f} - \vec{a} \cdot \vec{e} + \vec{a} \cdot \vec{f} = 0$ (iv)

and $BH \perp FD \Rightarrow (\vec{r} - \vec{b}) \cdot (\vec{f} - \vec{d}) = 0$

$\Rightarrow \vec{r} \cdot \vec{f} - \vec{r} \cdot \vec{d} - \vec{b} \cdot \vec{f} + \vec{b} \cdot \vec{d} = 0$ (v)

Adding (iv) and (v), we get

$$\vec{r}\cdot\vec{e}-\vec{a}\cdot\vec{e}+\vec{a}\cdot\vec{f}-\vec{r}\cdot\vec{d}-\vec{b}\cdot\vec{f}+\vec{b}\cdot\vec{d}=0$$

$$\Rightarrow \vec{r}\cdot(\vec{e}-\vec{d})-\vec{e}\cdot\vec{c}+\vec{d}\cdot\vec{c}=0 \quad \text{(using (i), (ii) and (iii))}$$

$$\Rightarrow (\vec{r}-\vec{c})\cdot(\vec{e}-\vec{d})=0$$

$$\Rightarrow \overrightarrow{CH}\cdot\overrightarrow{ED}=0 \Rightarrow CH \perp ED$$

2. $\overrightarrow{OA_1}\ \overrightarrow{OA_2},\ldots,\overrightarrow{OA_n}$. All vectors are of same magnitude, say a, and angle between any two consecutive vectors is the same, that is, $2\pi/n$. Let $\hat{p}$ be the unit vector parallel to the plane of the polygon.

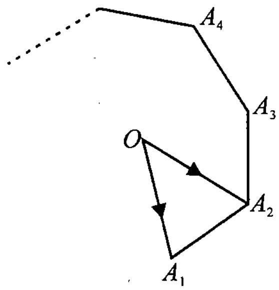


Fig. 2.50

$$\therefore \text{ Let } \overrightarrow{OA_1}\times\overrightarrow{OA_2}=a^2\sin\frac{2\pi}{n}\hat{p} \quad \text{(i)}$$

$$\text{Now, } \sum_{i=1}^{n-1}\overrightarrow{OA_i}\times\overrightarrow{OA_{i+1}}=\sum_{i=1}^{n-1}a^2\sin\frac{2\pi}{n}\hat{p}$$

$$=(n-1)\,a^2\sin\frac{2\pi}{n}\hat{p}$$

$$=(n-1)\,[-\overrightarrow{OA_2}\times\overrightarrow{OA_1}] \quad \text{(Using (i))}$$

$$=(1-n)\,[\overrightarrow{OA_2}\times\overrightarrow{OA_1}]=\text{R.H.S.}$$

3. $\vec{A}\times\vec{X}=\vec{B}$

$$\Rightarrow (\vec{A}\times\vec{X})\times\vec{A}=\vec{B}\times\vec{A}$$

$$\Rightarrow (\vec{A}\cdot\vec{A})\vec{X}-(\vec{X}\cdot\vec{A})\vec{A}=\vec{B}\times\vec{A}$$

$$\Rightarrow (\vec{A}\cdot\vec{A})\,\vec{X}-c\vec{A}=\vec{B}\times\vec{A}$$

$$\Rightarrow \vec{X}=\frac{\vec{B}\times\vec{A}+c\vec{A}}{(\vec{A}\cdot\vec{A})}$$

4. Let the position vectors of points A, B, C, D be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, respectively, with respect to some origin.

$|\overrightarrow{AB} \times \overrightarrow{CD} + \overrightarrow{BC} \times \overrightarrow{AD} + \overrightarrow{CA} \times \overrightarrow{BD}|$

$= [|(\vec{b} - \vec{a}) \times (\vec{d} - \vec{c}) + (\vec{c} - \vec{b}) \times (\vec{d} - \vec{a}) + (\vec{a} - \vec{c}) \times (\vec{d} - \vec{b})|]$

$= 2|\vec{b} \times \vec{a} + \vec{c} \times \vec{b} + \vec{a} \times \vec{c}|$ (i)

$= 2(2 \times (\text{area of } \Delta ABC))$
$= 4 \times (\text{area of } \Delta ABC)$

5. Given that $\vec{a}, \vec{b}$ and $\vec{c}$ are three coplanar vectors. Therefore, there exist scalars x, y and z, not all zero, such that

$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ (i)

Taking dot product of $\vec{a}$ and (i), we get

$x\vec{a}\cdot\vec{a} + y\vec{a}\cdot\vec{b} + z\vec{a}\cdot\vec{c} = 0$ (ii)

Again taking dot product of $\vec{b}$ and (i), we get

$x\vec{b}\cdot\vec{a} + y\vec{b}\cdot\vec{b} + z\vec{b}\cdot\vec{c} = 0$ (iii)

Now Eqs. (i), (ii) and (iii) form a homogeneous system of equations, where x, y and z are not all zero, Therefore the system must have a non-trivial solution, and for this, the determinant of coefficient matrix should be zero, i.e.,

$$\begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \end{vmatrix} = 0$$

6. We are given that $\vec{A} = 2\hat{i} + \hat{k}$, $\vec{B} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{C} = 4\hat{i} - 3\hat{j} + 7\hat{k}$ and to determine a vector $\vec{R}$ such that $\vec{R} \times \vec{B} = \vec{C} \times \vec{B}$ and $\vec{R}\cdot\vec{A} = 0$, let $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

Then $\vec{R} \times \vec{B} = \vec{C} \times \vec{B}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -3 & 7 \\ 1 & 1 & 1 \end{vmatrix}$$

$\Rightarrow (y - z)\hat{i} - (x - z)\hat{j} + (x - y)\hat{k} = -10\hat{i} + (x - z)\hat{j} + 7\hat{k}$

$y - z = -10$ (i)
$x - z = -3$ (ii)
$x - y = 7$ (iii)
Also $\vec{R}\cdot\vec{A} = 0$

$\Rightarrow 2x + z = 0$ (iv)

Substituting $y = x - 7$ and $z = -2x$ from (iii) and (iv), respectively in (i), we get
$x - 7 + 2x = -10$
$\Rightarrow 3x = -3$
$\Rightarrow x = -1, y = -8$ and $z = 2$

7. We have, $\vec{a} = cx\,\hat{i} - 6\hat{j} - 3\hat{k}$

$\vec{b} = x\hat{i} + 2\hat{j} + 2cx\,\hat{k}$

Now we know that $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

As the angle between $\vec{a}$ and $\vec{b}$ is obtuse, $\cos\theta < 0 \Rightarrow \vec{a}\cdot\vec{b} < 0$

$\Rightarrow cx^2 - 12 + 6cx < 0$
$\Rightarrow -cx^2 - 6cx + 12 > 0,\ x \in R$
$\Rightarrow -c > 0$ and $D < 0$
$\Rightarrow c < 0$ and $36c^2 + 48c < 0$
$\Rightarrow c < 0$ and $(3c + 4) > 0$
$\Rightarrow c < 0$ and $c > -4/3$
$\Rightarrow -4/3 < c < 0$

8. $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) + (\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b}) + (\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$

Here $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = -(\vec{c}\times\vec{d}\cdot\vec{b})\,\vec{a} + (\vec{c}\times\vec{d}\cdot\vec{a})\,\vec{b}$

$= [\vec{a}\ \vec{c}\ \vec{d}]\,\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a}$ (i)

$(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b}) = -(\vec{d}\times\vec{b}\cdot\vec{c})\,\vec{a} + (\vec{d}\times\vec{b}\cdot\vec{a})\,\vec{c}$

$= [\vec{a}\ \vec{d}\ \vec{b}]\,\vec{c} - [\vec{c}\ \vec{d}\ \vec{b}]\,\vec{a}$ (ii)

$(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{d}\cdot\vec{c})\,\vec{b} - (\vec{a}\times\vec{d}\cdot\vec{b})\,\vec{c}$

$= -[\vec{a}\ \vec{c}\ \vec{d}]\,\vec{b} - [\vec{a}\ \vec{d}\ \vec{b}]\,\vec{c}$ (iii)

(Note : Here we have tried to write the given expression in such a way that we can get terms involving $\vec{a}$ and other similar terms which can get cancelled)

Adding (i), (ii) and (iii), we get

Given vector $= -2\,[\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a} = k\vec{a}$

$\Rightarrow$ Given vector is parallel to $\vec{a}$.

9.

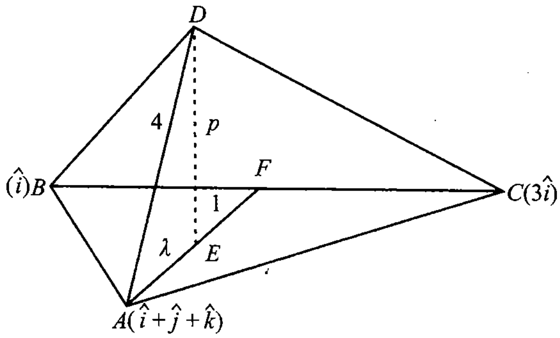


Fig. 2.51

We are given $AD = 4$

Volume of tetrahedron $= \dfrac{2\sqrt{2}}{3}$

$\Rightarrow \dfrac{1}{3}$ (Area of ΔABC) $p = \dfrac{2\sqrt{2}}{3}$

$\therefore \dfrac{1}{2} |\overrightarrow{BA} \times \overrightarrow{BC}|\, p = 2\sqrt{2}$

$\dfrac{1}{2} |(\hat{j}+\hat{k}) \times 2\hat{i}|\, p = 2\sqrt{2}$

$\Rightarrow |\hat{j}-\hat{k}|\, p = 2\sqrt{2}$

$\Rightarrow \sqrt{2}\, p = 2\sqrt{2},\ p = 2$

We have to find the P.V. of point E. Let it divide median AF in the ratio $\lambda : 1$.

P.V. of E is $\dfrac{\lambda \cdot 2\hat{i} + (\hat{i}+\hat{j}+\hat{k})}{\lambda+1}$. Therefore, (i)

$\overrightarrow{AE}$ = P.V. or E − P.V. of $A = \dfrac{\lambda(\hat{i}-\hat{j}-\hat{k})}{\lambda+1}$

$|\overrightarrow{AE}|^2 = 3\left(\dfrac{\lambda}{\lambda+1}\right)^2$ (ii)

Now, $4 + 3\left(\dfrac{\lambda}{\lambda+1}\right)^2 = 16$

$\left(\dfrac{\lambda}{\lambda+1}\right) = \pm 2$

$\lambda = -2$ or $-2/3$

Putting the value of λ in (i), we get the P.V. of possible positions of E as $-\hat{i}+3\hat{j}+3\hat{k}$ or $3\hat{i}-\hat{j}-\hat{k}$.

10. Given that $\vec{a}, \vec{b}$ and $\vec{c}$ are three unit vectors inclined at an angle θ with each other.

Also $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar. Therefore, $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$.

Also given that $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\vec{a} + q\vec{b} + r\vec{c}$.

Taking dot product on both sides with $\vec{a}$, we get

$$p + q\cos\theta + r\cos\theta = [\vec{a}\ \vec{b}\ \vec{c}] \quad \text{(i)}$$

Similarly, taking dot product on both sides with $\vec{b}$ and $\vec{c}$, we get, respectively,

$$p\cos\theta + q + r\cos\theta = 0 \quad \text{(ii)}$$

and $p\cos\theta + q\cos\theta + r = [\vec{a}\ \vec{b}\ \vec{c}]$ (iii)

Adding (i), (ii) and (iii), we get

$$p + q + r = \frac{2[\vec{a}\ \vec{b}\ \vec{c}]}{2\cos\theta + 1} \quad \text{(iv)}$$

Multiplying (iv) by $\cos\theta$ and subtracting (i) from it, we get

$$p(\cos\theta - 1) = \frac{2[\vec{a}\ \vec{b}\ \vec{c}]\cos\theta}{2\cos\theta + 1} - [\vec{a}\ \vec{b}\ \vec{c}]$$

or $p(\cos\theta - 1) = \dfrac{-[\vec{a}\ \vec{b}\ \vec{c}]}{2\cos\theta + 1}$

$$\Rightarrow p = \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{(1 - \cos\theta)(1 + 2\cos\theta)}$$

Similarly, (iv) $\times \cos\theta -$ (ii) gives

$$q = \frac{-2[\vec{a}\ \vec{b}\ \vec{c}]\cos\theta}{(1 + 2\cos\theta)(1 - \cos\theta)}$$

and (iv) $\times \cos\theta -$ (iii) gives

$$r(\cos\theta - 1) = \frac{2[\vec{a}\ \vec{b}\ \vec{c}]\cos\theta}{2\cos\theta + 1} - [\vec{a}\ \vec{b}\ \vec{c}]$$

$$\Rightarrow r = \frac{-[\vec{a}\ \vec{b}\ \vec{c}]}{(2\cos\theta + 1)(\cos\theta - 1)}$$

But we have to find p, q and r in terms of θ only.

So let us find the value of $[\vec{a}\ \vec{b}\ \vec{c}]$

We know that

$$[\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}.$$

$$= \begin{vmatrix} 1 & \cos\theta & \cos\theta \\ \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1 \end{vmatrix}$$

On operating $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} 1+2\cos\theta & \cos\theta & \cos\theta \\ 1+2\cos\theta & 1 & \cos\theta \\ 1+2\cos\theta & \cos\theta & 1 \end{vmatrix}$$

$$= (1+2\cos\theta)\begin{vmatrix} 1 & \cos\theta & \cos\theta \\ 1 & 1 & \cos\theta \\ 1 & \cos\theta & 1 \end{vmatrix}$$

Operating $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$= (1+2\cos\theta)\begin{vmatrix} 0 & \cos\theta-1 & 0 \\ 0 & 1-\cos\theta & \cos\theta-1 \\ 1 & \cos\theta & 1 \end{vmatrix}$$

Expanding along C_1

$= (1+2\cos\theta)(1-\cos\theta)^2$

$\therefore [\vec{a}\,\vec{b}\,\vec{c}] = (1-\cos\theta)\sqrt{1+2\cos\theta}$

Thus, we get

$$p = \frac{1}{\sqrt{1+2\cos\theta}}, q = \frac{-2\cos\theta}{\sqrt{1+2\cos\theta}}, r = \frac{1}{\sqrt{1+2\cos\theta}}$$

11. We have, $(\vec{A}+\vec{B})\times(\vec{A}+\vec{C})$

$= \vec{A}\times\vec{A}+\vec{B}\times\vec{A}+\vec{A}\times\vec{C}+\vec{B}\times\vec{C}$

$= \vec{B}\times\vec{A}+\vec{A}\times\vec{C}+\vec{B}\times\vec{C}$ $(\because \vec{A}\times\vec{A}=\vec{0})$

Thus $[(\vec{A}+\vec{B})\times(\vec{A}+\vec{C})]\times(\vec{B}\times\vec{C})$

$= [\vec{B}\times\vec{A}+\vec{A}\times\vec{C}+\vec{B}\times\vec{C}]\times(\vec{B}\times\vec{C})$

$= (\vec{B}\times\vec{A})\times(\vec{B}\times\vec{C})+(\vec{A}\times\vec{C})\times(\vec{B}\times\vec{C})$ $(\because x\times x=0)$

$= \{(\vec{B}\times\vec{A})\cdot\vec{C}\}\vec{B}-\{(\vec{B}\times\vec{A})\cdot\vec{B}\}\vec{C}+\{(\vec{A}\times\vec{C})\cdot\vec{C}\}\vec{B}-\{(\vec{A}\times\vec{C})\cdot\vec{B}\}\vec{C}$

$= [\vec{B}\,\vec{A}\,\vec{C}]\vec{B}-[\vec{A}\,\vec{C}\,\vec{B}]\vec{C}$

$= [\vec{A}\,\vec{C}\,\vec{B}]\{\vec{B}-\vec{C}\}$

Thus, L.H.S. of the given expression

$= [\vec{A}\,\vec{C}\,\vec{B}](\vec{B}-\vec{C})\cdot(\vec{B}+\vec{C})$

$= [\vec{A}\,\vec{C}\,\vec{B}]\{(\vec{B}-\vec{C})\cdot(\vec{B}+\vec{C})\}$

$= [\vec{A}\,\vec{C}\,\vec{B}]\{|\vec{B}|^2-|\vec{C}|^2\}=0 \quad (\because |B|=|C|)$

Alternative method:

Since $[(\vec{A}+\vec{B})\times(\vec{A}+\vec{C})]\times(\vec{B}+\vec{C})\cdot(\vec{B}+\vec{C})$ is scalar triple product of $(\vec{A}+\vec{B})\times(\vec{A}+\vec{C}), \vec{B}+\vec{C}$ and $\vec{B}+\vec{C}$, its value is 0.

12. **a.** We have $\vec{u}\cdot\vec{v}=|\vec{u}||\vec{v}|\cos\theta$

and $\vec{u}\times\vec{v} = |\vec{u}||\vec{v}|\sin\theta\,\hat{n}$

(where θ is the angle between $\vec{u}$ and $\vec{v}$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{u}$ and $\vec{v}$)

$\Rightarrow (\vec{u}\cdot\vec{v})^2+|\vec{u}\times\vec{v}|^2=|\vec{u}|^2|\vec{v}|^2(\cos^2\theta+\sin^2\theta)=|\vec{u}|^2|\vec{v}|^2$

b. $(1-\vec{u}\cdot\vec{v})^2+|\vec{u}+\vec{v}+(\vec{u}\times\vec{v})|^2$

$=1-2\vec{u}\cdot\vec{v}+(\vec{u}\cdot\vec{v})^2+|\vec{u}|^2+|\vec{v}|^2+|\vec{u}\times\vec{v}|^2+2\vec{u}\cdot\vec{v}$

$(\because \vec{u}\cdot(\vec{u}\times\vec{v})=\vec{v}\cdot(\vec{u}\times\vec{v})=0)$

$=1+|\vec{u}|^2+|\vec{v}|^2+(\vec{u}\cdot\vec{v})^2+|\vec{u}\times\vec{v}|^2$

$=1+|\vec{u}|^2+|\vec{v}|^2+|\vec{u}|^2|\vec{v}|^2$

$=(\vec{1}+|\vec{u}|^2)(\vec{1}+|\vec{v}|^2)$

13. $[\vec{u}\,\vec{v}\,\vec{w}]=(\vec{u}\times\vec{v})\cdot(\vec{v}-\vec{w}\times\vec{u})=(\vec{u}\times\vec{v})\cdot(\vec{u}\times\vec{w})$

$$=\begin{vmatrix}\vec{u}\cdot\vec{u} & \vec{u}\cdot\vec{w}\\ \vec{v}\cdot\vec{u} & \vec{v}\cdot\vec{w}\end{vmatrix}$$

Now, $\vec{u}\cdot\vec{u}=1$

$\vec{u}\cdot\vec{w}=\vec{u}\cdot(\vec{v}-\vec{w}\times\vec{u})=\vec{u}\cdot\vec{v}-[\vec{u}\,\vec{w}\,\vec{u}]=\vec{u}\cdot\vec{v}$

$\vec{v}\cdot\vec{w}=\vec{v}\cdot(\vec{v}-\vec{w}\times\vec{u})=1-[\vec{v}\,\vec{w}\,\vec{u}]=1-[\vec{u}\,\vec{v}\,\vec{w}]$

$\therefore [\vec{u}\,\vec{v}\,\vec{w}]=\begin{vmatrix}1 & \cos\theta\\ \cos\theta & 1-[\vec{u}\,\vec{v}\,\vec{w}]\end{vmatrix}$ (θ is the angle between $\vec{u}$ and $\vec{v}$)

$=1-[\vec{u}\,\vec{v}\,\vec{w}]-\cos^2\theta$

$\therefore [\vec{u}\,\vec{v}\,\vec{w}]=\frac{1}{2}\sin^2\theta\le\frac{1}{2}$

Equality holds when $\sin^2\theta=1$, i.e., $\theta=\pi/2$, i.e., $\vec{u}\perp\vec{v}$.

14. Given data are insufficient to uniquely determine the three vectors as there are only six equations involving nine variables.

Therefore, we can obtain infinite number of sets of three vectors, $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$, satisfying these conditions.

From the given data, we get

$\vec{v}_1 \cdot \vec{v}_1 = 4 \Rightarrow |\vec{v}_1| = 2$

$\vec{v}_2 \cdot \vec{v}_2 = 2 \Rightarrow |\vec{v}_2| = \sqrt{2}$

$\vec{v}_3 \cdot \vec{v}_3 = 29 \Rightarrow |\vec{v}_3| = \sqrt{29}$

Also $\vec{v}_1 \cdot \vec{v}_2 = -2$

$\Rightarrow |v_1||v_2| \cos\theta = -2$ (where θ is the angle between $\vec{v}_1$ and $\vec{v}_2$)

$\Rightarrow \cos\theta = \dfrac{-1}{\sqrt{2}}$

$\Rightarrow \theta = 135°$

Since any two vectors are always coplanar, let us suppose that $\vec{v}_1$ and $\vec{v}_2$ are in the x–y plane. Let $\vec{v}_1$ be along the positive direction of the x-axis. Then $\vec{v}_1 = 2\hat{i}$. $(\because |\vec{v}_1| = 2)$

As $\vec{v}_2$ makes an angle 135° with $\vec{v}_1$ and lies in the x–y plane, also keeping in mind $|\vec{v}_2| = \sqrt{2}$, we obtain $\vec{v}_2 = -\hat{i} \pm \hat{j}$

Again let $\vec{v}_3 = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$

$\because \vec{v}_3 \cdot \vec{v}_1 = 6 \Rightarrow 2\alpha = 6 \Rightarrow \alpha = 3$

and $\vec{v}_3 \cdot \vec{v}_2 = -5 \Rightarrow -\alpha \pm \beta = -5 \Rightarrow \beta = \pm 2$

Also $|\vec{v}_3| = \sqrt{29} \Rightarrow \alpha^2 + \beta^2 + \gamma^2 = 29$

$\Rightarrow \gamma = \pm 4$

Hence $\vec{v}_3 = 3\hat{i} \pm 2\hat{j} \pm 4\hat{k}$

15. Given that $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_2\hat{k}$ where a_r, b_r, c_r $(r = 1, 2, 3)$ are all non-negative real numbers

Also $\sum_{r=1}^{3}(a_r + b_r + c_r) = 3L$

To prove $V \le L^3$, where V is the volume of the parallolepiped formed by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$, we have

$$V = [\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\Rightarrow V = (a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2) - (a_1 b_3 c_2 + a_2 b_1 c_3 + a_3 b_2 c_1) \quad \text{(i)}$$

Now we know that A.M. ≥ G.M., therefore

$$\frac{(a_1+b_1+c_1)+(a_2+b_2+c_2)+(a_3+b_3+c_3)}{3} \geq [(a_1+b_1+c_1)(a_2+b_2+c_2)(a_3+b_3+c_3)]^{1/3}$$

$$\Rightarrow \frac{3L}{3} \geq [(a_1+b_1+c_1)(a_2+b_2+c_2)(a_3+b_3+c_3)]^{1/3}$$

$$\Rightarrow L^3 \geq (a_1+b_1+c_1)(a_2+b_2+c_2)(a_3+b_3+c_3)$$

$$= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 + 24 \text{ more such terms}$$

$$\geq a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 \quad (\because a_r, b_r, c_r \geq \text{or } r = 1, 2, 3)$$

$$\geq (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_1b_3c_2 + a_2b_1c_2 + a_3b_2c_1) \quad \text{(same reason)}$$

$$= V \text{ (from (i))}$$

Thus, $L^3 \geq V$

16. We know that $[\vec{x}\times\vec{y}\ \vec{y}\times\vec{z}\ \vec{z}\times\vec{x}] = [\vec{x}\ \vec{y}\ \vec{z}]^2$

Also a vector along the bisector of given two unit vectors $\vec{u}, \vec{v}$ is $\vec{u}+\vec{v}$.

A unit vector along the bisector is $\dfrac{\vec{u}+\vec{v}}{|\vec{u}+\vec{v}|}$

$$|\vec{u}+\vec{v}|^2 = 1+1+2\vec{u}\cdot\vec{v} = 2+2\cos\alpha = 4\cos^2\frac{\alpha}{2}$$

$$\Rightarrow \vec{x} = \frac{\vec{u}+\vec{v}}{2\cos\dfrac{\alpha}{2}}$$

Similarly, $\vec{y} = \dfrac{\vec{v}+\vec{w}}{2\cos\beta/2}$ and $\vec{z} = \dfrac{\vec{u}+\vec{w}}{2\cos\gamma/2}$

$$\Rightarrow [\vec{x}\ \vec{y}\ \vec{z}] = \frac{1}{8}[\vec{u}+\vec{v}\ \ \vec{v}+\vec{w}\ \ \vec{u}+\vec{w}]\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}$$

$$= \frac{1}{8}2[\vec{u}\ \vec{v}\ \vec{w}]\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}$$

$$= \frac{1}{4}[\vec{u}\ \vec{v}\ \vec{w}]\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}$$

$$\Rightarrow [\vec{x}\times\vec{y}\ \vec{y}\times\vec{z}\ \vec{z}\times\vec{x}] = [\vec{x}\ \vec{y}\ \vec{z}]^2$$

$$= \frac{1}{16}[\vec{u}\ \vec{v}\ \vec{w}]^2\sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}$$

17. Given that $\vec{a}\times\vec{c} = \vec{b}\times\vec{d}$ (i)

and $\vec{a}\times\vec{b} = \vec{c}\times\vec{d}$ (ii)

Subtracting (ii) from (i), we get

$$\vec{a}\times(\vec{c}-\vec{b}) = (\vec{b}-\vec{c})\times\vec{d}$$

$\Rightarrow \vec{a}\times(\vec{c}-\vec{b}) = \vec{d}\times(\vec{c}-\vec{b})$

$\Rightarrow \vec{a}\times(\vec{c}-\vec{b}) - \vec{d}\times(\vec{c}-\vec{b}) = 0$

$\Rightarrow (\vec{a}-\vec{d})\times(\vec{c}-\vec{d}) = 0$

$\Rightarrow (\vec{a}-\vec{d}) \,||\, (\vec{c}-\vec{d})$ $(\because \vec{a}-\vec{d}\neq 0, \vec{c}-\vec{b}\neq 0)$

$\Rightarrow$ Angle between $\vec{a}-\vec{d}$ and $\vec{c}-\vec{b}$ is either 0 or 180°.

$\Rightarrow (\vec{a}-\vec{d})\cdot(\vec{c}-\vec{b}) = |\vec{a}-\vec{d}||\vec{c}-\vec{b}|\cos 0 \neq 0$ as $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ all are different.

18. The following figure shows the possible situation for planes P_1 and P_2 and the lines L_1 and L_2:

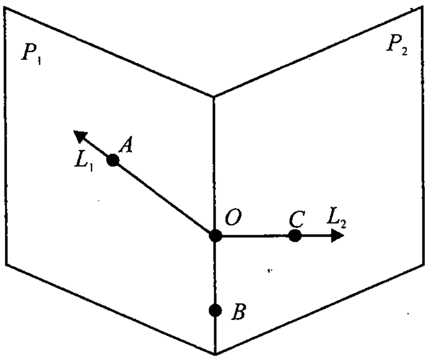


Fig. 2.52

Now if we choose points A, B and C as A on L_1, B on the line of intersection of P_1 and P_2 but other than the origin and C on L_2 again other than the origin, then we can consider

A corresponds to one of A', B', C'

B corresponds to one of the remaining of A', B' and C'

C corresponds to third of A', B' and C', e.g., $A'\equiv C$; $B'\equiv B$; $C'\equiv A$

Hence one permutation of $[A\ B\ C]$ is $[CBA]$. Hence proved.

19. Given that the incident ray is along $\hat{v}$, the reflected ray is along $\hat{w}$ and the normal is along $\hat{a}$, outwards. The given figure can be redrawn as shown.

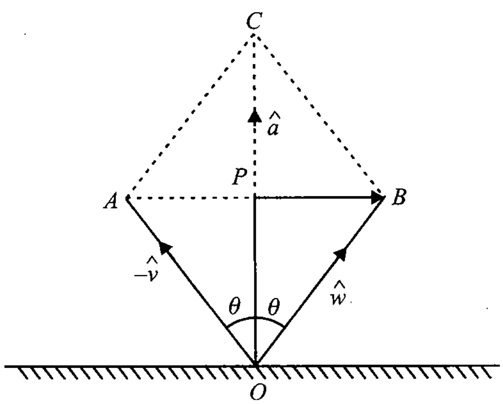


Fig. 2.53

We know that the incident ray, the reflected ray, and the normal lie in a plane, and the angle of incidence = angle of reflection.

Therefore, $\hat{a}$ will be along the angle bisector of $\hat{w}$ and $-\hat{v}$, i.e.,

$$\hat{a} = \frac{\hat{w} + (-\hat{v})}{|\hat{w} - \hat{v}|} \qquad \text{(i)}$$

But $\hat{a}$ is a unit vector

where $|\hat{w} - \hat{v}| = OC = 2OP$

$= 2|\hat{w}|\cos\theta = 2\cos\theta$

Substituting this value in (i),

$$\hat{a} = \frac{\hat{w} - \hat{v}}{2\cos\theta}$$

$\Rightarrow \hat{w} = \hat{v} + (2\cos\theta)\hat{a}$

$\Rightarrow \hat{a} = \hat{v} - 2(\hat{a}\cdot\hat{v})\hat{a} \quad (\hat{a}\cdot\hat{v} = -\cos\theta)$

Objective Type

Fill in the blanks

1. Given that $|\vec{A}| = 3;\ |\vec{B}| = 4;\ |\vec{C}| = 5$

$\vec{A} \perp (\vec{B} + \vec{C}) \Rightarrow \vec{A}\cdot(\vec{B} + \vec{C}) = 0 \Rightarrow \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C} = 0 \qquad \text{(i)}$

$\vec{B} \perp (\vec{C} + \vec{A}) \Rightarrow \vec{B}\cdot(\vec{C} + \vec{A}) = 0 \Rightarrow \vec{B}\cdot\vec{C} + \vec{B}\cdot\vec{A} = 0 \qquad \text{(ii)}$

$\vec{C} \perp (\vec{A} + \vec{B}) \Rightarrow \vec{C}\cdot(\vec{A} + \vec{B}) = 0 \Rightarrow \vec{C}\cdot\vec{A} + \vec{C}\cdot\vec{B} = 0 \qquad \text{(iii)}$

Adding (i), (ii) and (iii), we get

$2(\vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{C} + \vec{C}\cdot\vec{A}) = 0 \qquad \text{(iv)}$

Now, $|\vec{A} + \vec{B} + \vec{C}|^2$

$= (\vec{A} + \vec{B} + \vec{C})\cdot(\vec{A} + \vec{B} + \vec{C})$

$= |\vec{A}|^2 + |\vec{B}|^2 + |\vec{C}|^2 + 2(\vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{C} + \vec{C}\cdot\vec{A})$

$= 9 + 16 + 25 + 0$

$= 50$

$\therefore |\vec{A} + \vec{B} + \vec{C}| = 5\sqrt{2}$

2. Required unit vector

$$\hat{a} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|}$$

$$\overrightarrow{PQ} = \hat{i} + \hat{j} - 3\hat{k};\ \overrightarrow{PR} = -\hat{i} + 3\hat{j} - \hat{k}$$

$$\therefore\ \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix}$$

$$= 8\hat{i} + 4\hat{j} + 4\hat{k}$$

$$\therefore\ |\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{64+16+16} = \sqrt{96} = 4\sqrt{6}$$

$$\therefore\ \hat{n} = \frac{8\hat{i} + 4\hat{j} + 4\hat{k}}{4\sqrt{6}} = \frac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{6}}$$

3. Area of $\Delta ABC = \frac{1}{2}\ |\overrightarrow{BA} \times \overrightarrow{BC}|$

$$\overrightarrow{BA} = -\hat{i} - 2\hat{j} + 3\hat{k}$$

$$\overrightarrow{BC} = \hat{i} - 2\hat{j} + 3\hat{k}$$

$$\therefore\ \text{Area} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & 3 \\ 1 & -2 & 3 \end{vmatrix} = \frac{1}{2}\,|6\hat{j} + 4\hat{k}|$$

$$= |3\hat{j} + 2\hat{k}|$$

$$= \sqrt{9+4} = \sqrt{13}$$

4. $$\frac{\vec{A}\cdot\vec{B}\times\vec{C}}{\vec{C}\times\vec{A}\cdot\vec{B}} + \frac{\vec{B}\cdot\vec{A}\times\vec{C}}{\vec{C}\cdot\vec{A}\times\vec{B}}$$

$$= \frac{[\vec{A}\,\vec{B}\,\vec{C}]}{[\vec{A}\,\vec{B}\,\vec{C}]} + \frac{-[\vec{A}\,\vec{B}\,\vec{C}]}{[\vec{A}\,\vec{B}\,\vec{C}]} = 0$$

5. Given $\vec{A} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{C} = \hat{j} - \hat{k}$

Let $\vec{B} = x\hat{i} + y\hat{j} + z\hat{k}$

Given that $\vec{A} \times \vec{B} = \vec{C} \Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = \hat{j} - \hat{k}$

$\Rightarrow (z-y)\,i+(x-z)\,\hat{j}+(y-x)\,\hat{k} = \hat{j}-\hat{k}$

$\Rightarrow z-y=0, x-z=1$ and $y-x=-1$ (i)

Also, $\vec{A}\cdot\vec{B} = 3$

$\Rightarrow x+y+z=3$ (ii)

Using (i) and (ii), we get

$y=2/3, x=5/3, z=2/3$

$\therefore \vec{B} = \frac{5}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}$

6. Let $\vec{c} = \alpha\,\hat{i}+\beta\,\hat{j}$

Given that $\vec{b}\perp\vec{c}$

$\therefore \vec{b}\cdot\vec{c} = 0.$

$\Rightarrow (4\hat{i}+3\hat{j})\cdot(\alpha\hat{i}+\beta\,\hat{j}) = 0$

$\Rightarrow 4\,\alpha+3\beta=0$

$\Rightarrow \frac{\alpha}{3} = \frac{\beta}{-4} = \lambda$

$\Rightarrow \alpha=3\,\lambda, \beta=-4\,\lambda$ (i)

Now let $\vec{a} = x\hat{i}+y\,\hat{j}$ be the required vectors.

Given that projection of $\vec{a}$ along $\vec{b}$

$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$= \frac{4x+3y}{\sqrt{4^2+3^2}} = 1$

$\Rightarrow 4x+3y=5$ (ii)

Also projection of $\vec{a}$ along $\vec{c}$

$\Rightarrow \frac{\vec{a}\cdot\vec{c}}{|\vec{c}|} = 2$

$\Rightarrow \frac{\alpha x+\beta y}{\sqrt{\alpha^2+\beta^2}} = 2$

$\Rightarrow 3\lambda x-4\lambda y=10\,\lambda$

$\Rightarrow 3x-4y=10$ (iii)

Solving (ii) and (iii), we get $x=2, y=-1$

$\therefore$ The required vector is $2\hat{i}-\hat{j}$.

7.

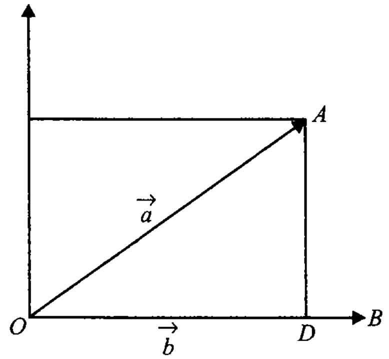


Fig. 2.54

Component of $\vec{a}$ along $\vec{b}$

$$\overrightarrow{OD} = OA\cos\theta \cdot \frac{\vec{b}}{|\vec{b}|}$$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\frac{\vec{b}}{|\vec{b}|} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

Component of $\vec{a}$ perpendicular to $\vec{b}$

$$\overrightarrow{DA} = \vec{a} - \overrightarrow{OD}$$

$$= \vec{a} - \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

8. Let $x\hat{i} + y\hat{j} + z\hat{k}$ be a unit vector coplanar with $\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$ and also perpendicular to $\hat{i} + \hat{j} + \hat{k}$

Then, $\begin{vmatrix} x & y & z \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 0$

$\Rightarrow -3x + y + z = 0$ (i)

and $x + y + z = 0$ (ii)

Solving the above by cross-product method, we get $\frac{x}{0} = \frac{y}{4} = \frac{z}{-4}$ or $\frac{x}{0} = \frac{y}{1} = \frac{z}{-1} = \lambda$ (say)

$\Rightarrow x = 0, y = \lambda, z = -\lambda$

As $x\hat{i} + y\hat{j} + z\hat{k}$ is a unit vector,

$\Rightarrow 0 + \lambda^2 + \lambda^2 = 1$

$\Rightarrow \lambda^2 = \frac{1}{2} \Rightarrow \lambda = \pm \frac{1}{\sqrt{2}}$

$\therefore$ The required vector is $\frac{\hat{j} - \hat{k}}{\sqrt{2}}$ or $\frac{-\hat{j} + \hat{k}}{\sqrt{2}}$.

9. A vector normal to the plane containing vectors $\hat{i}$ and $\hat{i} + \hat{j}$ is

$$\vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = \hat{k}$$

A vector normal to the plane containing vectors $\hat{i} - \hat{j}, \hat{i} + \hat{k}$ is

$$\vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -\hat{i} - \hat{j} + \hat{k}.$$

Vector $\vec{a}$ is parallel to vector $\vec{p} \times \vec{q}$.

$$\vec{p} \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ -1 & -1 & 1 \end{vmatrix} = \hat{i} - \hat{j}$$

$\therefore$ A vector in direction of $\vec{a}$ is $\hat{i} - \hat{j}$

Now if θ is the angle between $\vec{a}$ and $\hat{i} - 2\hat{j} + 2\hat{k}$, then

$$\cos\theta = \pm \frac{1 \cdot 1 + (-1) \cdot (-2)}{\sqrt{1+1}\ \sqrt{1+4+4}} = \pm \frac{3}{\sqrt{2} \cdot 3}$$

$$\Rightarrow \cos\theta = \pm \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

10. Let $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ be any three mutually perpendicular non-coplanar unit vectors and $\vec{a}$ be any vector, then

$$\vec{a} = (\vec{a} \cdot \vec{\alpha})\vec{\alpha} + (\vec{a} \cdot \vec{\beta})\vec{\beta} + (\vec{a} \cdot \vec{\gamma})\vec{\gamma}$$

Here $\vec{b}, \vec{c}$ are two mutually perpendicular vectors, therefore $\vec{b}, \vec{c}$ and $\frac{\vec{b} \times \vec{c}}{|\vec{b} \times \vec{c}|}$ are three mutually perpendicular non-coplanar unit vectors.

Hence $$\vec{a} = \left(\vec{a} \cdot \vec{b}\right)\vec{b} + \left(\vec{a} \cdot \vec{c}\right)\vec{c} + \left(\vec{a} \cdot \frac{\vec{b} \times \vec{c}}{|\vec{b} \times \vec{c}|}\right)\frac{\vec{b} \times \vec{c}}{|\vec{b} \times \vec{c}|}$$

$$= \left(\vec{a} \cdot \vec{b}\right)\vec{b} + \left(\vec{a} \cdot \vec{c}\right)\vec{c} + \frac{\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)}{|\vec{b} \times \vec{c}|^2}\left(\vec{b} \times \vec{c}\right)$$

11. $\vec{a}\times(\vec{a}\times\vec{c})+\vec{b}=\vec{0}$

$\Rightarrow (\vec{a}\cdot\vec{c})\,\vec{a}-(\vec{a}\cdot\vec{a})\,\vec{c}+\vec{b}=\vec{0}$

$\Rightarrow 2\cos\theta\cdot\vec{a}-\vec{c}+\vec{b}=\vec{0}$ (using $|\vec{a}|=1, |\vec{b}|=1, |\vec{c}|=2$)

$\Rightarrow (2\cos\theta\,\vec{a}-\vec{c})^2=(-\vec{b})^2$

$\Rightarrow 4\cos^2\theta\cdot|\vec{a}|^2+|\vec{c}|^2-2\cdot 2\cos\theta\cdot\vec{a}\cdot\vec{c}=|\vec{b}|^2$

$\Rightarrow 4\cos^2\theta+4-8\cos\theta\cdot\cos\theta=1$

$\Rightarrow 4\cos^2\theta-8\cos^2\theta+4=1$

$\Rightarrow 4\cos^2\theta=3$

$\Rightarrow \cos\theta=\pm\sqrt{3}/2$

For θ to be acute, $\cos\theta=\dfrac{\sqrt{3}}{2}\Rightarrow\theta=\dfrac{\pi}{6}$

12. Given that $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ are position vectors of points A, B, C and D, respectively, such that

$(\vec{a}-\vec{d})\cdot(\vec{b}-\vec{c})=(\vec{b}-\vec{d})\cdot(\vec{c}-\vec{a})=0$

$\Rightarrow \overrightarrow{DA}\cdot\overrightarrow{CB}=\overrightarrow{DB}\cdot\overrightarrow{AC}=0$

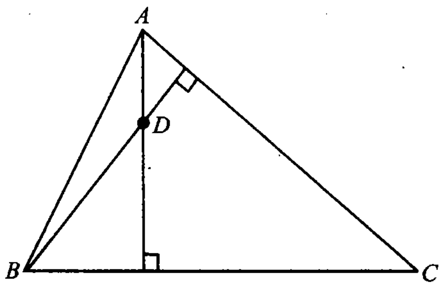


Fig. 2.55

$\Rightarrow \overrightarrow{DA}\perp\overrightarrow{CB}$ and $\overrightarrow{DB}\perp\overrightarrow{AC}$

Clearly, D is the orthocentre of ΔABC.

13. q = area of parallelogram with $\overrightarrow{OA}$ and $\overrightarrow{OC}$ as adjacent sides

$=|\overrightarrow{OA}\times\overrightarrow{OC}|$

$=|\vec{a}\times\vec{b}|$

p = area of quadrilateral $OABC$

$=\dfrac{1}{2}|\overrightarrow{OA}\times\overrightarrow{OB}|+\dfrac{1}{2}|\overrightarrow{OB}\times\overrightarrow{OC}|=\dfrac{1}{2}[|\vec{a}\times(10\vec{a}+2\vec{b})|+|(10\vec{a}+2\vec{b})\times\vec{b}|]$

$=\dfrac{1}{2}|(12\,\vec{a}\times\vec{b})|=6|\vec{a}\times\vec{b}|\Rightarrow k=6$

14. $\vec{a}\cdot\vec{b} = -1+3 = 2$

$|\vec{a}| = 2, |\vec{b}| = 2$

$\cos\theta = \frac{2}{2\times 2} = \frac{1}{2}$

$\theta = \frac{\pi}{3}$ but its value is $\frac{2\pi}{3}$ as it is opposite to the side of maximum length.

True or false

1. $\vec{A}, \vec{B}$ and $\vec{C}$ are three unit vectors such that $\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{C} = 0$ (i) and the angle between $\vec{B}$ and $\vec{C}$ is $\pi/3$. Now Eq. (i) shows that $\vec{A}$ is perpendicular to both $\vec{B}$ and $\vec{C}$.

$\Rightarrow \vec{B}\times\vec{C} = \lambda\vec{A}$, where λ is any scalar.

$\Rightarrow |\vec{B}\times\vec{C}| = |\lambda\vec{A}|$

$\Rightarrow \sin\pi/3 = \pm\lambda$ (as $\pi/3$ is the angle between $\vec{B}$ and $\vec{C}$)

$\Rightarrow \lambda = \pm\sqrt{3}/2$

$\Rightarrow \vec{B}\times\vec{C} = \pm\frac{\sqrt{3}}{2}\vec{A}$

$\Rightarrow \vec{A} = \pm\frac{2}{\sqrt{3}}(\vec{B}\times\vec{C})$

Therefore, the given statement is false.

2. $\vec{X}\cdot\vec{A} = 0 \Rightarrow$ either $\vec{A} = 0$ or $\vec{X}\perp\vec{A}$

$\vec{X}\cdot\vec{B} = 0 \Rightarrow$ either $\vec{B} = 0$ or $\vec{X}\perp\vec{B}$

$\vec{X}\cdot\vec{C} = 0 \Rightarrow$ either $\vec{C} = 0$ $\vec{X}\perp\vec{C}$

In any of the three cases, $\vec{A}, \vec{B}, \vec{C} = 0 \Rightarrow [\vec{A}\,\vec{B}\,\vec{C}] = 0$

Otherwise if $\vec{X}\perp\vec{A}, \vec{X}\perp\vec{B}$ and $\vec{X}\perp\vec{C}$, then $\vec{A}, \vec{B}$ and $\vec{C}$ are coplanar.

$\Rightarrow [\vec{A}\,\vec{B}\,\vec{C}] = 0$

Therefore, the statement is true.

3. Clearly vectors $\vec{a}-\vec{b}, \vec{b}-\vec{c}, \vec{c}-\vec{a}$ are coplanar

$\Rightarrow [\vec{a}-\vec{b}\;\; \vec{b}-\vec{c}\;\; \vec{c}-\vec{a}] = 0$

Therefore, the given statement is false.

Multiple choice questions with one correct answer

1. **a.** $\vec{A}\cdot(\vec{B}+\vec{C})\times(\vec{A}+\vec{B}+\vec{C}) = \vec{A}\cdot[\vec{B}\times\vec{A}+\vec{B}\times\vec{B}+\vec{B}\times\vec{C}+\vec{C}\times\vec{A}+\vec{C}\times\vec{B}+\vec{C}\times\vec{C}]$

$= \vec{A}\cdot\vec{B}\times\vec{A}+\vec{A}\cdot\vec{B}\times\vec{C}+\vec{A}\cdot\vec{C}\times\vec{A}+\vec{A}\cdot\vec{C}\times\vec{B}$ (using $\vec{a}\times\vec{a}=0$)

$=0+[\vec{A}\,\vec{B}\,\vec{C}]+0+[\vec{A}\,\vec{C}\,\vec{B}]$

$=[\vec{A}\,\vec{B}\,\vec{C}]-[\vec{A}\,\vec{B}\,\vec{C}]$

$=0$

2. **d.** $|(\vec{a}\times\vec{b})\cdot\vec{c}| = |\vec{a}||\vec{b}||\vec{c}|$

$\Rightarrow ||\vec{a}||\vec{b}|\sin\theta\,\hat{n}\cdot\vec{c}| = |\vec{a}||\vec{b}||\vec{c}|$

$\Rightarrow |\vec{a}||\vec{b}||\vec{c}||\sin\theta\cos\alpha| = |\vec{a}||\vec{b}||\vec{c}|$

$\Rightarrow |\sin\theta||\cos\alpha| = 1$

$\Rightarrow \theta=\pi/2$ and $\alpha=0$

$\Rightarrow \vec{a}\perp\vec{b}$ and $\vec{c}\,||\,\hat{n}$ or perpendicular to both $\vec{a}$ and $\vec{b}$

$\Rightarrow \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0$

3. **d.** Volume of parallelopiped $= [\vec{a}\;\vec{b}\;\vec{c}]$

$$= \begin{vmatrix} 2 & -2 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix} = 2(-1)+2(-1+3)=2$$

4. **d.** Given that $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar. Therefore,

$[\vec{a}\;\vec{b}\;\vec{c}]\neq 0$

Also $\vec{p}=\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\;\vec{b}\;\vec{c}]},\ \vec{q}=\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\;\vec{b}\;\vec{c}]},\ \vec{r}=\dfrac{\vec{a}\times\vec{b}}{[\vec{a}\;\vec{b}\;\vec{c}]}$ (i)

Now, $(\vec{a}+\vec{b})\cdot\vec{p}+(\vec{b}+\vec{c})\cdot\vec{q}+(\vec{c}+\vec{a})\cdot\vec{r}$

$= (\vec{a}+\vec{b})\cdot\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\;\vec{b}\;\vec{c}]}+(\vec{b}+\vec{c})\cdot\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\;\vec{b}\;\vec{c}]}+(\vec{c}+\vec{a})\cdot\dfrac{\vec{a}\times\vec{b}}{[\vec{a}\;\vec{b}\;\vec{c}]}$

$= \dfrac{\vec{a}\cdot\vec{b}\times\vec{c}}{[\vec{a}\;\vec{b}\;\vec{c}]}+\dfrac{\vec{b}\cdot\vec{c}\times\vec{a}}{[\vec{a}\;\vec{b}\;\vec{c}]}+\dfrac{\vec{c}\cdot\vec{a}\times\vec{b}}{[\vec{a}\;\vec{b}\;\vec{c}]}$ $[\because \vec{b}\cdot\vec{b}\times\vec{c} = \vec{c}\cdot\vec{c}\times\vec{a} = \vec{a}\cdot\vec{a}\times\vec{b} = 0]$

$= \dfrac{[\vec{a}\;\vec{b}\;\vec{c}]}{[\vec{a}\;\vec{b}\;\vec{c}]}+\dfrac{[\vec{a}\;\vec{b}\;\vec{c}]}{[\vec{a}\;\vec{b}\;\vec{c}]}+\dfrac{[\vec{a}\;\vec{b}\;\vec{c}]}{[\vec{a}\;\vec{b}\;\vec{c}]}$

$=1+1+1$

$=3$

5. **a.** Let $\vec{d} = x\hat{i} + y\hat{j} + z\hat{k}$

where $x^2 + y^2 + z^2 = 1$ (i)

($\vec{d}$ being a unit vector)

$\therefore \vec{a}.\vec{d} = 0$

$\Rightarrow x - y = 0 \Rightarrow x = y$ (ii)

$[\vec{b}\ \vec{c}\ \vec{d}] = 0$

$$\Rightarrow \begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ x & y & z \end{vmatrix} = 0$$

$\Rightarrow x + y + z = 0$

$\Rightarrow 2x + z = 0$ (using (ii))

$\Rightarrow z = -2x$ (iii)

From (i), (ii) and (iii)

$x^2 + x^2 + 4x^2 = 1$

$x = \pm \dfrac{1}{\sqrt{6}}$

$$\therefore \vec{d} = \pm\left(\frac{1}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j} - \frac{2}{\sqrt{6}}\vec{k}\right) = \pm\left(\frac{\hat{i} + \hat{j} - 2\hat{k}}{\sqrt{6}}\right)$$

6. **a.** Since $\vec{a} \times (\vec{b} \times \vec{c}) = \dfrac{\vec{b} + \vec{c}}{\sqrt{2}}$

$$\therefore (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c} = \frac{1}{\sqrt{2}}\vec{b} + \frac{1}{\sqrt{2}}\vec{c}$$

Since $\vec{b}$ and $\vec{c}$ are non-coplanar

$\Rightarrow \vec{a}\cdot\vec{c} = \dfrac{1}{\sqrt{2}}$ and $\vec{a}\cdot\vec{b} = -\dfrac{1}{\sqrt{2}}$

$\Rightarrow \cos\theta = -\dfrac{1}{\sqrt{2}}$ (because $\vec{a}$ and $\vec{b}$ are unit vectors)

$\Rightarrow \theta = \dfrac{3\pi}{4}$

7. **b.** Since $\vec{u} + \vec{v} + \vec{w} = 0$,

$|\vec{u} + \vec{v} + \vec{w}|^2 = 0$

$\Rightarrow |\vec{u}|^2 + |\vec{v}|^2 + |\vec{w}|^2 + 2(\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{u}) = 0$

$\Rightarrow 9 + 16 + 25 + 2\,(\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{u}) = 0$

$\Rightarrow \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{u} = -25$

8. **d.** $(\vec{a}+\vec{b}+\vec{c})\cdot[(\vec{a}+\vec{b})\times(\vec{a}+\vec{c})]$

$= (\vec{a}+\vec{b}+\vec{c})\cdot[\vec{a}\times\vec{a}+\vec{a}\times\vec{c}+\vec{b}\times\vec{a}+\vec{b}\times\vec{c}]$

$= (\vec{a}+\vec{b}+\vec{c})\cdot[\vec{a}\times\vec{c}+\vec{b}\times\vec{a}+\vec{b}\times\vec{c}]$

$= \vec{a}\cdot\vec{b}\times\vec{c}+\vec{b}\cdot\vec{a}\times\vec{c}+\vec{c}\cdot\vec{b}\times\vec{a}$

$= [\vec{a}\,\vec{b}\,\vec{c}]-[\vec{a}\,\vec{b}\,\vec{c}]-[\vec{a}\,\vec{b}\,\vec{c}]$

$= -[\vec{a}\,\vec{b}\,\vec{c}]$

9. **b.** As $\vec{p}, \vec{q}$ and $\vec{r}$ are three mutually perpendicular vectors of same magnitude, so let us consider $\vec{p}=a\hat{i}, \vec{q}=a\hat{j}, \vec{r}=a\hat{k}$

Also let $\vec{x}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$

Given that $\vec{x}$ satisfies the equation

$\vec{p}\times[(\vec{x}-\vec{q})\times\vec{p}]+\vec{q}\times[(\vec{x}-\vec{r})\times\vec{q}]+\vec{r}\times[(\vec{x}-\vec{p})\times\vec{r}]=0$ (i)

Now $\vec{p}\times[(\vec{x}-\vec{q})\times\vec{p}] = \vec{p}\times[\vec{x}\times\vec{p}-\vec{q}\times\vec{p}]$

$= \vec{p}\times(\vec{x}\times\vec{p})-\vec{p}\times(\vec{q}\times\vec{p})$

$= (\vec{p}\cdot\vec{p})\,\vec{x}-(\vec{p}\cdot\vec{x})\,\vec{p}-(\vec{p}\cdot\vec{p})\,\vec{q}+(\vec{p}\cdot\vec{q})\,\vec{p}$

$= a^2\vec{x}-a^2x_1\hat{i}-a^3\hat{j}+0$

Similarly,

$\vec{q}\times[(\vec{x}-\vec{r})\times\vec{q}] = a^2\vec{x}-a^2y_1\hat{j}-a^3\hat{k}$

and $\vec{r}\times[(\vec{x}-\vec{p})\times\vec{r}]=a^2\vec{x}-a^2z_1\hat{k}-a^3\hat{i}$

Substituting these values in the equation, we get

$3a^2\vec{x}-a^2(x_1\hat{i}+y_1\hat{j}+z_1\hat{k})-a^2(a\hat{i}+a\hat{j}+a\hat{k})=0$

$\Rightarrow 3a^2\vec{x}-a^2\vec{x}-a^2(\vec{p}+\vec{q}+\vec{r})=\vec{0}$

$\Rightarrow 2a^2\vec{x}=(\vec{p}+\vec{q}+\vec{r})\,a^2$

$\Rightarrow \vec{x}=\frac{1}{2}(\vec{p}+\vec{q}+\vec{r})$

10. **b.** $|(\vec{a}\times\vec{b})\times\vec{c}|=|\vec{a}\times\vec{b}||\vec{c}|\sin 30°$

$=\frac{1}{2}|\vec{a}\times\vec{b}||\vec{c}|$ (i)

We have, $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$

$\Rightarrow \vec{a} \times \vec{b} = 2\hat{i} - 2\hat{j} + \hat{k}$

$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{9} = 3$

Also given $|\vec{c} - \vec{a}| = 2\sqrt{2}$

$\Rightarrow |\vec{c} - \vec{a}|^2 = 8$

$\Rightarrow |\vec{c}|^2 + |\vec{a}|^2 - 2\,\vec{a}\cdot\vec{c} = 8$

Given $|\vec{a}| = 3$ and $\vec{a} \cdot \vec{c} = |\vec{c}|$, using these we get

$|\vec{c}|^2 - 2|\vec{c}| + 1 = 0$

$\Rightarrow (|\vec{c}| - 1)^2 = 0$

$\Rightarrow |\vec{c}| = 1$

Substituting values of $|\vec{a} \times \vec{b}|$ and $|\vec{c}|$ in (i), we get

$|(\vec{a} \times \vec{b}) \times \vec{c}| = \frac{1}{2} \times 3 \times 1 = \frac{3}{2}$

11. **a.** As $\vec{c}$ is coplanar with $\vec{a}$ and $\vec{b}$, we take $\vec{c} = \alpha\,\vec{a} + \beta\,\vec{b}$ (i)

where α and β are scalars.

As $\vec{c}$ is perpendicular to $\vec{a}$, using (i), we get,

$0 = \alpha\,\vec{a} \cdot \vec{a} + \beta\,\vec{b} \cdot \vec{a}$

$\Rightarrow 0 = \alpha(6) + \beta(2 + 2 - 1) = 3\,(2\alpha + \beta)$

$\Rightarrow \beta = -2\alpha$

Thus, $\vec{c} = \alpha\,(\vec{a} - 2\vec{b}) = \alpha\,(-3j + 3k) = 3\alpha\,(-j + k)$

$\Rightarrow |\vec{c}|^2 = 18\alpha^2$

$\Rightarrow 1 = 18\alpha^2$

$\Rightarrow \alpha = \pm\,\frac{1}{3\sqrt{2}}$

$\therefore\ \vec{c} = \pm\,\frac{1}{\sqrt{2}}\,(-j + k)$

12. **b.** Given $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ (by triangle law). Therefore,

$\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0}$

$\vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}$

$\vec{a} \times \vec{b} = \vec{c} \times \vec{a}$

Similarly by taking cross product with $\vec{b}$, we get $\vec{a} \times \vec{b} = \vec{b} \times \vec{c}$

$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

13. **a.** Given that $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are vectors such that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{0}$ (i)

P_1 is the plane determined by vectors $\vec{a}$ and $\vec{b}$. Therefore, normal vectors $\vec{n_1}$ to P_1 will be given by

$\vec{n_1} = \vec{a} \times \vec{b}$

Similarly, P_2 is the plane determined by vectors $\vec{c}$ and $\vec{d}$. Therefore, normal vectors $\vec{n_2}$ to P_2 will be given by

$\vec{n_2} = \vec{c} \times \vec{d}$

Substituting the values of $\vec{n_1}$ and $\vec{n_2}$ in (i), we get

$\vec{n_1} \times \vec{n_2} = \vec{0}$

Hence, $\vec{n_1} \,||\, \vec{n_2}$

Hence, the planes will also be parallel to each other.

Thus angle between the planes = 0.

14. **a.** $\vec{a}, \vec{b}$ and $\vec{c}$ are unit coplanar vectors, $2\vec{a} - \vec{b}, 2\vec{b} - 2\vec{c}$ and $2\vec{c} - \vec{a}$ are also coplanar vectors, being linear combination of $\vec{a}, \vec{b}$ and $\vec{c}$.

Thus, $[2\vec{a} - \vec{b}\ \ 2\vec{b} - \vec{c}\ \ 2\vec{c} - \vec{a}] = 0$

15. **b.** $\hat{a}, \hat{b}$ and $\hat{c}$ are unit vectors.

Now $x = |\hat{a} - \hat{b}|^2 + |\hat{b} - \hat{c}|^2 + |\hat{c} - \hat{a}|^2$

$= \frac{1}{2}(\hat{a}\cdot\hat{a} + \hat{b}\cdot\hat{b} + \hat{c}\cdot\hat{c}) - 2\hat{a}\cdot\hat{b} - 2\hat{b}\cdot\hat{c} - 2\hat{c}\cdot\hat{a}$

$\Rightarrow 6 - 2(\hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{c} + \hat{c}\cdot\hat{a})$ (i)

Also, $|\hat{a} + \hat{b} + \hat{c}| \geq 0$

$\Rightarrow \hat{a}\cdot\hat{a} + \hat{b}\cdot\hat{b} + \hat{c}\cdot\hat{c} + 2(\hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{c} + \hat{c}\cdot\hat{a}) \geq 0$

$\Rightarrow 3 + 2(\hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{c} + \hat{c}\cdot\hat{a}) \geq 0$

$\Rightarrow 2(\hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{c} + \hat{c}\cdot\hat{a}) \geq -3$

$\Rightarrow -2(\hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{c} + \hat{c}\cdot\hat{a}) \leq 3$

$\Rightarrow 6 - 2(\hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{c} + \hat{c}\cdot\hat{a}) \leq 9$ (ii)

From (i) and (ii), $x \leq 9$

Therefore, x does not exceed 9.

16. **b.** Given that $\vec{a}$ and $\vec{b}$ are two unit vectors.

$\therefore |\vec{a}| = 1$ and $|\vec{b}| = 1$

Also given that $(\vec{a} + 2\vec{b}) \cdot (5\vec{a} - 4\vec{b}) = 0$

$\Rightarrow 5|\vec{a}|^2 - 8|\vec{b}|^2 - 4\vec{a}\cdot\vec{b} + 10\vec{b}\cdot\vec{a} = 0$

$\Rightarrow 5 - 8 + 6\vec{a}\cdot\vec{b} = 0$

$\Rightarrow 6|\vec{a}||\vec{b}|\cos\theta = 3$ (where θ is the angle between $\vec{a}$ and $\vec{b}$)

$\Rightarrow \cos\theta = 1/2$

$\Rightarrow \theta = 60°$

17. **c.** Given that $\vec{V} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{W} = \hat{i} + 3\hat{k}$ and $\vec{U}$ is a unit vector

$|\vec{U}| = 1$

Now, $[\vec{U}\,\vec{V}\,\vec{W}] = \vec{U}\cdot(\vec{V}\times\vec{W})$

$= \vec{U}\cdot(2\hat{i} + \hat{j} - \hat{k})\times(\hat{i} + 3\hat{k})$

$= \vec{U}\cdot(3\hat{i} - 7\hat{j} - \hat{k})$

$= \sqrt{3^2 + 7^2 + 1^2}\,\cos\theta$ which is maximum when $\cos\theta = 1$

Therefore, maximum value of $[\vec{U}\,\vec{V}\,\vec{W}] = \sqrt{59}$

18. **c.** Volume of parallelopiped formed by $\vec{u} = \hat{i} + a\hat{j} + \hat{k}$, $\vec{v} = \hat{j} + a\hat{k}$, $\vec{w} = a\hat{i} + \hat{k}$ is

$$V = [\vec{u}\,\vec{v}\,\vec{w}] = \begin{vmatrix} 1 & a & 1 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix}$$

$= 1(1 - 0) - a(0 - a^2) + 1(0 - a)$

$= 1 + a^3 - a$

For V to be minimum, $\dfrac{dV}{da} = 0$

$\Rightarrow 3a^2 - 1 = 0$

$\Rightarrow a = \pm\dfrac{1}{\sqrt{3}}$

But $a > 0 \Rightarrow a = \dfrac{1}{\sqrt{3}}$

19. **c.** $(\vec{a}\times\vec{b})\times\vec{a} = (\vec{a}\cdot\vec{a})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{a}$

$(\hat{j} - \hat{k})\times(\hat{i} + \hat{j} + \hat{k}) = (\sqrt{3})^2\,\vec{b} - (\hat{i} + \hat{j} + \hat{k})$

$\Rightarrow 3\vec{b} = 3\hat{i} \Rightarrow \vec{b} = \hat{i}$

20. **c.** Any vector coplanar to $\vec{a}$ and $\vec{b}$ can be written as $\vec{r} = \mu\vec{a} + \lambda\vec{b}$

$\vec{r} = (\mu+2\lambda)\ \hat{i} + (-\mu+\lambda)\ \hat{j} + (\mu+\lambda)\ \hat{k}$ since $\vec{r}$ is orthogonal to $5\hat{j} + 2\hat{j} + 6\hat{k}$

$\Rightarrow 5(\mu+2\lambda) + 2(-\mu+\lambda) + 6(\mu+\lambda) = 0$

$\Rightarrow 9\mu + 18\lambda = 0$

$\Rightarrow \lambda = -\dfrac{1}{2}\mu$

$\therefore\ \vec{r} = \lambda(3\hat{j} - \hat{k})$

Since $\hat{r}$ is a unit vector, $\hat{r} = \dfrac{3\hat{i} - \hat{k}}{\sqrt{10}}$

21. **c.** We observe that $\vec{a}\cdot\vec{b}_1 = \vec{a}\cdot\vec{b} - \left(\dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\right)\vec{a}\cdot\vec{a} = \vec{a}\cdot\vec{b} - \vec{a}\ \vec{b} = 0$

$$\vec{a}\cdot\vec{c}_2 = \vec{a}\left(\vec{c} - \frac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a} - \frac{\vec{c}\cdot\vec{b}_1}{|\vec{b}_1|^2}\vec{b}_1\right)$$

$$= \vec{a}\cdot\vec{c} - \frac{\vec{a}\cdot\vec{c}}{|\vec{a}|^2}|\vec{a}|^2 - \frac{\vec{c}\cdot\vec{b}_1}{|\vec{b}_1|^2}(\vec{a}\cdot\vec{b}_1)$$

$$= \vec{a}\cdot\vec{c} - \vec{a}\cdot\vec{c} - 0 \quad (\because \vec{a}\cdot\vec{b}_1 = 0)$$

And $\vec{b}_1\cdot\vec{c}_2 = \vec{b}_1\cdot\left(\vec{c} - \dfrac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a} - \dfrac{\vec{c}\cdot\vec{b}_1}{|\vec{b}_1|^2}\vec{b}_1\right)$

$$= \vec{b}_1\cdot\vec{c} - \frac{(\vec{c}\cdot\vec{a})(\vec{b}_1\cdot\vec{a})}{|\vec{a}|^2} - \frac{\vec{c}\cdot\vec{b}_1}{|\vec{b}_1|^2}\vec{b}_1\cdot\vec{b}_1$$

$$= \vec{b}_1\cdot\vec{c} - 0 - b_1\cdot\vec{c} \quad (\text{using } \vec{b}_1\cdot\vec{a} = 0)$$

$$= 0$$

22. **a.** A vector in the plane of $\vec{a}$ and $\vec{b}$ is $\vec{u} = \mu\vec{a} + \lambda\vec{b} = (\mu+\lambda)\ \hat{i} + (2\mu-\lambda)\ \hat{j} + (\mu+\lambda)\ \hat{k}$

Projection of $\vec{u}$ on $\vec{c} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \dfrac{\vec{u}\cdot\vec{c}}{|\vec{c}|} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \vec{u}\cdot\vec{c} = 1$

$\Rightarrow |\mu + \lambda + 2\mu - \lambda - \mu - \lambda| = 1$

$\Rightarrow |2\mu - \lambda| = 1$

$\Rightarrow \lambda = 2\mu \pm 1$

$\Rightarrow \vec{u} = 2\hat{i} + \hat{j} + 2\hat{k}$ or $4\hat{i} - \hat{j} + 4\hat{k}$

23. **a.** $|\overrightarrow{OP}| = |\hat{a}\cos t + \hat{b}\sin t|$

$= (\cos^2 t + \sin^2 t + 2\cos t \sin t\ \hat{a}\cdot\hat{b})^{1/2}$

$= (1 + 2\cos t \sin t\ \hat{a}\cdot\hat{b})^{1/2}$

$= (1 + \sin 2t\ \hat{a}\cdot\hat{b})^{1/2}$

$\therefore |\overrightarrow{OP}|_{\max} = (1 + \hat{a}\cdot\hat{b})^{1/2}$ when $t = \pi/4$

$$\hat{u} = \frac{\hat{a}+\hat{b}}{\sqrt{2}\,\dfrac{|\hat{a}+\hat{b}|}{\sqrt{2}}}$$

$$= \frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$$

24. **c.** $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = 1$ is possible only when $|\vec{a}\times\vec{b}| = |\vec{c}\times\vec{d}| = 1$ and $(\vec{a}\times\vec{b}) \parallel (\vec{c}\times\vec{d})$.

Since $\vec{a}\cdot\vec{c} = \frac{1}{2}$ and if $\vec{b}\parallel\vec{d}$, then $|\vec{c}\times\vec{d}| \neq 1$

25. **b.** Angle between vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ is given by

$$\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{AD}}{|\overrightarrow{AB}|\cdot|\overrightarrow{AD}|} = \frac{-2+20+22}{\sqrt{4+100+121}\sqrt{1+4+4}} = \frac{8}{9}$$

$$\Rightarrow \cos\alpha = \cos(90° - \theta) = \sin\theta = \frac{\sqrt{17}}{9}$$

26. **a.**

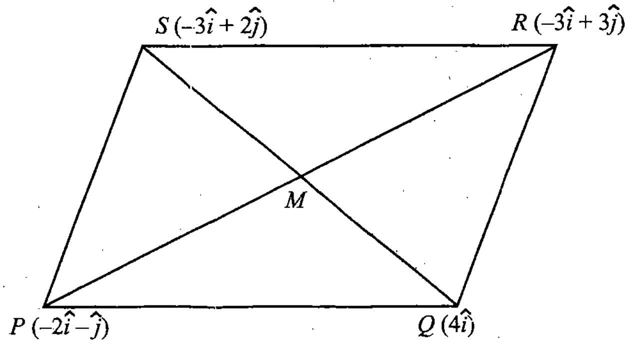


Fig. 2.56

Evaluating midpoint of PR and QS which gives $M \equiv \left[\frac{\hat{i}}{2} + \hat{j}\right]$, same for both.

$\overrightarrow{PQ} = \overrightarrow{SR} = 6\hat{i} + \hat{j}$

$\overrightarrow{PS} = \overrightarrow{QR} = -\hat{i} + 3\hat{j}$

$\Rightarrow \overrightarrow{PQ} \cdot \overrightarrow{PS} \neq 0$

$\overrightarrow{PQ} \parallel \overrightarrow{SR}, \overrightarrow{PS} \parallel \overrightarrow{QR}$ and $|\overrightarrow{PQ}| = |\overrightarrow{SR}|, |\overrightarrow{PS}| = |\overrightarrow{QR}|$

Hence, $PQRS$ is a parallelogram but not rhombus or rectangle.

27. **c.** $\vec{v} = \lambda\vec{a} + \mu\vec{b}$

$= \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - \hat{j} + \hat{k})$

Projection of $\vec{v}$ on $\vec{c}$

$$\frac{\vec{v} \cdot \vec{c}}{|\vec{c}|} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \frac{[(\lambda + \mu)\hat{i} + (\lambda - \mu)\hat{j} + (\lambda + \mu)\hat{k}] \cdot (\hat{i} - \hat{j} - \hat{k})}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$\Rightarrow \lambda + \mu - \lambda + \mu - \lambda - \mu = 1$

$\Rightarrow \mu - \lambda = 1$

$\Rightarrow \lambda = \mu - 1$

$\Rightarrow \vec{v} = (\mu - 1)(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - \hat{j} + \hat{k})$

$\Rightarrow \vec{v} = (2\mu - 1)\hat{i} - \hat{j} + (2\mu - 1)\hat{k}$

At $\mu = 2$, $\boldsymbol{\vec{v} = 3\hat{i} - \hat{j} + 3\hat{k}}$

Multiple choice questions with one or more than one correct answer

1. **c.** We are given that $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$$\text{Then} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 = [\vec{a}\ \vec{b}\ \vec{c}]^2$$

$$= (\vec{a} \times \vec{b} \cdot \vec{c})^2$$

$= (|\vec{a}\times\vec{b}|\cdot 1\cos 0°)^2$ (since $\vec{c}$ is $\perp$ to $\vec{a}$ and $\vec{b}$, $\vec{c}$ is $\perp$ to $\vec{a}\times\vec{b}$)

$= (|\vec{a}\times\vec{b}|)^2$

$= \left(|\vec{a}||\vec{b}|\cdot\sin\frac{\pi}{6}\right)^2$

$= \left(\frac{1}{2}\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}\right)^2$

$= \frac{1}{4}\left(a_1^2+a_2^2+a_3^2\right)\left(b_1^2+b_2^2+b_3^2\right)$

2. **b.** We know that if $\hat{n}$ is perpendicular to $\vec{a}$ as well as $\vec{b}$, then

$$\hat{n} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} \text{ or } \frac{\vec{b}\times\vec{a}}{|\vec{b}\times\vec{a}|}$$

As $\vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$ represent two vectors in opposite directions, we have two possible values of $\hat{n}$

3. **a., c.** We have $\vec{a} = 2\hat{i}-\hat{j}+\hat{k}$, $\vec{b} = \hat{i}+2\hat{j}-\hat{k}$, $\vec{c} = \hat{i}+\hat{j}-2\hat{k}$

Any vector in the plane of $\vec{b}$ and $\vec{c}$ is

$\vec{u} = \mu\vec{b}+\lambda\vec{c}$

$= \mu(\hat{i}+2\hat{j}-\hat{k})+\lambda(\hat{i}+\hat{j}-2\hat{k})$

$= (\mu+\lambda)\hat{i}+(2\mu+\lambda)\hat{j}-(\mu+2\lambda)\hat{k}$

Given that the magnitude of projection of $\vec{u}$ on $\vec{a}$ is $\sqrt{2/3}$

$\Rightarrow \sqrt{\frac{2}{3}} = \left|\frac{\vec{u}\cdot\vec{a}}{|\vec{a}|}\right|$

$\Rightarrow \sqrt{\frac{2}{3}} = \left|\frac{2(\mu+\lambda)-(2\mu+\lambda)-(\mu+2\lambda)}{\sqrt{6}}\right|$

$\Rightarrow |-\lambda-\mu| = 2$

$\Rightarrow \lambda+\mu = 2$ or $\lambda+\mu = -2$

Therefore, the required vector is either $2\hat{i}+3\hat{j}-3\hat{k}$ or $-2\hat{i}-\hat{j}+5\hat{k}$.

4. **c.** $[\vec{u}\ \vec{v}\ \vec{w}] = [\vec{v}\ \vec{w}\ \vec{u}] = [\vec{w}\ \vec{u}\ \vec{v}]$

but $[\vec{v}\ \vec{u}\ \vec{w}] = -[\vec{u}\ \vec{v}\ \vec{w}]$

5. **a., c.** Dot product of two vectors gives a scalar quantity.

6. **a., c.** We have $\vec{v} = \vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n} = \sin\theta\, \hat{n}$, where $\vec{a}$ and $\vec{b}$ are unit vectors. Therefore, $|\vec{v}| = \sin\theta$

Now, $\vec{u} = \vec{a} - (\vec{a}\cdot\vec{b})\vec{b}$

$= \vec{a} - \vec{b}\cos\theta$ (where $\vec{a}\cdot\vec{b} = \cos\theta$)

$\therefore |\vec{u}|^2 = |\vec{a} - \vec{b}\cos\theta|^2$

$= 1 + \cos^2\theta - 2\cos\theta\cdot\cos\theta$

$= 1 - \cos^2\theta = \sin^2\theta = |v|^2$

$\Rightarrow |\vec{u}| = |\vec{v}|$

Also, $\vec{u}\cdot\vec{b} = \vec{a}\cdot\vec{b} - (\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{b})$

$= \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{b}$

$= 0$

$\therefore |\vec{u}\cdot\vec{b}| = 0$

$\therefore |\vec{v}| = |\vec{u}| + |\vec{u}\cdot\vec{b}|$ is also correct.

7. **a., c., d.**

$\vec{a} = \frac{1}{3}(2\hat{i} - 2\hat{j} + \hat{k})$

$|\vec{a}|^2 = \frac{1}{9}(4+4+1) = 1 \Rightarrow |\vec{a}| = 1$

Let $\vec{b} = 2\hat{i} - 4\hat{j} + 3\hat{k}$. Then

$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{5}{\sqrt{29}} \Rightarrow \theta \neq \frac{\pi}{3}$

Let $\vec{c} = -\hat{i} + \hat{j} - \frac{1}{2}\hat{k} = \frac{-3}{2}\hat{a} \Rightarrow \vec{c} \parallel \vec{a}$

Let $\vec{d} = 3\hat{i} + 2\hat{j} + 2\hat{k}$. Then $\vec{a}\cdot\vec{d} = 0 \Rightarrow \vec{a} \perp \vec{d}$

8. **b., d.** Normal to plane P_1 is

$\vec{n}_1 = (2\hat{j} + 3\hat{k}) \times (4\hat{j} - 3\hat{k}) = -18\hat{i}$

Normal to plane P_2 is

$\vec{n}_2 = (\hat{j} - \hat{k}) \times (3\hat{i} + 3\hat{j}) = 3\hat{i} - 3\hat{j} - 3\hat{k}$

$\therefore \vec{A}$ is parallel to $\pm(\vec{n}_1 \times \vec{n}_2) = \pm(-54\hat{j} + 54\hat{k})$

Now, the angle between $\vec{A}$ and $2\hat{i}+\hat{j}-2\hat{k}$ is given by

$$\cos\theta = \pm\frac{(-54\hat{j}+54\hat{k})\cdot(2\hat{i}+\hat{j}-2\hat{k})}{54\sqrt{2}\cdot 3} = \pm\frac{1}{\sqrt{2}}$$

$\theta = \pi/4$ or $3\pi/4$

9. **a., d.** Any vector in the plane of $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+\hat{k}$ is

$\vec{r}=\lambda(\hat{i}+\hat{j}+2\hat{k})+\mu(\hat{i}+2\hat{j}+\hat{k})$

$=(\lambda+\mu)\hat{i}+(\lambda+2\mu)\hat{j}+(2\lambda+\mu)\hat{k}$

Also $\vec{r}$ is perpendicular to the vector $\hat{i}+\hat{j}+\hat{k}$

$\Rightarrow \vec{r}\cdot\vec{c}=0$

$\Rightarrow \lambda+\mu=0$

Possible vectors are $\hat{j}-\hat{k}$ or $-\hat{j}+\hat{k}$

Integer Answer Type

1. **(5)** $E=(2\vec{a}+\vec{b})\cdot[|\vec{a}|^2\vec{b}-(\vec{a}\cdot\vec{b})\vec{a}-2(\vec{a}\cdot\vec{b})\vec{b}+2|\vec{b}|^2\vec{a}]$

$$\vec{a}\cdot\vec{b}=\frac{2-2}{\sqrt{70}}=0$$

$|\vec{a}|=1$

$|b|=1$

$$\Rightarrow \quad E=(2\vec{a}+\vec{b})\cdot[2|\vec{b}|^2\vec{a}+|\vec{a}|^2\vec{b}]$$

$$=4|\vec{a}|^2|\vec{b}|^2+|\vec{a}|^2(\vec{a}\cdot\vec{b})+2|\vec{b}|^2(\vec{b}\cdot\vec{a})+|\vec{a}|^2|\vec{b}|^2$$

$$=5|\vec{a}|^2|\vec{b}|^2=5$$

2. **(9)** $\vec{r}\times\vec{b}=\vec{c}\times\vec{b}$

taking cross product with $\vec{a}$

$\vec{a}\times(\vec{r}\times\vec{b})=\vec{a}\times(\vec{c}\times\vec{b})$

$\Rightarrow (\vec{a}\cdot\vec{b})\vec{r}-(\vec{a}\cdot\vec{r})\vec{b}=\vec{a}\times(\vec{c}\times\vec{b})$

$\Rightarrow \vec{r}=-3\hat{i}+6\hat{j}+3\hat{k}$

$\Rightarrow \vec{r}\cdot\vec{b}=3+6=9$

CHAPTER

3

Three-Dimensional Geometry

- Direction Cosines and Direction Ratios
- Equation of Straight Line Passing through a Given Point and Parallel to a Given Vector
- Equation of Line Passing through Two Given Points
- Angle between Two Lines
- Perpendicular Distance of a Point from a Line
- Shortest Distance between Two Lines
- Plane
- Angle Between Two Planes
- Line of Intersection of Two Planes
- Angle between a Line and a Plane
- Equation of a Plane Passing Through the Line of Intersection of Two Planes
- Distance of a Point from a Plane
- Distance between Parallel Planes
- Equation of a Plane Bisecting the Angle between Two Planes
- Two Sides of a Plane
- Spheres

DIRECTION COSINES AND DIRECTION RATIOS

From Chapter 1, recall that if a directed line L, passing through the origin, makes angles α, β and γ with the x-, y- and z-axes, respectively, called direction angles, then the cosines of these angles, namely, $\cos\alpha$, $\cos\beta$ and $\cos\gamma$, are called the direction cosines of the directed line L.

If we reverse the direction of L, the direction angles are replaced by their supplements, i.e., $\pi-\alpha$, $\pi-\beta$ and $\pi-\gamma$. Thus, the signs of the direction cosines are reversed.

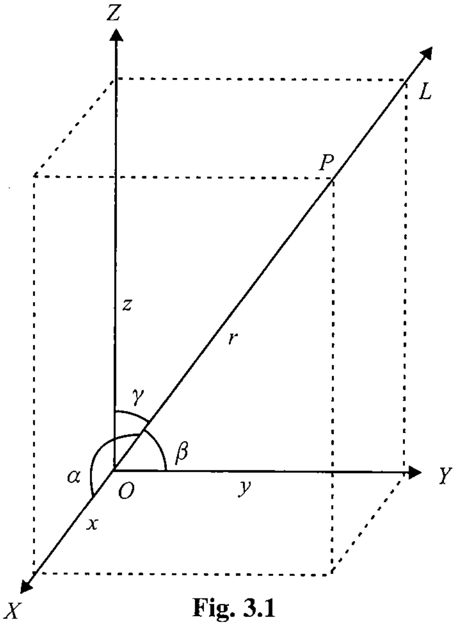


Fig. 3.1

Note that a given line in space can be extended in two opposite directions, and so it has two sets of direction cosines. In order to have a unique set of direction cosines for a given line in space, we must take the given line as a directed line. These unique direction cosines are denoted by l, m and n.

If the given line in space does not pass through the origin, then in order to find its direction cosines, we draw a line through the origin and parallel to the given line. Now take one of the directed lines from the origin and find its direction cosines as two parallel lines have same set of direction cosines.

Any three numbers which are proportional to the direction cosines of a line are called the *direction ratios* of the line. If l, m and n are direction cosines and a, b and c are the direction ratios of a line, then $a=\lambda l$, $b=\lambda m$ and $c=\lambda n$ for any non-zero $\lambda \in R$.

Notes:

1. *Direction cosines of the x-axis are* (1, 0, 0).

   Direction cosines of the y-axis are (0, 1, 0).

   Direction cosines of the z-axis are (0, 0, 1).

2. *Let OP be any line passing through the origin O which has direction cosines* $\cos\alpha$, $\cos\beta$ *and* $\cos\gamma$, *i.e.,* (l, m, n) *where distance* $OP = r \Rightarrow$ *Coordinates of P are* $(r\cos\alpha, r\cos\beta, r\cos\gamma)$.

3. *If* l, m *and* n *are the direction cosines of a vector, then* $l^2+m^2+n^2=1$.

4. $\vec{r} = |\vec{r}|\,(l\hat{i}+m\hat{j}+n\hat{k})$ *and* $\hat{r} = l\hat{i}+m\hat{j}+n\hat{k}$.

Direction Ratios

Let l, m and n be the direction cosines of a vector $\vec{r}$ and a, b and c be three numbers such that a, b, c are proportional to l, m and n. Therefore,

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k \text{ or } (l, m, n) = (ka, kb, kc)$$

Hence, a, b and c are direction ratios.

For example, if $(1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$ are direction cosines of a vector $\vec{r}$, then its direction ratios are $(1, -1, 1)$ or $(-1, 1, -1)$ or $(2, -2, 2)$ or $(\lambda, -\lambda, \lambda)$ ...

It is evident from the above definition that to obtain the direction ratios of a vector from its direction cosines, we just multiply them by a common number.

"That shows there can be an infinite number of direction ratios for a given vector, but the direction cosines are unique."

Direction ratios of a line joining two points

For points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$,

Vector $\overrightarrow{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$,

Then the direction ratios of PQ are $< (x_2 - x_1), (y_2 - y_1), (z_2 - z_1) >$.

To obtain direction cosines from direction ratios

Let a, b and c be the direction ratios of a vector $\vec{r}$ having direction cosines l, m and n.

Then, $l = \lambda a$, $m = \lambda b$, $n = \lambda c$ (by definition)

$\therefore \quad l^2 + m^2 + n^2 = 1$

$\Rightarrow \quad a^2\lambda^2 + b^2\lambda^2 + c^2\lambda^2 = 1$

$\Rightarrow \quad \lambda = \pm\dfrac{1}{\sqrt{a^2+b^2+c^2}}$

$\Rightarrow \quad l = \pm\dfrac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\dfrac{b}{\sqrt{a^2+b^2+c^2}}, n = \pm\dfrac{c}{\sqrt{a^2+b^2+c^2}}$

Example:

Let the direction ratios of a line be 3, 1 and –2.

Direction cosines are

$$\left(\frac{3}{\sqrt{3^2+1^2+(-2)^2}}, \frac{1}{\sqrt{3^2+1^2+(-2)^2}}, \frac{-2}{\sqrt{3^2+1^2+(-2)^2}}\right) \Rightarrow \left(\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}\right)$$

Notes:

1. *If $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ is a vector having direction cosines l, m and n, then* $l = \dfrac{a}{|\vec{r}|}, m = \dfrac{b}{|\vec{r}|}, n = \dfrac{c}{|\vec{r}|}$.

2. *Direction cosines of parallel vectors:*

   Let $\vec{a}$ and $\vec{b}$ be two parallel vectors. Then $\vec{b} = \lambda\vec{a}$ for some λ.

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, *then* $\vec{b} = \lambda\vec{a} \Rightarrow \vec{b} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$

This shows that $\vec{b}$ *has direction ratios* λa_1, λa_2 *and* λa_3, *i.e.,* a_1, a_2 *and* a_3 *because* $\lambda a_1 : \lambda a_2 : \lambda a_3 = a_1 : a_2 : a_3$. *Thus,* $\vec{a}$ *and* $\vec{b}$ *have equal direction ratios and hence equal direction cosines too.*

3. *If the direction ratios of* $\vec{r}$ *are a, b and c* $\Rightarrow \vec{r} = \dfrac{|\vec{r}|}{\sqrt{a^2+b^2+c^2}}(a\hat{i} + b\hat{j} + c\hat{k})$.
4. *Projections of* $\vec{r}$ *on the coordinate axes are:* $l|\vec{r}|$, $m|\vec{r}|$ and $n|\vec{r}|$.
5. *The projection of a segment joining points* $P(x_1, y_1, z_1)$ *and* $Q(x_2, y_2, z_2)$ on *a line with direction cosines l, m* and *n* is $(x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)\,n$.
6. *If* l_1, m_1, n_1 *and* l_2, m_2, n_2 *are the direction cosines of two concurrent lines, then the direction cosines of the lines bisecting the angles between them are proportional to* $l_1 \pm l_2$, $m_1 \pm m_2$ and $n_1 \pm n_2$.
7. *Acute angle* θ *between the two lines having direction cosines* l_1, m_1, n_1 and l_2, m_2, n_2 *is given by*

   $\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$, $\sin\theta = \sqrt{(l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2}$
8. *If* a_1, b_1, c_1 *and* a_2, b_2, c_2 *be the direction ratios of two lines, then the acute angle* θ *between them is given by* $\cos\theta = \dfrac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$,

   $$\sin\theta = \frac{\sqrt{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$
9. *Two lines having direction cosines* l_1, m_1, n_1 *and* l_2, m_2, n_2 *are*
   - **a.** *perpendicular if and only if* $l_1l_2 + m_1m_2 + n_1n_2 = 0$.
   - **b.** *parallel if and only if* $\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$
10. *Two lines having direction ratios* a_1, b_1, c_1 and a_2, b_2, c_2 *are*
    - **a.** *perpendicular if and only if* $a_1a_2 + b_1b_2 + c_1c_2 = 0$
    - **b.** *parallel if and only if* $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

Direction ratio of line along the bisector of two given lines

If l_1, m_1 and n_1 and l_2, m_2 and n_2 are the direction cosines of the two lines inclined to each other at an angle θ, then the direction cosines of the

a. internal bisector of the angle between these lines are $\dfrac{l_1 + l_2}{2\cos(\theta/2)}$, $\dfrac{m_1 + m_2}{2\cos(\theta/2)}$ and $\dfrac{n_1 + n_2}{2\cos(\theta/2)}$, and

b. external bisector of the angle between the lines are $\dfrac{l_1 - l_2}{2\sin(\theta/2)}$, $\dfrac{m_1 - m_2}{2\sin(\theta/2)}$ and $\dfrac{n_1 - n_2}{2\sin(\theta/2)}$.

Proof:

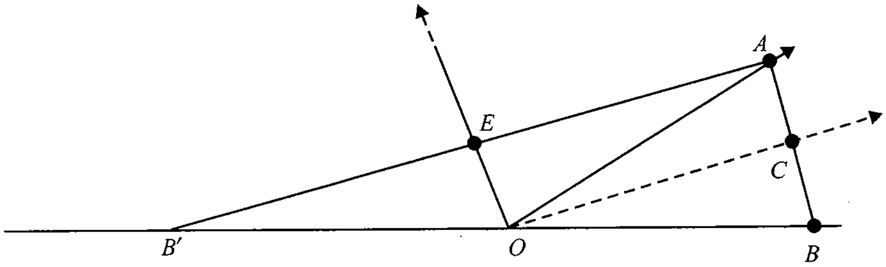


Fig. 3.2

Let OA and OB be two lines with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2. Let $OA = OB = 1$. Then the coordinates of A and B are (l_1, m_1, n_1) and (l_2, m_2, n_2), respectively. Let OC be the bisector of $\angle AOB$. Then C is the midpoint of AB and so its coordinates are

$$\left(\frac{l_1+l_2}{2}, \frac{m_1+m_2}{2}, \frac{n_1+n_2}{2}\right)$$

Therefore, the direction ratios of OC are $\frac{l_1+l_2}{2}, \frac{m_1+m_2}{2}$ and $\frac{n_1+n_2}{2}$.

$$\text{We have } OC = \sqrt{\left(\frac{l_1+l_2}{2}\right)^2+\left(\frac{m_1+m_2}{2}\right)^2+\left(\frac{n_1+n_2}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{(l_1^2+m_1^2+n_1^2)+(l_2^2+m_2^2+n_2^2)+2(l_1l_2+m_1m_2+n_1n_2)}$$

$$= \frac{1}{2}\sqrt{2+2\cos\theta} \quad (\because \cos\theta = l_1l_2+m_1m_2+n_1n_2)$$

$$= \frac{1}{2}\sqrt{2(1+\cos\theta)} = \cos\left(\frac{\theta}{2}\right)$$

Therefore, the direction cosines of $\overrightarrow{OC}$ are $\frac{l_1+l_2}{2(OC)}, \frac{m_1+m_2}{2(OC)}, \frac{n_1+n_2}{2(OC)}$

or $\frac{l_1+l_2}{2\cos(\theta/2)}, \frac{m_1+m_2}{2\cos(\theta/2)}, \frac{n_1+n_2}{2\cos(\theta/2)}$

In Fig. 3.2, OE is the external bisector.

The coordinates of E are $\frac{l_1-l_2}{2}, \frac{m_1-m_2}{2}$ and $\frac{n_1-n_2}{2}$.

Therefore, direction ratios of OE are $\frac{l_1-l_2}{2}, \frac{m_1-m_2}{2}$ and $\frac{n_1-n_2}{2}$.

$$\text{Also, } OE = \frac{1}{2}\sqrt{2-2\cos\theta}$$

$$= \frac{1}{2}\sqrt{2(1-\cos\theta)}$$

$$= \sin(\theta/2)$$

Therefore, the direction cosines of $\overrightarrow{OE}$ are $\frac{l_1-l_2}{2\sin(\theta/2)}, \frac{m_1-m_2}{2\sin(\theta/2)}$ and $\frac{n_1-n_2}{2\sin(\theta/2)}$.

Example 3.1 **If α, β and γ are the angles which a directed line makes with the positive directions of the co-ordinates axes, then find the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$.**

Sol. The direction cosines of the line are $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$.

Since $l^2 + m^2 + n^2 = 1$, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow 1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma = 1$

$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

Example 3.2 **A line OP through origin O is inclined at 30° and 45° to OX and OY, respectively. Then find the angle at which it is inclined to OZ.**

Sol. Let l, m and n be the direction cosines of the given vector. Then $l^2 + m^2 + n^2 = 1$.

If $l = \cos 30° = \sqrt{3}/2$, $m = \cos 45° = 1/\sqrt{2}$, then $\frac{3}{4} + \frac{1}{2} + n^2 = 1$.

$\Rightarrow n^2 = -1/4$, which is not possible. So, such a line cannot exist.

Example 3.3 **ABC is a triangle and $A = (2, 3, 5)$, $B = (-1, 3, 2)$ and $C = (\lambda, 5, \mu)$. If the median through A is equally inclined to the axes, then find the value of λ and μ.**

Sol.

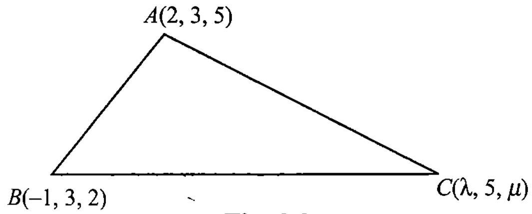


Fig. 3.3

Midpoint of BC is $\left(\frac{\lambda - 1}{2}, 4, \frac{2 + \mu}{2}\right)$

Direction ratios of the median through A are $\frac{\lambda - 1}{2} - 2$, $4 - 3$ and $\frac{2 + \mu}{2} - 5$, i.e., $\frac{\lambda - 5}{2}$, 1 and $\frac{\mu - 8}{2}$.

The median is equally inclined to the axes; so the direction ratios must be equal. Therefore,

$$\frac{\lambda - 5}{2} = 1 = \frac{\mu - 8}{2} \Rightarrow \lambda = 7, \mu = 10$$

Example 3.4 **A line passes through the points $(6, -7, -1)$ and $(2, -3, 1)$. Find the direction cosines of the line if the line makes an acute angle with the positive direction of the x-axis.**

Sol. Let l, m and n be the direction cosines of the given line. As it makes an acute angle with the x-axis, $l > 0$. The line passes through $(6, -7, -1)$ and $(2, -3, 1)$; therefore, its direction ratios are $(6 - 2, -7 + 3, -1 - 1)$ or $(4, -4, -2)$. Hence the direction cosines of the given line are $2/3$, $-2/3$ and $-1/3$.

Example 3.5 **Find the ratio in which the y-z plane divides the join of the points $(-2, 4, 7)$ and $(3, -5, 8)$**

Sol. Let the y-z plane divide the join of $P(-2, 4, 7)$ and $Q(3, -5, 8)$ in the ratio $\lambda : 1$.

$\left(\frac{3\lambda - 2}{\lambda + 1}, \frac{-5\lambda + 4}{\lambda + 1}, \frac{8\lambda + 7}{\lambda + 1}\right)$ is in the y-z plane. Then its x-coordinate is zero.

$\frac{3\lambda - 2}{\lambda + 1} = 0$ or $3\lambda - 2 = 0$

$\therefore \lambda = 2/3$

Example 3.6 **If $A(3, 2, -4)$, $B(5, 4, -6)$ and $C(9, 8, -10)$ are three collinear points, then find the ratio in which point C divides AB.**

Sol. Let C divide AB in the ratio $\lambda : 1$. Then

$$C \equiv \left(\frac{5\lambda+3}{\lambda+1}, \frac{4\lambda+2}{\lambda+1}, \frac{-6\lambda-4}{\lambda+1}\right) = (9, 8\ -10)$$

Comparing, $5\lambda + 3 = 9\lambda + 9$ or $4\lambda = -6$

$\therefore\ \lambda = -3/2$

Also, from $4\lambda + 2 = 8\lambda + 8$ and $-6\lambda - 4 = -10\lambda - 10$, we get the same value of λ.

Example 3.7 **If the sum of the squares of the distance of a point from the three coordinate axes is 36, then find its distance from the origin.**

Sol. Let $P(x, y, z)$ be the point. Now under the given condition,

$$[\sqrt{x^2+y^2}]^2 + [\sqrt{y^2+z^2}]^2 + [\sqrt{z^2+x^2}]^2 = 36$$

$\Rightarrow x^2 + y^2 + z^2 = 18$

Then distance from the origin to point (x, y, z) is

$$\sqrt{x^2+y^2+z^2} = \sqrt{18} = 3\sqrt{2}$$

Example 3.8 **A line makes angles α, β, γ and δ with the diagonals of a cube; show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = 4/3$.**

Sol. The four diagonals of a cube are AL, BM, CN and OP.

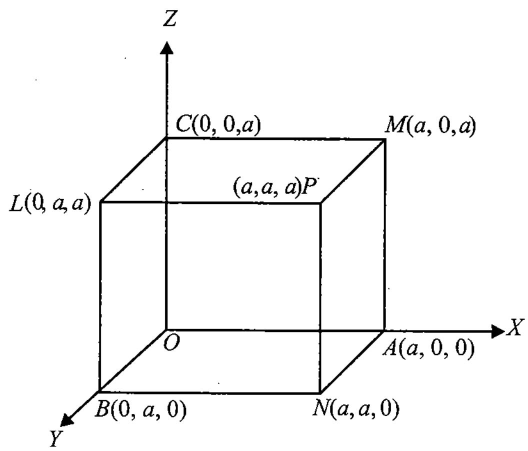


Fig. 3.4

Direction cosines of OP are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$.

Direction cosines of AL are $\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$.

Direction cosines of BM are $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$.

Direction cosines of CN are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and $\frac{-1}{\sqrt{3}}$.

Let l, m and n be the direction cosines of a line which is inclined at angles α, β, γ and δ, respectively, to the four diagonals; then

$$\cos\alpha = l\cdot\frac{1}{\sqrt{3}} + m\cdot\frac{1}{\sqrt{3}} + n\cdot\frac{1}{\sqrt{3}}$$

$$= \frac{l+m+n}{\sqrt{3}}$$

Similarly, $\cos\beta = \frac{-l+m+n}{\sqrt{3}}$

$$\cos\gamma = \frac{l-m+n}{\sqrt{3}}$$

$$\cos\delta = \frac{l+m-n}{\sqrt{3}}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{1}{3}[(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2]$$

$$= \frac{1}{3}\cdot 4(l^2+m^2+n^2) = \frac{4}{3}$$

Example 3.9 **Find the angle between the lines whose direction cosines are given by $l + m + n = 0$ and $2l^2 + 2m^2 - n^2 = 0$.**

Sol. $l^2 + m^2 + n^2 = 1$ (i)

$l + m + n = 0$ (ii)

$2l^2 + 2m^2 - n^2 = 0$ (iii)

$2(1 - n^2) = n^2 \Rightarrow 3n^2 = 2 \Rightarrow n = \pm\sqrt{2/3}$ (iv)

$2(l^2 + m^2) = n^2 = (-(l+m))^2 \Rightarrow l = m$ (v)

$l + m = \pm\sqrt{2/3} \Rightarrow 2l = \pm\sqrt{2/3}$

$l = \pm 1/\sqrt{6}$, $m = \pm 1/\sqrt{6}$

Direction cosines are

$$\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\right) \text{and} \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}\right)$$

or

$$\left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\right) \text{and} \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}\right)$$

The angle between these lines in both the cases is $\cos^{-1}\left(-\frac{1}{3}\right)$.

Example 3.10 **A mirror and a source of light are situated at the origin O and at a point on OX, respectively. A ray of light from the source strikes the mirror and is reflected. If the direction ratios of the normal to the plane are 1, –1, 1, then find the DCs of the reflected ray.**

Sol. Let the source of light be situated at $A(a, 0, 0)$, where $a \neq 0$.
Let OA be the incident ray and OB the reflected ray.
ON is the normal to the mirror at O. Therefore,
$\angle AON = \angle NOB = \theta/2$ (say)

Direction ratios of OA are a, 0 and 0 and so its direction cosines are 1, 0 and 0.

Direction ratios of ON are $1/\sqrt{3}, -1/\sqrt{3}$ and $1/\sqrt{3}$. Therefore,

$\angle AON = \angle NOB = (\theta/2)$ (say)

$\cos(\theta/2) = 1/\sqrt{3}$

Let l, m and n be the direction cosines of the reflected ray OB.

$$\frac{l+1}{2\cos(\theta/2)} = \frac{1}{\sqrt{3}}, \frac{m+0}{2\cos(\theta/2)} = -\frac{1}{\sqrt{3}} \text{ and } \frac{n+0}{2\cos(\theta/2)} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow l = \frac{2}{3} - 1, m = \frac{-2}{3}, n = \frac{2}{3}$$

$$\Rightarrow l = -\frac{1}{3}, m = -\frac{2}{3}, n = \frac{2}{3}$$

Concept Application Exercise 3.1

1. If the x-coordinate of a point P on the join of $Q(2, 2, 1)$ and $R(5, 1, -2)$ is 4, then find its z-coordinate.
2. Find the distance of the point $P(a, b, c)$ from the x-axis.
3. If $\vec{r}$ is a vector of magnitude 21 and has direction ratios 2, –3 and 6, then find $\vec{r}$.
4. If $P(x, y, z)$ is a point on the line segment joining $Q(2, 2, 4)$ and $R(3, 5, 6)$ such that the projections of $\overrightarrow{OP}$ on the axes are 13/5, 19/5 and 26/5, respectively, then find the ratio in which P divides QR.
5. If O is the origin, $OP = 3$ with direction ratios –1, 2 and –2, then find the coordinates of P.
6. A line makes angles α, β and γ with the coordinate axes. If $\alpha + \beta = 90°$, then find γ.
7. The line joining the points (–2, 1, –8) and (a, b, c) is parallel to the line whose direction ratios are 6, 2 and 3. Find the values of a, b and c.
8. If a line makes angles α, β and γ with three-dimensional coordinate axes, respectively, then find the value of $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$.
9. A parallelopiped is formed by planes drawn through the points $P(6, 8, 10)$ and $Q(3, 4, 8)$ parallel to the coordinate planes. Find the length of edges and diagonal of the parallolepiped.
10. Find the angle between any two diagonals of a cube.
11. Direction ratios of two lines are a, b, c and $1/bc$, $1/ca$, $1/ab$. Then the lines are_________.
12. Find the angle between the lines whose direction cosines are connected by the relations $l + m + n = 0$ and $2lm + 2nl - mn = 0$.

EQUATION OF STRAIGHT LINE PASSING THROUGH A GIVEN POINT AND PARALLEL TO A GIVEN VECTOR

Vector Form

Line Passing through Point A($\vec{a}$) and Parallel to Vector $\vec{b}$

Let A be the given point and let AP be the given line through A.

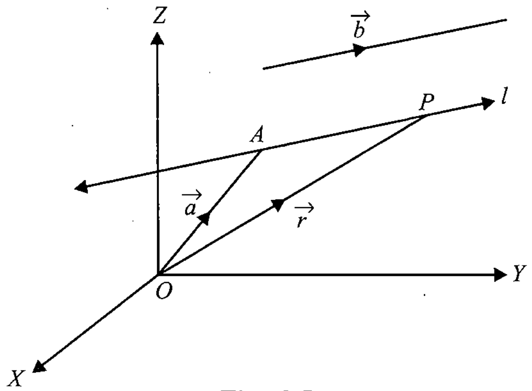


Fig. 3.5

Let $\vec{b}$ be any vector parallel to the given line.
Position vector of point A is $\vec{a}$.
Let P be any point on line AP, and let its position vector be $\vec{r}$.

Then, we have $\vec{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \vec{a} + \lambda\vec{b}$ (where, $\overrightarrow{AP} = \lambda\vec{b}$).

Hence, the vector equation of straight line; $\vec{r} = \vec{a} + \lambda\vec{b}$. (i)

Here, $\vec{r}$ is the position vector of any point $P(x, y, z)$ on the line. So $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

In particular, the equation of straight line through origin and parallel to $\vec{b}$ is $\vec{r} = \lambda\vec{b}$.

Cartesian Form

Let the coordinates of the given point A be (x_1, y_1, z_1) and the direction ratios of the line be a, b and c. Consider the coordinates of any point P be (x, y, z). Then

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$; $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$.

Substituting these values in (i) and equating the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$x = x_1 + \lambda a;\ y = y_1 + \lambda b;\ z = z_1 + \lambda c$$

These are parametric equations of the line.

Eliminating the parameter λ, we get $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$.

Notes:

1. *Here any point on the line is* $(x, y, z) \equiv (x_1 + \lambda a, y_1 + \lambda b, z_1 + \lambda c)$ *(λ being a parameter).*
2. *Since the x-, y- and z-axes pass through the origin and have direction cosines* (1, 0, 0), (0, 1, 0) *and* (0, 0, 1), *their equations are*

Equation of x-axis : $\frac{x-0}{1} = \frac{y-0}{0} = \frac{z-0}{0}$ *or* $y = 0, z = 0$

Equation of y-axis : $\frac{x-0}{0} = \frac{y-0}{1} = \frac{z-0}{0}$ *or* $x = 0, z = 0$

Equation of z-axis : $\frac{x-0}{0} = \frac{y-0}{0} = \frac{z-0}{1}$ *or* $x = 0, y = 0$

EQUATION OF LINE PASSING THROUGH TWO GIVEN POINTS

Vector Form

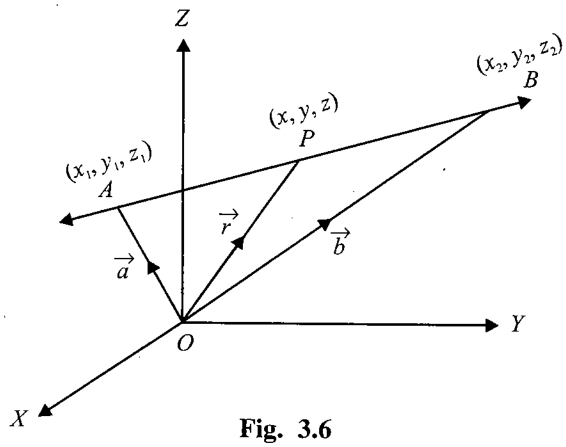


Fig. 3.6

From the figure, $\overrightarrow{OP} = \vec{r}$, $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.

Since $\overrightarrow{AP}$ is collinear with $\overrightarrow{AB}$, $\overrightarrow{AP} = \lambda \overrightarrow{AB}$ for some scalar λ.

$\Rightarrow \overrightarrow{OP} - \overrightarrow{OA} = \lambda(\overrightarrow{OB} - \overrightarrow{OA})$

$\Rightarrow \vec{r} - \vec{a} = \lambda(\vec{b} - \vec{a})$

$\Rightarrow \vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$ (i)

Therefore, the equation of a straight line passing through $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$.

Cartesian Form

We have $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$.

Substituting these values in (i), we get

$$x\hat{i}+y\hat{j}+z\hat{k} = x_1\hat{i}+y_1\hat{j}+z_1\hat{k} + \lambda[(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}]$$

Equating the coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$, we get

$$x = x_1+\lambda(x_2-x_1);\ y = y_1+\lambda(y_2-y_1);\ z = z_1+\lambda(z_2-z_1)$$

On eliminating λ, we obtain $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = \lambda$

which is the equation of the line in Cartesian form.

Example 3.11 **The Cartesian equation of a line is $\frac{x-3}{2}=\frac{y+1}{-2}=\frac{z-3}{5}$. Find the vector equation of the line.**

Sol. The given line is $\frac{x-3}{2}=\frac{y+1}{-2}=\frac{z-3}{5}$.

Note that it passes through (3, –1, 3) and is parallel to the line whose direction ratios are 2, –2 and 5. Therefore, its vector equation is $\vec{r}=3\hat{i}-\hat{j}+3\hat{k}+\lambda(2\hat{i}-2\hat{j}+5\hat{k})$, where λ is a parameter.

Example 3.12 **The Cartesian equations of a line are $6x-2=3y+1=2z-2$. Find its direction ratios and also find a vector equation of the line.**

Sol. The given line is $6x-2=3y+1=2z-2$ (i)

To put it in the symmetrical form, we must make the coefficients of x, y and z as 1. To do this, we divide each of the expressions in (i) by 6 and obtain $\frac{x-(1/3)}{1}=\frac{y+(1/3)}{2}=\frac{z-1}{3}$.

This shows that the given line passes through (1/3, –1/3, 1) and is parallel to the line whose direction ratios are 1, 2 and 3.

Therefore, its vector equation is $\vec{r}=\frac{1}{3}\hat{i}-\frac{1}{3}\hat{j}+\hat{k}+\lambda(\hat{i}+2\hat{j}+3\hat{k})$.

Example 3.13 **A line passes through the point with position vector $2\hat{i}-3\hat{j}+4\hat{k}$ and is in the direction of $3\hat{i}+4\hat{j}-5\hat{k}$. Find the equations of the line in vector and Cartesian forms.**

Sol. Since the line passes through $2\hat{i}-3\hat{j}+4\hat{k}$ and has direction of $3\hat{i}+4\hat{j}-5\hat{k}$, its vector equation is

$$\vec{r}=\hat{a}+\lambda\hat{b} \Rightarrow \vec{r}=2\hat{i}-3\hat{j}+4\hat{k}+\lambda(3\hat{i}+4\hat{j}-5\hat{k}),$$ where λ is a parameter. (i)

Cartesian equivalent of (i) is $\frac{x-2}{3}=\frac{y+3}{4}=\frac{z-4}{-5}$

Example 3.14 **Find the vector equation of line passing through $A(3, 4, -7)$ and $B(1, -1, 6)$. Also find its cartesian equations.**

Sol. Since the line passes through $A(3\hat{i}+4\hat{j}-7\hat{k})$ and $B(\hat{i}-\hat{j}+6\hat{k})$, its vector equation is

$$\vec{r}=3\hat{i}+4\hat{j}-7\hat{k}+\lambda[(\hat{i}-\hat{j}+6\hat{k})-(3\hat{i}+4\hat{j}-7\hat{k})]$$

or $\vec{r}=3\hat{i}+4\hat{j}-7\hat{k}-\lambda(2\hat{i}+5\hat{j}-13\hat{k})$ (i)

where λ is a parameter.

The Cartesian equivalent of (i) is $\frac{x-3}{2}=\frac{y-4}{5}=\frac{z+7}{-13}$.

Example 3.15 **Find the vector equation of a line passing through (2, –1, 1) and parallel to the line whose equation is $\frac{x-3}{2}=\frac{y+1}{7}=\frac{z-2}{-3}$.**

Sol. Since the required line is parallel to $\frac{x-3}{2}=\frac{y+1}{7}=\frac{z-2}{-3}$, it follows that the required line passing through $A(2\hat{i}-\hat{j}+\hat{k})$ has the direction of $2\hat{i}+7\hat{j}-3\hat{k}$. Hence, the vector equation of the required line is $\vec{r}=2\hat{i}-\hat{j}+\hat{k}+\lambda(2\hat{i}+7\hat{j}-3\hat{k})$ where λ is a parameter.

Example 3.16 **Find the equation of a line which passes through the point (2, 3, 4) and which has equal intercepts on the axes.**

Sol. Since line has equal intercepts on axes, it is equally inclined to axes.

$\Rightarrow$ line is along the vector $a(\hat{i}+\hat{j}+\hat{k})$

$\Rightarrow$ Equation of line is $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{1}$

Example 3.17 **Find the points where line $\frac{x-1}{2}=\frac{y+2}{-1}=\frac{z}{1}$ intersects xy, yz and zx planes.**

Sol. Line meets xy-plane where $z=0$

Hence from the given equation of line, $\frac{x-1}{2}=\frac{y+2}{-1}=\frac{0}{1}$

$\Rightarrow$ $x=1$ and $y=-2$.

$\Rightarrow$ Line meets xy-plane at $(1, -2, 0)$.

Line meets yz-plane where $x=0$

Hence from the given equation of line, $\frac{0-1}{2}=\frac{y+2}{-1}=\frac{z}{1}$

$\Rightarrow$ $z=\frac{-1}{2}$ and $y=-\frac{3}{2}$

$\Rightarrow$ Line meets yz-plane at $\left(0, -\frac{3}{2}, \frac{-1}{2}\right)$

Line meets zx-plane where $y=0$

Hence from the given equation of line $\frac{x-1}{2}=\frac{0+2}{-1}=\frac{z}{1}$

$\Rightarrow$ $z=-2, x=-3$

$\Rightarrow$ Line meets zx-plane at $(-3, 0, -2)$

Example 3.18 **Find the equation of line $x+y-z-3=0=2x+3y+z+4$ in symmetric form. Find the direction ratios of the line.**

Sol. In the section of planes we will see that equation of the form $ax+by+cz+d=0$ is the equation of the plane in the space.

Now equation of line in the form $x+y-z-3=0=2x+3y+z+4$ means set of those points in space which are common to the planes $x+y-z-3=0$ and $2x+3y+z+4=0$, which lie on the line of intersection of planes.

For example, equation of x-axis is $y = z = 0$ where xy-plane ($z = 0$) and xz-plane ($y = 0$) intersect.
Now to get the equation of line in symmetric form, in above equations, first of all we eliminate any one of the variables, say z.
Then adding $x + y - z - 3 = 0$ and $2x + 3y + z + 4 = 0$,
$3x + 4y + 1 = 0$ or $3x = -4y - 1 = \lambda$ (say)

$$\Rightarrow \quad x = \frac{\lambda}{3},\ y = \frac{\lambda + 1}{-4}$$

Putting these values in $x + y - z - 3 = 0$, we have $\frac{\lambda}{3} + \frac{\lambda + 1}{-4} - z - 3 = 0$

$\Rightarrow \quad \lambda = 39 + 12z$

Comparing values of λ, we have equation of line as

$$3x = -4y - 1 = 12z + 39$$

or $\quad \frac{3x}{12} = \frac{-4y - 1}{12} = \frac{12z + 39}{12} \quad$ or $\quad \frac{x}{4} = \frac{y + \frac{1}{4}}{-3} = \frac{z + \frac{13}{4}}{1}$

Hence the line is passing through point $\left(0, -\frac{1}{4}, -\frac{13}{4}\right)$ and having direction ratios 4, – 3, 1.

If we eliminate x or y first we will get equation of line having same direction ratio but with different point on the line.

Example 3.19 **Find the equation of a line which passes through point $A(1, 0, -1)$ and is perpendicular to the straight lines $\vec{r} = 2\hat{i} - \hat{j} + \hat{k} + \lambda(2\hat{i} + 7\hat{j} - 3\hat{k})$ and $\vec{r} = 3\hat{i} - \hat{j} + 3\hat{k} + \lambda(2\hat{i} - 2\hat{j} + 5\hat{k})$.**

Sol. Since the line to be determined is perpendicular to the given two straight lines, it is directed towards vector

$$(2\hat{i} + 7\hat{j} - 3\hat{k}) \times (2\hat{i} - 2\hat{j} + 5\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 7 & -3 \\ 2 & -2 & 5 \end{vmatrix} = 29\hat{i} - 16\hat{j} - 18\hat{k}$$

Hence, the equation of the line passing through point $A(1, 0, -1)$ and along vector $29\hat{i} - 16\hat{j} - 18\hat{k}$ is $\frac{x-1}{29} = \frac{y}{-16} = \frac{z+1}{-18}$.

Example 3.20 **Find the coordinates of a point on the line $\frac{x-1}{2} = \frac{y+1}{-3} = z$ at a distance $4\sqrt{14}$ from the point (1, – 1, 0).**

Sol. Any point on the given line is $(2r + 1, -3r - 1, r)$, its distance from $(1, -1, 0)$

$\Rightarrow (2r)^2 + (-3r)^2 + r^2 = \left(4\sqrt{14}\right)^2$

$\Rightarrow r = \pm 4$

$\Rightarrow$ Coordinates are $(9, -13, 4)$ and $(-7, 11, -4)$ and the point nearer to the origin is $(-7, 11, -4)$.

ANGLE BETWEEN TWO LINES

Let the given lines be

$$\left.\begin{array}{ll}\vec{r}=\vec{a}+\lambda\vec{b} & \text{(i)}\\ \vec{r}=\vec{a}'+\lambda\vec{b}' & \text{(ii)}\end{array}\right\}\rightarrow \text{Vector form}$$

$$\left.\begin{array}{l}\dfrac{x-a_1}{b_1}=\dfrac{y-a_2}{b_2}=\dfrac{z-a_3}{b_3}\\ \dfrac{x-a_1'}{b_1'}=\dfrac{y-a_2'}{b_2'}=\dfrac{z-a_3'}{b_3'}\end{array}\right\}\rightarrow \text{Cartesian form}$$

Clearly (i) and (ii) are straight lines in the directions of $\vec{b}$ and $\vec{b}'$, respectively.
Let θ be the angle between the straight lines (i) and (ii).
Then θ is the angle between vectors $\vec{b}$ and $\vec{b}'$. Therefore,

$$\cos\theta = \frac{\vec{b}\cdot\vec{b}'}{|\vec{b}||\vec{b}'|}$$

$$\vec{b} = b\hat{i}+b_2\hat{j}+b_3\hat{k},\ \vec{b}' = b_1'\hat{i}+b_2'\hat{j}+b_3'\hat{k}$$

$$\therefore\ \vec{b}\cdot\vec{b}' = b_1b_1'+b_2b_2'+b_3b_3'$$

and $|\vec{b}| = \sqrt{b_1^2+b_2^2+b_3^3}$, $|\vec{b}'| = \sqrt{b_1'^2+b_2'^2+b_3'^2}$

$$\Rightarrow \cos\theta = \frac{b_1b_1'+b_2b_2'+b_3b_3'}{\sqrt{b_1^2+b_2^2+b_3^3}\sqrt{b_1'^2+b_2'^2+b_3'^2}}$$

Notes:

1. *If the lines are perpendicular, then* $\vec{b}\cdot\vec{b}'=0 \Rightarrow b_1b_1'+b_2b_2'+b_3b_3'=0$.
2. *If the lines are parallel, then* $\vec{b}=\lambda\vec{b}'$ *for some scalar* $\lambda \Rightarrow \dfrac{b_1}{b_2'}=\dfrac{b_2}{b_2'}=\dfrac{b_3}{b_3'}$.

Example 3.21 **Find the angle between each of the following pairs of lines:**

i. $\vec{r}=3\hat{i}+2\hat{j}-4\hat{k}+\lambda(\hat{i}+2\hat{j}+2\hat{k});\ \vec{r}=5\hat{i}-2\hat{k}+\mu(3\hat{i}+2\hat{j}+6\hat{k})$, **where λ and μ are parameters.**

ii. $\dfrac{x+4}{3}=\dfrac{y-1}{5}=\dfrac{z+3}{4};\ \dfrac{x+1}{1}=\dfrac{y-4}{1}=\dfrac{z-5}{2}$

Sol. **i.** Lines are along vectors, $\vec{b}_1 = \hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b}_2 = 3\hat{i} + 2\hat{j} + 6\hat{k}$

If θ is the angle between the two given lines, then

$$\cos\theta = \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|} = \frac{(1)(3)+(2)(2)+(2)(6)}{\sqrt{1^2+2^2+2^2}\sqrt{3^2+2^2+6^2}} = \frac{19}{(3)(7)} = \frac{19}{21} \Rightarrow \theta = \cos^{-1}\left(\frac{19}{21}\right)$$

ii. Lines are along vectors $\vec{b}_1 = 3\hat{i} + 5\hat{j} + 4\hat{k}$ and $\vec{b}_2 = \hat{i} + \hat{j} + 2\hat{k}$

If θ is the angle between the two given lines, then

$$\cos\theta = \frac{(3)(1)+(5)(1)+(4)(2)}{\sqrt{3^2+5^2+4^2}\sqrt{1^2+1^2+2^2}} = \frac{3+5+8}{\sqrt{9+25+16}\sqrt{1+1+4}}$$

$$= \frac{16}{5\sqrt{2}\sqrt{6}} = \frac{16}{5\sqrt{2}\sqrt{2}\sqrt{3}} = \frac{8\sqrt{3}}{15} \Rightarrow \theta = \cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$$

Example 3.22 **Find the condition if lines $x = ay + b, z = cy + d$ and $x = a'y + b', z = c'y + d'$ are perpendicular.**

Sol. The equations of straight lines can be rewritten as

$$x = ay + b,\ z = cy + d \Rightarrow \frac{x-b}{a} = \frac{y-0}{1} = \frac{z-d}{c}$$

$$\text{and } x = a'y + b',\ z = c'y + d' \Rightarrow \frac{x-b'}{a'} = \frac{y-0}{1} = \frac{z-d'}{c'}$$

The above lines are perpendicular if $aa' + 1\cdot 1 + c\cdot c' = 0$.

PERPENDICULAR DISTANCE OF A POINT FROM A LINE

Foot of Perpendicular from a Point on the Given Line

Cartesian form

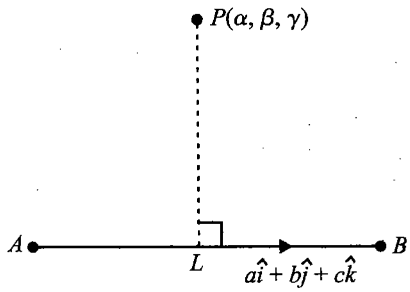


Fig. 3.7

Here, the equation of line AB is $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$.

Let L be the foot of the perpendicular drawn from $P(\alpha, \beta, \gamma)$ on the line $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$.

Let the coordiantes of L be $(x_1+a\lambda, y_1+b\lambda, z_1+c\lambda)$.

Then the direction ratios of PL are $(x_1+a\lambda-\alpha, y_1+b\lambda-\beta, z_1+c\lambda-\gamma)$.

Direction ratios of AB are (a, b, c).

Since PL is perpendicular to AB,

$$a(x_1+a\lambda-\alpha)+b(y_1+b\lambda-\beta)+c(z_1+c\lambda-\gamma)=0$$

$$\lambda=\frac{a(\alpha-x_1)+b(\beta-y_1)+c(\gamma-z_1)}{a^2+b^2+c^2}$$

Putting the value of λ in $(x_1+a\lambda, y_1+b\lambda, z_1+c\lambda)$, we get the foot of the perpendicular. Now we can get distance PL using distance formula.

Vector form

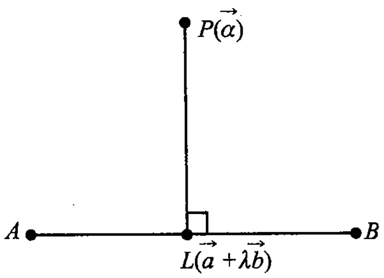


Fig. 3.8

Let L be the foot of the perpendicular drawn from $P(\vec{\alpha})$ on the line $\vec{r}=\vec{a}+\lambda\vec{b}$.

Since $\vec{r}$ denotes the position vector of any point on the line $\vec{r}=\vec{a}+\lambda\vec{b}$, the position vector of L will be $(\vec{a}+\lambda\vec{b})$

Directions ratios of $PL=\vec{a}-\vec{\alpha}+\lambda\vec{b}$

Since $\overrightarrow{PL}$ is perpendicular to $\vec{b}$,

$$(\vec{a}-\vec{\alpha}+\lambda\vec{b})\cdot\vec{b}=0$$

$$\Rightarrow (\vec{a}-\vec{\alpha})\cdot\vec{b}+\lambda\vec{b}\cdot\vec{b}=0$$

$$\Rightarrow \lambda=\frac{-(\vec{a}-\vec{\alpha})\cdot\vec{b}}{|\vec{b}|^2}$$

$\Rightarrow$ Position vector of L is $\vec{a}-\left(\frac{(\vec{a}-\vec{\alpha})\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$, which is the foot of the perpendicular.

Image of a Point in the Given Line

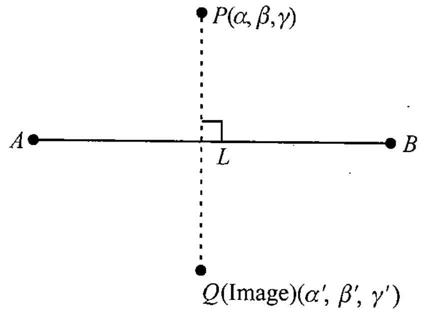


Fig. 3.9

Since L (foot of perpendicular) is the midpoint of P and Q (image of a point P in the line), we can get Q if L is found out.

Example 3.23 **Find the coordinates of the foot of the perpendicular drawn from point $A(1, 0, 3)$ to the join of points $B(4, 7, 1)$ and $C(3, 5, 3)$.**

Sol. Let D be the foot of the perpendicular and let it divide BC in the ratio $\lambda : 1$. Then the coordinates of D are $\frac{3\lambda+4}{\lambda+1}, \frac{5\lambda+7}{\lambda+1}$ and $\frac{3\lambda+1}{\lambda+1}$.

Now, $\overrightarrow{AD} \perp \overrightarrow{BC} \Rightarrow \overrightarrow{AD}\cdot\overrightarrow{BC} = 0$

$\Rightarrow (2\lambda + 3) + 2(5\lambda + 7) + 4 = 0$

$\Rightarrow \lambda = -\frac{7}{4}$

$\Rightarrow$ Coordinates of D are $\frac{5}{3}, \frac{7}{3}$ and $\frac{17}{3}$

Example 3.24 **Find the length of the perpendicular drawn from point (2, 3, 4) to line $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$.**

Sol. Let P be the foot of the perpendicular from $A(2, 3, 4)$ to the given line l whose equation is

$\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$ or $\frac{x-4}{-2} = \frac{y}{6} = \frac{z-1}{-3} = k$ (say). Therefore, (i)

$x = 4 - 2k, y = 6k, z = 1 - 3k$

As P lies on (i), coordinates of P are $(4 - 2k, 6k, 1 - 3k)$ for some value of k.

The direction ratios of AP are

$(4 - 2k - 2, 6k - 3, 1 - 3k - 4)$ or $(2 - 2k, 6k - 3, -3 - 3k)$.

Also, the direction ratios of l are -2, 6 and -3.

Since $AP \perp l$,

$\Rightarrow -2(2 - 2k) + 6(6k - 3) - 3(-3 - 3k) = 0$

$\Rightarrow -4 + 4k + 36k - 18 + 9 + 9k = 0$ or $49k - 13 = 0$ or $k = 13/49$

We have $AP^2 = (4-2k-2)^2 + (6k-3)^2 + (1-3k-4)^2$

$$= (2-2k)^2 + (6k-3)^2 + (-3-3k)^2$$
$$= 4 - 8k + 4k^2 + 36k^2 - 36k + 9 + 9 + 18k + 9k^2$$
$$= 22 - 26k + 49k^2$$
$$= 22 - 26\left(\frac{13}{49}\right) + 49\left(\frac{13}{49}\right)^2$$
$$= \frac{22\times 49 - 26\times 13 + 13^2}{49} = \frac{909}{49}$$
$$AP = \frac{3}{7}\sqrt{101}$$

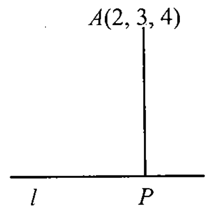


Fig. 3.10

SHORTEST DISTANCE BETWEEN TWO LINES

If two lines in space intersect at a point, then the shortest distance between them is zero. Also, if two lines in space are parallel, then the shortest distance between them will be the perpendicular distance, i.e., the length of the perpendicular drawn from any point on one line onto the other line. Further, in a space, there are lines which are neither intersecting nor parallel. In fact, such pair of lines are *non-coplanar* and are called *skew lines.*

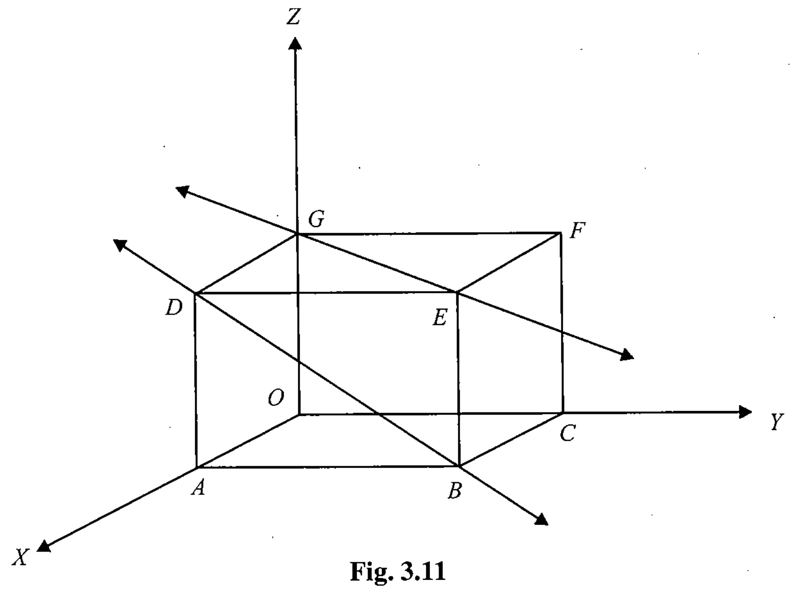


Fig. 3.11

Line GE goes diagonally across the ceiling and line DB passes through one corner of the ceiling directly above A and goes diagonally down the wall. These lines are skew because they are not parallel and also never meet.

By the shortest distance between two lines, we mean the join of a point in one line with one point on the other line so that the length of the segment so obtained is the smallest.

For skew lines, the line of the shortest distance will be perpendicular to both the lines.

Shortest Distance between Two Non-Coplanar Lines

Vector form

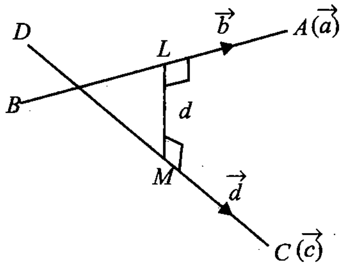


Fig. 3.12

Let the given lines be $\vec{r} = \vec{a} + t\,\vec{b}$ and $\vec{r} = \vec{c} + t_1\,\vec{d}$.

If two lines AB and CD do not intersect, there is always a line intersecting both the lines perpendicularly. The intercept on this line made by AB and CD is called the shortest distance between lines AB and CD. In Fig. 3.12, the shortest distance $= LM$, where $\angle ALM = \angle CML = 90°$. In the figure, the shortest distance $LM = |\text{projection of } \overrightarrow{AC} \text{ along } \overrightarrow{ML}|$

$$= \left|\overrightarrow{AC}\cdot\frac{\overrightarrow{ML}}{|\overrightarrow{ML}|}\right| = \frac{|(\vec{a}-\vec{c})\cdot\overrightarrow{LM}|}{|\overrightarrow{LM}|}$$

Now $\overrightarrow{LM}$ is perpendicular to both $\vec{b}$ and $\vec{d}$.

$$\Rightarrow \overrightarrow{LM} = \vec{b}\times\vec{d}$$

$$= \frac{|(\vec{a}-\vec{c})\cdot(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|}$$

$$= \frac{|[\vec{b}\ \vec{d}\ \vec{a}-\vec{c}]|}{|\vec{b}\times\vec{d}|}$$

Cartesian form

Let the two skew lines be $\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$ and $\frac{x-c_1}{d_1} = \frac{y-c_2}{d_2} = \frac{z-c_3}{d_3}$

Then the shortest distance $= \dfrac{\left\|\begin{matrix} c_1-a_1 & c_2-a_2 & c_3-a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{matrix}\right\|}{\left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{matrix}\right\|}$

Condition for Lines to Intersect

Two lines $\vec{r} = \vec{a} + t\vec{b}$ and $\vec{r} = \vec{c} + t_1\vec{d}$ are intersecting if

$$\left|\frac{(\vec{a}-\vec{c})\cdot(\vec{b}-\vec{d})}{\vec{b}\times\vec{d}}\right| = 0$$

$$\Rightarrow \begin{vmatrix} c_1-a_1 & c_2-a_2 & c_3-a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

Distance Between Two Parallel Lines

If two lines l_1 and l_2 are parallel, then they are coplanar. Let the lines be given by

$$\vec{r} = \vec{a_1} + \lambda\vec{b} \quad \text{(i)}$$

$$\vec{r} = \vec{a_2} + \mu\vec{b} \quad \text{(ii)}$$

where $\vec{a_1}$ is the position vector of a point S on l_1 and $\vec{a_2}$ is the position vector of a point T on l_2.

As l_1 and l_2 are coplanar, if the foot of the perpendicular from T on line l_1 is P, then the distance between the lines l_1 and $l_2 = |TP|$.

Let θ be the angle between vectors $\overrightarrow{ST}$ and $\vec{b}$.

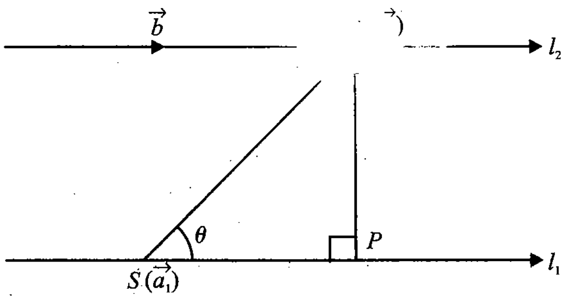


Fig. 3.13

Then $\vec{b} \times \overrightarrow{ST} = (|\vec{b}||\overrightarrow{ST}|\sin\theta)\hat{n}$ (iii)

where $\hat{n}$ is the unit vector perpendicular to the plane of the lines l_1 and l_2.

But $\overrightarrow{ST} = \vec{a_2} - \vec{a_1}$

Therefore, from (iii), we get

$\vec{b} \times (\vec{a_2} - \vec{a_1}) = |\vec{b}|\, PT\, \hat{n}$ (since $PT = ST\sin\theta$)

i.e., $|\vec{b} \times (\vec{a_2} - \vec{a_1})| = |\vec{b}|\, PT \cdot 1$ (as $|\hat{n}| = 1$)

Hence, the distance between the given parallel lines is

$$d = |\overrightarrow{PT}| = \left|\frac{\vec{b}\times(\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$$

Example 3.25 **Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$. Also obtain the equation of the line of the shortest distance.**

Sol. (i) The two given lines are $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = r_1$ (say) (i)

and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5} = r_2$ (say) (ii)

Any point on (i) is given by $P\,(2r_1 + 1, 3r_1 + 2, 4r_1 + 3)$

And any point on (ii) is given by $Q\,(3r_2 + 2, 4r_2 + 4, 5r_2 + 5)$

Direction ratios of PQ are given by $3r_2 - 2r_1 + 1$, $4r_2 - 3r_1 + 2$ and $5r_2 - 4r_1 + 2$

Since PQ is perpendicular to (i), we get

$2\,(3r_2 - 2r_1 + 1) + 3(4r_2 - 3r_1 + 2) + 4\,(5r_2 - 4r_1 + 2) = 0$

or $38r_2 - 29r_1 + 16 = 0$ (iii)

Also PQ is perpendicular to (ii), we get

$3(3r_2 - 2r_1 + 1) + 4\,(4r_2 - 3r_1 + 2) + 5(5r_2 - 4r_1 + 2) = 0$

or $50r_2 - 38r_1 + 21 = 0$ (iv)

Solving (iii) and (iv), we obtain $r_2 = -(1/6)$, $r_1 = (1/3)$.

Therefore, coordinates of P and Q are $\left(\frac{5}{3}, 3, \frac{13}{3}\right)$ and $\left(\frac{3}{2}, \frac{10}{3}, \frac{25}{6}\right)$, respectively.

Thus, $PQ^2 = \left(\frac{3}{2} - \frac{5}{3}\right)^2 + \left(\frac{10}{3} - 3\right)^2 + \left(\frac{25}{6} - \frac{13}{3}\right)^2 = \left(-\frac{1}{6}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(-\frac{1}{6}\right)^2 = \frac{1}{6}$

$\Rightarrow \quad PQ = \frac{1}{\sqrt{6}}$

The equation of the line of the shortest distance is given by

$$\frac{x-(5/3)}{(3/2)-(5/3)} = \frac{y-3}{(10/3)-3} = \frac{z-(13/3)}{(25/6)-(13/3)}$$

$$\frac{x-(5/3)}{-(1/6)} = \frac{y-3}{(1/3)} = \frac{z-(13/3)}{-(1/6)}$$

$$\frac{x-(5/3)}{1} = \frac{y-3}{-2} = \frac{z-(13/3)}{1}$$

Alternative method for finding the shortest distance:

Line (i) is passing through the point $(x_1, y_1, z_1) \equiv (1, 2, 3)$ and is parallel to vector

$a_1\hat{i}+b_1\hat{j}+c_1\hat{k}\equiv 2\hat{i}+3\hat{j}+4\hat{k}$.

Line (ii) is passing through the point $(x_2, y_2, z_2)\equiv(2, 4, 5)$ and is parallel to the vector

$a_2\hat{i}+b_2\hat{j}+c_2\hat{k}\equiv 3\hat{i}+4\hat{j}+5\hat{k}$.

Hence the shortest distance between the lines using the formula

$$\frac{\left\|\begin{matrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix}\right\|}{\left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix}\right\|} \text{ is } \frac{\left\|\begin{matrix} 2-1 & 4-2 & 5-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{matrix}\right\|}{\left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{matrix}\right\|} = \frac{1}{\sqrt{6}}$$

Example 3.26 **Determine whether the following pair of lines intersect or not.**

i. $\vec{r}=\hat{i}-\hat{j}+\lambda(2\hat{i}+\hat{k});\ \vec{r}=2\hat{i}-\hat{j}+\mu(\hat{i}+\hat{j}-\hat{k})$

ii. $\vec{r}=\hat{i}+\hat{j}-\hat{k}+\lambda(3\hat{i}-\hat{j});\ \vec{r}=4\hat{i}-\hat{k}+\mu(2\hat{i}+3\hat{k})$

Sol. **i.** Here $\vec{a}_1=\hat{i}-\hat{j}$, $\vec{a}_2=2\hat{i}-\hat{j}$, $\vec{b}_1=2\hat{i}+\hat{k}$ and $\vec{b}_2=\hat{i}+\hat{j}-\hat{k}$

$$\text{Now } [\vec{a}_2-\vec{a}_1\ \vec{b}_1\ \vec{b}_2]=\begin{vmatrix} 2-1 & -1+1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$=-1\neq 0$$

Thus, the two given lines do not intersect.

ii. Here $\vec{a}_1=\hat{i}+\hat{j}-\hat{k}$, $\vec{a}_2=4\hat{i}-\hat{k}$, $\vec{b}_1=3\hat{i}-\hat{j}$ and $\vec{b}_2=2\hat{i}+3\hat{k}$

$$\Rightarrow [\vec{a}_2-\vec{a}_1\ \vec{b}_1\ \vec{b}_2]=\begin{vmatrix} 4-1 & 0-1 & -1+1 \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix}$$

$$=\begin{vmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix}=0$$

Thus, the two given lines intersect. Let us obtain the point of intersection of the two given lines. For some values of λ and μ, the two values of $\vec{r}$ must coincide.

Thus, $\hat{i}+\hat{j}-\hat{k}+\lambda(3\hat{i}-\hat{j}) = 4\hat{i}-\hat{k}+\mu(2\hat{i}+3\hat{k})$

$\Rightarrow (3+2\mu-3\lambda)\hat{i}+(\lambda-1)\hat{j}+3\mu\hat{k}=0$

$\Rightarrow 3+2\mu-3\lambda=0, \lambda-1=0, 3\mu=0$

Solving, we obtain $\lambda = 1$ and $\mu = 0$

Therefore, the point of intersection is $\vec{r}=4\hat{i}-\hat{k}$ (by putting $\mu = 0$ in the second equation).

Example 3.27 **Find the shortest distance between lines $\vec{r}=(\hat{i}+2\hat{j}+\hat{k})+\lambda(2\hat{i}+\hat{j}+2\hat{k})$ and $\vec{r}=2\hat{i}-\hat{j}-\hat{k}+\mu(2\hat{i}+\hat{j}+2\hat{k})$.**

Sol. Here lines (i) ane (ii) are passing through the points $\vec{a}_1=\hat{i}+2\hat{j}+\hat{k}$ and $\vec{a}_2=2\hat{i}-\hat{j}-\hat{k}$, respectively, and are parallel to the vector $\vec{b}=2\hat{i}+\hat{j}+2\hat{k}$.

Hence, the distance between the lines using the formula

$$\frac{|\vec{b}\times(\vec{a}_2-\vec{a}_1)|}{|\vec{b}|}=\frac{\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & 2\\ 1 & -3 & -2\end{vmatrix}}{3}=\frac{|4\hat{i}-6\hat{j}-7\hat{k}|}{3}=\frac{\sqrt{16+36+49}}{3}=\frac{\sqrt{101}}{3}$$

Example 3.28 **If the straight lines $x=-1+s$, $y=3-\lambda s$, $z=1+\lambda s$ and $x=\frac{t}{2}$, $y=1+t$, $z=2-t$, with parameters s and t, respectively, are coplanar, then find λ.**

Sol. The given lines $\frac{x+1}{1}=\frac{y-3}{-\lambda}=\frac{z-1}{\lambda}=s$ and

$\frac{x-0}{1/2}=\frac{y-1}{1}=\frac{y-2}{-1}=t$ are coplanar if $\begin{vmatrix}0+1 & 1-3 & 2-1\\ 1 & -\lambda & \lambda\\ 1/2 & 1 & -1\end{vmatrix}=0$

$$\begin{vmatrix}1 & -2 & 1\\ 1 & -\lambda & \lambda\\ 1/2 & 1 & -1\end{vmatrix}=0$$

$$\Rightarrow 1(\lambda-\lambda)+2\left(-1-\frac{\lambda}{2}\right)+1\left(1+\frac{\lambda}{2}\right)=0$$

$$\Rightarrow \lambda=-2$$

Example 3.29 **Find the equation of a line which passes through the point (1, 1, 1) and intersects the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x+2}{1}=\frac{y-3}{2}=\frac{z+1}{4}$.**

Sol. Any line passing through the point (1, 1, 1) is $\frac{x-1}{a}=\frac{y-1}{b}=\frac{z-1}{c}$ (i)

This line intersects the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$.

If $a:b:c \neq 2:3:4$ and $\begin{vmatrix} 1-1 & 2-1 & 3-1 \\ a & b & c \\ 2 & 3 & 4 \end{vmatrix} = 0$

$\Rightarrow \quad a - 2b + c = 0$ (ii)

Again, line (i) intersects line $\frac{x-(-2)}{1} = \frac{y-3}{2} = \frac{z-(-1)}{4}$.

If $a:b:c \neq 1:2:4$ and $\begin{vmatrix} -2-1 & 3-1 & -1-1 \\ a & b & c \\ 1 & 2 & 4 \end{vmatrix} = 0$

$\Rightarrow 6a + 5b - 4c = 0$ (iii)

From (ii) and (iii) by cross multiplication, we have $\frac{a}{8-5} = \frac{b}{6+4} = \frac{c}{5+12}$

$\Rightarrow \frac{a}{3} = \frac{b}{10} = \frac{c}{17}$

So, the required line is $\frac{x-1}{3} = \frac{y-1}{10} = \frac{z-1}{17}$

Concept Application Exercise 3.2

1. Find the point where line which passes through point (1, 2, 3) and is parallel to line $\vec{r} = \hat{i} - \hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 3\hat{k})$ meets the xy-plane.
2. Find the equation of the line passing through the points (1, 2, 3) and (– 1, 0, 4).
3. Find the equation of the line passing through the point (2, –1, –1) and parallel to the line $-6x - 2 = 3y + 1 = 2z - 2$.
4. Find the equation of the line passing through the point (–1, 2, 3) and perpendicular to the lines $\frac{x}{2} = \frac{y-1}{-3} = \frac{z+2}{-2}$ and $\frac{x+3}{-1} = \frac{y+3}{2} = \frac{z-1}{3}$.
5. Find the equation of the line passing through the intersection of $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ and also through the point (2, 1, – 2).
6. The straight line $\frac{x-3}{3} = \frac{y-2}{1} = \frac{z-1}{0}$ is
   - **a.** parallel to the x-axis
   - **b.** parallel to the y-axis
   - **c.** parallel to the z-axis
   - **d.** perpendicular to the z-axis
7. Find the angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$.
8. If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-5}{1} = \frac{z-6}{-5}$ are at right angle, then find the value of k.

9. The equations of motion of a rocket are $x = 2t$, $y = -4t$ and $z = 4t$, where time t is given in seconds, and the coordinates of a moving point in kilometres. What is the path of the rocket? At what distance will be the rocket from the starting point $O(0, 0, 0)$ in 10?

10. Find the length of the perpendicular drawn from the point (5, 4, –1) to the line $\vec{r} = \hat{i} + \lambda(2\hat{i} + 9\hat{j} + 5\hat{k})$, where λ is a parameter.

11. Find the image of point (1, 2, 3) in the line $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$.

12. Find the shortest distance between the lines $\vec{r} = (1-\lambda)\hat{i} + (\lambda-2)\hat{j} + (3-2\lambda)\hat{k}$ and $r = (\mu+1)\hat{i} + (2\mu-1)\hat{j} - (2\mu+1)\hat{k}$.

13. If the lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then find the value of k.

PLANE

A plane is a surface such that if any two points are taken on it, the line segment joining them lies completely on the surface.

A plane is determined uniquely if:

i. The normal to the plane and its distance from the origin is given, i.e., the equation of a plane in normal form.
ii. It passes through a point and is perpendicular to a given direction.
iii. It passes through three given non-collinear points.

Equation of a Plane in Normal Form

Consider a plane whose perpendicular distance from the origin is d ($d \neq 0$). If $\overrightarrow{ON}$ is the normal from the origin to the plane, and $\hat{n}$ is the unit normal vector along $\overrightarrow{ON}$, then $\overrightarrow{ON} = d\hat{n}$. Let P be any point on the plane. Therefore, $\overrightarrow{NP}$ is perpendicular to $\overrightarrow{ON}$.

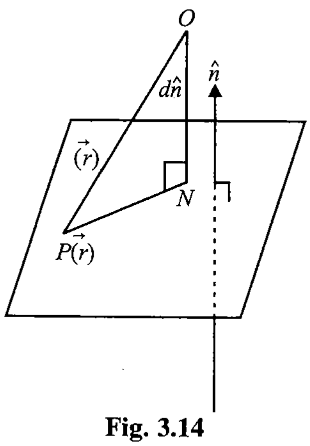


Fig. 3.14

Therefore, $\overrightarrow{NP} \cdot \overrightarrow{ON} = 0$ (i)

Let $\vec{r}$ be the position vector of the point P. Then $\overrightarrow{NP} = \vec{r} - d\hat{n}$ (as $\overrightarrow{ON} + \overrightarrow{NP} = \overrightarrow{OP}$)

Therefore, (i) becomes

$(\vec{r} - d\hat{n}) \cdot d\hat{n} = 0$

$\Rightarrow (\vec{r} - d\hat{n}) \cdot \hat{n} = 0 (d \neq 0)$

$\Rightarrow (\vec{r} \cdot \hat{n}) - d\hat{n} \cdot \hat{n} = 0$

$\Rightarrow \vec{r} \cdot \hat{n} = d (\text{as } \hat{n} \cdot \hat{n} = 1)$ (ii)

This is the vector form of the equation of the plane.

Cartesian form

Equation (ii) gives the vector equation of a plane, where $\hat{n}$ is the unit vector normal to the plane. Let $P(x, y, z)$ be any point on the plane. Then

$\overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Let l, m and n be the direction cosines of $\hat{n}$.

Then $\hat{n} = l\hat{i} + m\hat{j} + n\hat{k}$

Therefore, (ii) gives

$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (l\hat{i} + m\hat{j} + n\hat{k}) = d$

or $lx + my + nz = d$ (iii)

This is the Cartesian equation of the plane in the normal form.

Note: *Equation (iii) shows that if $\vec{r} \cdot (a\hat{i} + b\hat{j} + c\hat{k}) = d$ is the vector equation of a plane, then $ax + by + cz = d$ is the Cartesian equation of the plane, where a, b and c are the direction ratios of the normal to the plane.*

Example 3.30 **Find the equation of plane which is at a distance $\frac{4}{\sqrt{14}}$ from the origin and is normal to vector $2\hat{i} + \hat{j} - 3\hat{k}$.**

Sol. Here $\vec{n} = 2\hat{i} + \hat{j} - 3\hat{k}$. Then $\frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i} + \hat{j} - 3\hat{k}}{\sqrt{2^2 + 1^2 + (-3)^2}} = \frac{2\hat{i} + \hat{j} - 3\hat{k}}{\sqrt{14}}$

Hence required equation of plane is $\vec{r} \cdot \frac{1}{\sqrt{14}}(2\hat{i} + \hat{j} - 3\hat{k}) = \pm \frac{1}{\sqrt{14}}$

or $\vec{r} \cdot (2\hat{i} + \hat{j} - 3\hat{k}) = \pm 1$ (vector form)

or $2x + y - 3z = \pm 1$ (cartesian form)

Example 3.31 **Find the unit vector perpendicular to the plane $\vec{r} \cdot (2\hat{i} + \hat{j} + 2\hat{k}) = 5$.**

Sol. Vector normal to the plane is $\vec{n} = 2\hat{i} + \hat{j} + 2\hat{k}$

Hence unit vector perpendicular to the plane is $\frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{2^2+1^2+2^2}} = \frac{1}{3}(2\hat{i}+\hat{j}+2\hat{k})$

Example 3.32 **Find the distance of the plane $2x - y - 2z - 9 = 0$ from the origin.**

Sol. The plane can be put in vector form as $\vec{r}\cdot(2\hat{i}-\hat{j}-2\hat{k}) = 9$ where $\vec{r} = 2\hat{i}-\hat{j}-2\hat{k}$.

Here $\vec{n} = 2\hat{i}-\hat{j}-2\hat{k}$

$$\Rightarrow \quad \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i}-\hat{j}-2\hat{k}}{3}$$

Dividing equation throughout by 3, we have equation of plane in normal form as

$\vec{r}\cdot\frac{(2\hat{i}-\hat{j}-2\hat{k})}{3} = 3$, in which 3 is the distance of the plane from the origin.

Example 3.33 **Find the vector equation of a line passing through $3\hat{i}-5\hat{j}+7\hat{k}$ and perpendicular to the plane $3x - 4y + 5z = 8$.**

Sol. The given plane $3x - 4y + 5z = 8$ or $(3\hat{i}-4\hat{j}+5\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k}) = 8$.

This shows that $\vec{d} = 3\hat{i}-4\hat{j}+5\hat{k}$ is normal to the given plane.

Therefore, the required line is parallel to $3\hat{i}-4\hat{j}+5\hat{k}$.

Since the required line passes through $3\hat{i}-5\hat{j}+7\hat{k}$, its equation is given by

$\vec{r} = 3\hat{i}-5\hat{j}+7\hat{k}+\lambda(3\hat{i}-4\hat{j}+5\hat{k})$, where λ is a parameter.

Vector Equation of a Plane Passing through a Given Point and Normal to a Given Vector

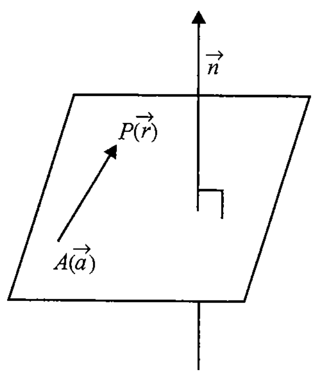


Fig. 3.15

Suppose the plane passes through a point having position vector $\vec{a}$ and is normal to vector $\vec{n}$.

Then for any position of point $P(\vec{r})$ on the plane, $\overrightarrow{AP} \perp \vec{n}$

$\Rightarrow \overrightarrow{AP} \cdot \vec{n} = 0$

$\Rightarrow (\vec{r} - \vec{a}) \cdot \vec{n} = 0 \quad (\because \overrightarrow{AP} = \vec{r} - \vec{a})$

Hence the required equation of the plane is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$.

Note:

The above equation can be written as $\vec{r} \cdot \vec{n} = d$, *where* $d = \vec{a} \cdot \vec{n}$ *(known as scalar product form of plane).*

The equation $\vec{r} \cdot \vec{n} = d$ *is in normal form if* $\vec{n}$ *is a unit vector and d is the distance of the plane from the origin. If* $\vec{n}$ *is not a unit vector, then we reduce the equation* $\vec{r} \cdot \vec{n} = d$ *to the normal form by dividing both sides by* $|\vec{n}|$; we get $\frac{\vec{r} \cdot \vec{n}}{|\vec{n}|} = \frac{d}{|\vec{n}|} \Rightarrow \vec{r} \cdot \hat{n} = \frac{d}{|n|} = p$ (distance from the origin).

Cartesian form

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$, then

$(\vec{r} - \vec{a}) = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$

Then equation of the plane can be written as

$((x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}) \cdot (a\hat{i} + b\hat{j} + c\hat{k}) = 0$

$\Rightarrow a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

Thus, the coefficients of x, y and z in the Cartesian equation of a plane are the direction ratios of the normal to the plane.

Example 3.34 **Find the equation of the plane passing through the point (2, 3, 1) having (5, 3, 2) as the direction ratios of the normal to the plane.**

Sol. The equation of the plane passing through (x_1, y_1, z_1) and perpendicular to the line with direction ratios a, b and c is given by $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

Now, since the plane passes through (2, 3, 1) and is perpendicular to the line having direction ratios (5, 3, 2), the equation of the plane is given by $5(x - 2) + 3(y - 3) + 2(z - 1) = 0$ or $5x + 3y + 2z = 21$.

Example 3.35 **The foot of the perpendicular drawn from the origin to a plane is (12, – 4, 3). Find the equation of the plane.**

Sol. Since $P(12, -4, 3)$ is the foot of the perpendicular from the origin to the plane, OP is normal to the plane. Thus, the direction ratios of normal to the plane are 12, – 4 and 3.

Now, since the plane passes through (12, –4, 3), its equation is given by

$12(x - 12) - 4(y + 4) + 3(z - 3) = 0$

or $12x - 4y + 3z - 169 = 0$.

Example 3.36 **Find the equation of the plane such that image of point (1, 2, 3) in it is (−1, 0, 1).**

Sol. Since the image of A (1, 2, 3) in the plane is B (−1, 0, 1), the plane passes through the midpoint (0, 1, 2) of AB and is normal to the vector $\overrightarrow{AB} \equiv -2\hat{i} - 2\hat{j} - 2\hat{k}$

Hence, the equation of the plane is $-2(x-0) - 2(y-1) - 2(z-2) = 0$ or $x + y + z = 3$

Equation of a Plane Passing through Three Given Points

Cartesian form

Let the plane be passing through points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$.

Let $P(x, y, z)$ be any point on the plane.

Then vectors $\overrightarrow{PA}$, $\overrightarrow{BA}$ and $\overrightarrow{CA}$ are coplanar.

$[\overrightarrow{PA}\ \overrightarrow{BA}\ \overrightarrow{CA}] = 0$

$$\Rightarrow \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0,$$ which is the required equation of the plane

Vector form

Vector form of the equation of the plane passing through three points A, B and C having position vectors $\vec{a}, \vec{b}$ and $\vec{c}$, respectively.

Let $\vec{r}$ be the position vector of any point P in the plane.

Hence vectors $\overrightarrow{AP} = \vec{r} - \vec{a}$ $\overrightarrow{AB} = \vec{b} - \vec{a}$ and $\overrightarrow{AC} = \vec{c} - \vec{a}$ are coplanar.

Hence, $(\vec{r} - \vec{a}) \cdot \{(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})\} = 0$

$\Rightarrow (\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a}) = 0$

$\Rightarrow (\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}) = 0$

$\Rightarrow \vec{r} \cdot (\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}) = \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{a} \times \vec{b}) + \vec{a} \cdot (\vec{c} \times \vec{a})$

$\Rightarrow [\vec{r}\ \vec{b}\ \vec{c}] + [\vec{r}\ \vec{a}\ \vec{b}] + [\vec{r}\ \vec{c}\ \vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]$

which is the required equation of the plane.

Notes:

1. *If p is the length of perpendicular from the origin on this plane, then* $p = [\vec{a}\ \vec{b}\ \vec{c}]/n$, *where* $n = |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$.

2. *Four points $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are coplanar if $\vec{d}$ lies on the plane containing $\vec{a}, \vec{b}$ and $\vec{c}$.*

$$or\ \vec{d}\cdot[\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}]=[\vec{a}\ \vec{b}\ \vec{c}]$$

$$or\ [\vec{d}\ \vec{a}\ \vec{b}]+[\vec{d}\ \vec{b}\ \vec{c}]+[\vec{d}\ \vec{c}\ \vec{a}]=[\vec{a}\ \vec{b}\ \vec{c}].$$

Example 3.37 **Find the equation of the plane passing through A (2, 2, –1), B (3, 4, 2) and C (7, 0, 6). Also find a unit vector perpendicular to this plane.**

Sol. Here $(x_1, y_1, z_1) \equiv (2, 2, -1)$, $(x_2, y_2, z_2) \equiv (3, 4, 2)$ and $(x_3, y_3, z_3) \equiv (7, 0, 6)$

Then the equation of the plane is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} x-2 & y-2 & z-(-1) \\ 3-2 & 4-2 & 2-(-1) \\ 7-2 & 0-2 & 6-(-1) \end{vmatrix} = 0$$

or

$5x + 2y - 3z = 17$

A normal vector to this plane is $\vec{d} = 5\hat{i} + 2\hat{j} - 3\hat{k}$ (i)

Therefore, a unit vector normal to (i) is given by

$$\hat{n} = \frac{\vec{d}}{|\vec{d}|} = \frac{5\hat{i}+2\hat{j}-3\hat{k}}{\sqrt{25+4+9}} = \frac{1}{\sqrt{38}}(5\hat{i}+2\hat{j}-3\hat{k})$$

Example 3.38 **Show that the line of intersection of the planes $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})=0$ and $\vec{r}\cdot(3\hat{i}+2\hat{j}+\hat{k}) = 0$ is equally inclined to $\hat{i}$ and $\hat{k}$. Also find the angle it makes with $\hat{j}$.**

Sol. The line of intersection of the two planes will be perpendicular to the normals to the planes. Hence it is parallel to the vector $(\hat{i}+2\hat{j}+3\hat{k})\times(3\hat{i}+2\hat{j}+\hat{k}) = (-4\hat{i}+8\hat{j}-4\hat{k})$.

Now, $(-4\hat{i}+8\hat{j}-4\hat{k}).\hat{i} = -4$ and $(-4\hat{i}+8\hat{j}-4\hat{k})\cdot\hat{k} = -4$

Hence the line is equally inclined to $\hat{i}$ and $\hat{k}$.

$$\text{Also, } \frac{(-4\hat{i}+8\hat{j}-4\hat{k})}{\sqrt{16+64+16}}.\hat{j} = \frac{8}{\sqrt{96}} = \sqrt{\frac{2}{3}}$$

If θ is the required angle, then $\cos\theta = \sqrt{\frac{2}{3}} \Rightarrow \theta = \cos^{-1}\sqrt{\frac{2}{3}}$

Equation of the Plane that Passes through Point A with Position Vector $\vec{a}$ and is Parallel to Given Vectors $\vec{b}$ and $\vec{c}$

Vector form

Let $\vec{r}$ be the position vector of any point P in the plane. Then

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \vec{r} - \vec{a}$$

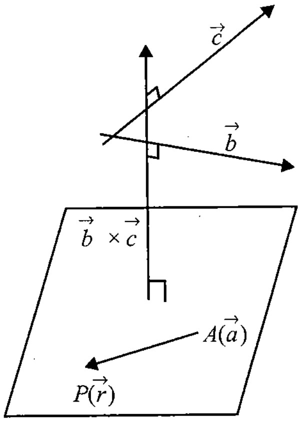


Fig. 3.16

Since vectors $\vec{r} - \vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar,

$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0$

$\Rightarrow \vec{r} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c}) \Rightarrow [\vec{r}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{b}\ \vec{c}]$

which is the required equation of the plane.

Cartesian form

From $(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0$, we have $[\vec{r} - \vec{a}\ \vec{b}\ \vec{c}]$

$$\Rightarrow \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$, which is the required equation of the plane,

where $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ and $\vec{c} = x_3\hat{i} + y_3\hat{j} + z_3\hat{k}$.

Example 3.39 **Find the vector equation of the following planes in cartesian form:**
$\vec{r} = \hat{i} - \hat{j} + \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - 2\hat{j} + 3\hat{k})$.

Sol. The equation of the plane is $\vec{r} = \hat{i} - \hat{j} + \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - 2\hat{j} + 3\hat{k})$.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Hence, the equation is $(x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} - \hat{j}) = \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - 2\hat{j} + 3\hat{k})$

Thus vectors $(x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} - \hat{j})$, $\hat{i} + \hat{j} + \hat{k}$, $\hat{i} - 2\hat{j} + 3\hat{k}$ are coplanar.

Therefore, the equation of the plane is $\begin{vmatrix} x-1 & y-(-1) & z-0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 0$ or $5x - 2y - 3z - 7 = 0$

Equation of a Plane Passing through a Given Point and Line

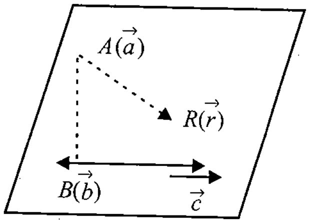


Fig. 3.17

Let the plane pass through a given point $A\ (\vec{a})$ and line $\vec{r} = \vec{b} + \lambda\,\vec{c}$.

For any position of point $R\ (\vec{r})$ on the plane, vectors $\overrightarrow{AB}$, $\overrightarrow{RA}$ and $\vec{c}$ are coplanar. Then $[\vec{r} - \vec{a}\ \ \vec{b} - \vec{a}\ \ \vec{c}] = 0$, which is required equation of the plane.

Example 3.40 **Prove that the plane $\vec{r}\cdot(\hat{i}+2\hat{j}-\hat{k})=3$ contains the line $\vec{r} = \hat{i} + \hat{j} + \lambda\,(2\hat{i} + \hat{j} + 4\hat{k})$.**

Sol. To show that $\vec{r} = \hat{i} + \hat{j} + \lambda\,(2\hat{i} + \hat{j} + 4\hat{k})$ (i)

lies in the plane $\vec{r}\cdot(\hat{i} + 2\hat{j} - \hat{k}) = 3$, (ii)

we must show that each point of (i) lies in (ii). In other words, we must show that $\vec{r}$ in (i) satisfies (ii) for every value of λ.

We have $[\hat{i} + \hat{j} + \lambda\,(2\hat{i} + \hat{j} + 4\hat{k})]\cdot(\hat{i} + 2\hat{j} - \hat{k})$

$= (\hat{i} + \hat{j})\cdot(\hat{i} + 2\hat{j} - \hat{k}) + \lambda\,(2\hat{i} + \hat{j} + 4\hat{k})\cdot(\hat{i} + 2\hat{j} - \hat{k})$

$= (1)\,(1) + (1)\,(2) + \lambda\,[(2)\,(1) + (1)\,(2) + 4\,(-1)] = 3 + \lambda\,(0) = 3$

Hence line (i) lies in plane (ii).

Example 3.41 **Find the equation of the plane which is parallel to the lines $\vec{r} = \hat{i} + \hat{j} + \lambda\,(2\hat{i} + \hat{j} + 4\hat{k})$ and $\dfrac{x+1}{-3} = \dfrac{y-3}{2} = \dfrac{z+2}{1}$ and is passing through the point (0, 1, –1).**

Sol. The plane is parallel to the given lines, which are directed along vectors $\vec{a} = 2\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{b} = -3\hat{i} + 2\hat{j} + 1\hat{k}$.

Then the plane is normal to vector $\vec{a}\times\vec{b} \equiv \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 4 \\ -3 & 2 & 1 \end{vmatrix} = -7\hat{i} - 14\hat{j} + 7\hat{k}$

Also, the plane passes through the point (0, 1, –1).

Therefore, the equation of the plane is $-7\,(x-0) - 14\,(y-1) + 7\,(z+1) = 0$ or $7x + 14y - 7z = 21$

Intercept Form of a Plane

Let O be the origin and let OX, OY and OZ be the coordinate axes.

Let the plane meet the coordinate axes at the points A, B and C, respectively, such that

$OA = a$, $OB = b$ and $OC = c$. Then, the coordinates of points are $A\ (a, 0, 0)$, $B\ (0, b, 0)$ and $C\ (0, 0, c)$.

Let the equation of the plane be $Ax + By + Cz + D = 0$ (i)

Since (i) passes through $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, we have

$$Aa + D = 0 \Rightarrow A = \frac{-D}{a},\ Bb + D = 0 \Rightarrow B = \frac{-D}{b},\ Cc + D = 0 \Rightarrow C = \frac{-D}{c}$$

Putting these values in (i), we get the required equation of the plane as

$$\frac{-D}{a}x - \frac{D}{b}y - \frac{D}{c}z = -D \Rightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Example 3.42 **If a plane meets the coordinate axes at A, B and C such that the centroid of the triangle is (1, 2, 4), then find the equation of the plane.**

Sol. Let the plane meet the coordinate axes at $A\ (a, 0, 0)$, $B\ (0, b, 0)$, and $C\ (0, 0, c)$. Then, $a = 3$, $b = 6$, $c = 12$.

Hence, the equation of required plane is $\frac{x}{3} + \frac{y}{6} + \frac{z}{12} = 1$ or $4x + 2y + z = 12$

Equation of a Plane Passing through Two Parallel Lines

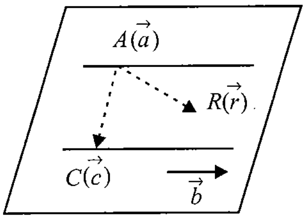


Fig. 3.18

Let the plane pass through parallel lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{c} + \mu\vec{b}$.

As shown in the diagram, for any position of R in the plane, vectors $\overrightarrow{RA}$, $\overrightarrow{AC}$ and $\vec{b}$ are coplanar. Then $[\vec{r} - \vec{a}\ \ \vec{c} - \vec{a}\ \ \vec{b}] = 0$, which is the required equation of the plane.

Equation of a Plane Parallel to a Given Plane

The general equation of the plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + k = 0$, where k is any scalar, as normal to both the planes is $a\hat{i} + b\hat{j} + c\hat{k}$.

Example 3.43 **Find the equation of the plane passing through (3, 4, –1), which is parallel to the plane $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 5\hat{k}) + 7 = 0$.**

Sol. The equation of any plane which is parallel to $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 5\hat{k}) + 7 = 0$ is

$\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})+\lambda=0$ (i)

or $2x-3y+5z+\lambda=0$

Further (i) will pass through (3, 4, −1) if $(2)(3)+(-3)(4)+5(-1)+\lambda=0$ or $-11+\lambda=0 \Rightarrow \lambda=11$

Thus equation of the required plane is $\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})+11=0$.

ANGLE BETWEEN TWO PLANES

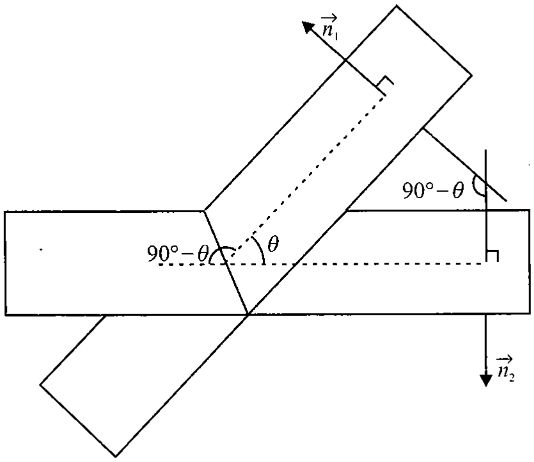


Fig. 3.19

The angle between two planes is defined as the angle between their normals.

Let θ be the angle between planes $\vec{r}\cdot\vec{n}_1=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$

then $\cos\theta=\dfrac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$

Condition for Perpendicularity

If the planes $\vec{r}\cdot\vec{n}_1=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$ are perpendicular, then $\vec{n}_1$ and $\vec{n}_2$ are perpendicular. Therefore,

$\vec{n}_1\cdot\vec{n}_2=0$

Condition for Parallelism

If the planes $\vec{r}\cdot\vec{n}_1=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$ are parallel, there exists the scalar λ such that $\vec{n}_1=\lambda\vec{n}_2$.

Cartesian form

If the planes are $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$

$$\Rightarrow \cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Condition for parallelism: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \lambda$

Condition for perpendicularity: $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Example 3.44 **Find the angle between the planes $2x + y - 2x + 3 = 0$ and $\vec{r} \cdot (6\hat{i} + 3\hat{j} + 2\hat{k}) = 5$.**

Sol. Normals along the given planes are $2\hat{i} + \hat{j} - 2\hat{k}$ and $6\hat{i} + 3\hat{j} + 2\hat{k}$

Then angle between planes, $\theta = \cos^{-1}\frac{(2\hat{i} + \hat{j} - 2\hat{k}) \cdot (6\hat{i} + 3\hat{j} + 2\hat{k})}{\sqrt{(2)^2 + (1)^2 + (-2)^2}\sqrt{(6)^2 + (3)^2 + (2)^2}} = \cos^{-1}\frac{11}{21}$

Example 3.45 **Show that $ax + by + r = 0$, $by + cz + p = 0$ and $cz + ax + q = 0$ are perpendicular to x-y, y-z and z-x planes, respectively.**

Sol. The planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are perpendicular to each other if and only if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

The equation of x-y, y-z and z-x planes are $z = 0$, $x = 0$ and $y = 0$, respectively.

Now we have to show that $z = 0$ is perpendicular to $ax + by + r = 0$.

It follows immediately, since a (0) + b (0) + (0) (1) = 0, other parts can be done similarly.

LINE OF INTERSECTION OF TWO PLANES

Let two non-parallel planes are $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$

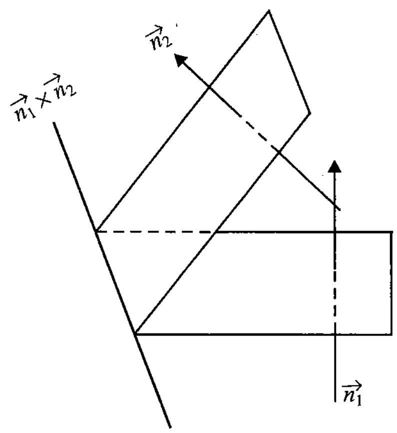


Fig. 3.20

Now line of intersection of planes is perpendicular to vectors $\vec{n}_1$ and $\vec{n}_2$.

$\therefore$ Line of intersection is parallel to vector $\vec{n}_1 \times \vec{n}_2$.

If we wish to find the equation of line of intersection of planes $a_1x + b_1y + c_1z - d_1 = 0$ and $a_2x + b_2y + c_2z - d_2 = 0$, then we find any point on the line by putting $z = 0$ (say), then we can find corresponding values of x and

y by solving equations $a_1x + b_1y - d_1 = 0$ and $a_2x + b_2y - d_2 = 0$. Thus by fixing the value of $z = \lambda$, we can find the corresponding value of x and y in terms of λ. After getting x, y and z in terms of λ, we can find the equation of line in symmetric form.

Example 3.46 **Reduce the equation of line $x - y + 2z = 5$ and $3x + y + z = 6$ in symmetrical form.**

or

Find the line of intersection of planes $x - y + 2z = 5$ and $3x + y + z = 6$.

Sol. Given $x - y + 2z = 5$, $3x + y + z = 6$.

Let $z = \lambda$.

Then $x - y = 5 - 2\lambda$ and $3x + y = 6 - \lambda$.

Solving these two equations, $4x = 11 - 3\lambda$ and $4y = 4x - 20 + 8\lambda = -9 + 5\lambda$.

The equation of the line is $\dfrac{4x-11}{-3} = \dfrac{4y+9}{5} = \dfrac{z-0}{1}$.

Example 3.47 **Find the equation of the plane passing through the points (–1, 1, 1) and (1, –1, 1) and perpendicular to the plane $x + 2y + 2z = 5$.**

Sol. The equation of any plane which passes through (–1, 1, 1) is

$a(x + 1) + b(y - 1) + c(z - 1) = 0$ (i)

This plane will pass through (1, –1, 1) if

$2a - 2b = 0$ or $a = b$ (ii)

Next, (i) will be perpendicular to $x + 2y + 2z = 5$ if

$a + 2b + 2c = 0$ (iii)

Using (ii), we can write (iii) as $a + 2a + 2c = 0$ or $c = -3a/2$.

Thus $a : b : c = a : a : \left(\dfrac{-3}{2}\right)a = 2 : 2 : -3$

Putting these values in (i), we get $2(x + 1) + 2(y - 1) - 3(z - 1) = 0$

or $2x + 2y - 3z + 3 = 0$, which is the equation of the required plane.

Alternative method:

The plane is passing through the points $A(-1, 1, 1)$ and $B(1, -1, 1)$.

Let any point on the plane be $P(x, y, z)$.

Then vector $\overrightarrow{AP} \times \overrightarrow{AB}$ is perpendicular to vector $\hat{i} + 2\hat{j} + 2\hat{k}$, which is normal to the plane $x + 2y + 2z = 5$.

Hence, the equation of the plane is $\begin{vmatrix} x-(-1) & y-1 & z-1 \\ 1-(-1) & -1-1 & 1-1 \\ 1 & 2 & 2 \end{vmatrix} = 0$ or $2x + 2y - 3z + 3 = 0$

Example 3.48 **Find the equation of the plane containing line $\dfrac{x+1}{-3} = \dfrac{y-3}{2} = \dfrac{z+2}{1}$ and point (0, 7, – 7).**

Sol. The equation of the plane containing line $\dfrac{x+1}{-3} = \dfrac{y-3}{2} = \dfrac{z+2}{1}$ is

$a(x + 1) + b(y - 3) + c(z + 2) = 0,$ (i)

where $-3a + 2b + c = 0$ (ii)

This passes through $(0, 7, -7)$.

$\therefore a + 4b - 5c = 0.$ (iii)

From (ii) and (iii), $\frac{a}{-14} = \frac{b}{-14} = \frac{c}{-14}$ or $\frac{a}{1} = \frac{b}{1} = \frac{c}{1}$

So, the required plane is $x + y + z = 0$.

Example 3.49 **Find the distance of the point P (3, 8, 2) from the line $\frac{1}{2}(x-1) = \frac{1}{4}(y-3) = \frac{1}{3}(z-2)$ measured parallel to the plane $3x + 2y - 2z + 15 = 0$.**

Sol.

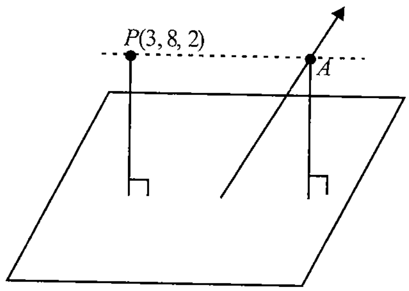


Fig. 3.21

Let the general point of the line be $A\ (2\lambda + 1, 4\lambda + 3, 3\lambda + 2)$.

Let this point lie on the line such that AP is parallel to the plane

$\Rightarrow \overrightarrow{AP} \perp (3\hat{i} + 2\hat{j} - 2\hat{k})$

$\Rightarrow 3 \cdot (2\lambda - 2) + 2(4\lambda - 5) - 2(3\lambda) = 0$

$\Rightarrow \lambda = 2$

Therefore, A is $(5, 11, 8)$.

$PA = \sqrt{(5-3)^2 + (11-8)^2 + (8-2)^2} = \sqrt{4 + 9 + 36} = 7$

Example 3.50 **Find the distance of the point (1, 0, −3) from the plane $x - y - z = 9$ measured parallel to the line $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$.**

Sol.

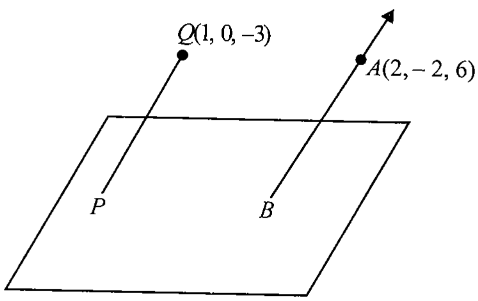


Fig. 3.22

The given plane is $x - y - z = 9$ (i)

The given line AB is $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$ (ii)

The equation of the line passing through (1, 0, – 3) and parallel to $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$ is

$$\frac{x-1}{2} = \frac{y-0}{3} = \frac{z+3}{-6} = r \quad \text{(iii)}$$

Coordinate of any point on (iii) may be given as $P(2r+1, 3r, -6r-3)$.

If P is the point of the intersection of (i) and (iii), then it must lie on (i). Therefore,

$(2r+1) - (3r) - (-6r-3) = 9$

$2r + 1 - 3r + 6r + 3 = 9 \Rightarrow r = 1$

Therefore, the coordinates of P are 3, 3, –9.

Distance between Q (1, 0, – 3) and P (3, 3, – 9) = $\sqrt{(3-1)^2 + (3-0)^2 + (-9+3)^2}$

$= \sqrt{4+9+36} = 7$

ANGLE BETWEEN A LINE AND A PLANE

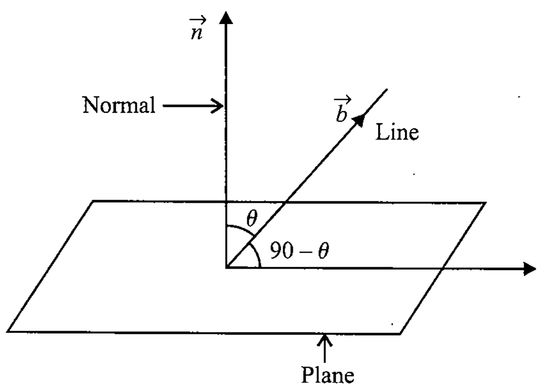


Fig. 3.23

The angle between a line and a plane is the complement of the angle between the line and the normal to the plane.

If the equation of the line is $\vec{r} = \vec{a} + \lambda\vec{b}$ and that of the plane is $\vec{r} \cdot \vec{n} = d$, then angle θ between the line and the normal to the plane is $\cos\theta = \left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right|$.

So the angle ϕ between the line and the plane is given by $90° - \theta$

$$\sin\phi = \left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right| \text{ or } \phi = \sin^{-1}\left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right|$$

Line $\vec{r}=\vec{a}+\lambda\vec{b}$ and plane $\vec{r}\cdot\vec{n}=d$ are perpendicular if $\vec{b}=\lambda\vec{n}$ or $\vec{b}\times\vec{n}=\vec{0}$ and parallel if $\vec{b}\perp\vec{n}$ or $\vec{b}\cdot\vec{n}=0$.

Example 3.51 **Find the angle between the line $\vec{r}=\hat{i}+2\hat{j}-\hat{k}+\lambda(\hat{i}-\hat{j}+\hat{k})$ and the plane $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})=4$.**

Sol. We know that if θ is the angle between the lines $\vec{r}=\vec{a}+\lambda\vec{b}$ and $\vec{r}\cdot\vec{n}=p$, then $\sin\theta=\left|\dfrac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right|$

Therefore, if θ is the angle between $\vec{r}=\hat{i}+2\hat{j}-\hat{k}+\lambda(\hat{i}-\hat{j}+\hat{k})$ and $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})=4$, then

$$\sin\theta=\left|\frac{(\hat{i}-\hat{j}+\hat{k})\cdot(2\hat{i}-\hat{j}+\hat{k})}{|\hat{i}-\hat{j}+\hat{k}||2\hat{i}-\hat{j}+\hat{k}|}\right|$$

$$=\frac{2+1+1}{\sqrt{1+1+1}\sqrt{4+1+1}}$$

$$=\frac{4}{\sqrt{3}\sqrt{6}}=\frac{4}{3\sqrt{2}}$$

$$\Rightarrow \qquad \theta=\sin^{-1}\left(\frac{4}{3\sqrt{2}}\right)$$

EQUATION OF A PLANE PASSING THROUGH THE LINE OF INTERSECTION OF TWO PLANES

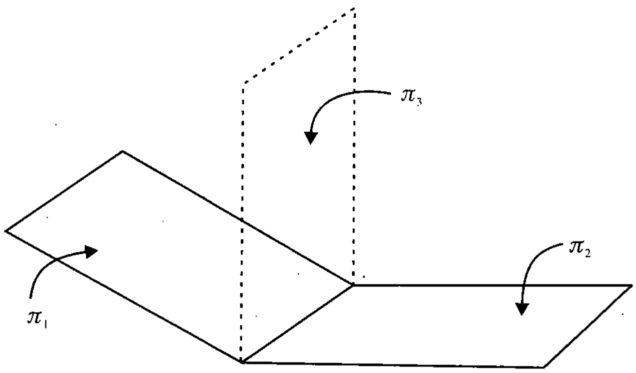


Fig. 3.24

Let π_1 and π_2 be two planes with equations $\vec{r}\cdot\hat{n}_1=d_1$ and $\vec{r}\cdot\hat{n}_2=d_2$, respectively. The position vector of any point on the line of intersection must satisfy both the equations.

If $\vec{t}$ is the position vector of a point on the line, then

$$\vec{t}\cdot\hat{n}_1=d_1 \text{ and } \vec{t}\cdot\hat{n}_2=d_2$$

Therefore, for all real values of λ, we have

$$\vec{t} \cdot (\hat{n}_1 + \lambda \hat{n}_2) = d_1 + \lambda d_2 \quad \text{(i)}$$

Since $\vec{t}$ is arbitrary, it satisfies for any point on the line.

Hence, the equation $\vec{r} \cdot (\vec{n}_1 + \lambda \vec{n}_2) = d_1 + \lambda d_2$ represents a plane π_3 which is such that if any vector $\vec{r}$ satisfies both the equations π_1 and π_2, it also satisfies the equation π_3.

Cartesian Form

In Cartesian system, let $\vec{n}_1 = A_1\hat{i} + B_1\hat{j} + C_1\hat{k}$, $\vec{n}_2 = A_2\hat{i} + B_2\hat{j} + C_2\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Then (i) becomes

$$x(A_1 + \lambda A_2) + y(B_1 + \lambda B_2) + z(C_1 + \lambda C_2) = d_1 + \lambda d_2$$

or $\quad (A_1x + B_1y + C_1z - d_1) + \lambda(A_2x + B_2y + C_2z - d_2) = 0 \quad$ (ii)

which is the required Cartesian form of the equation of the plane passing through the intersection of the given planes for each value of λ.

Example 3.52 **Find the plane passing through the intersection of planes $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 4\hat{k}) = 1$ and $\vec{r} \cdot (\hat{i} - \hat{j}) + 4 = 0$ and perpendicular to $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = -8$.**

Sol. The equation of any plane through the line of intersection of the given planes is

$$\{\vec{r} \cdot (2\hat{i} - 3\hat{j} + 4\hat{k}) - 1\} + \lambda\{\vec{r} \cdot (\hat{i} - \hat{j}) + 4\} = 0$$

$$\vec{r} \cdot \{(2+\lambda)\hat{i} - (3+\lambda)\hat{j} + 4\hat{k}\} = 1 - 4\lambda \quad \text{(i)}$$

If it is perpendicular to $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 8 = 0$, then

$$\{(2+\lambda)\hat{i} - (3+\lambda)\hat{j} + 4\hat{k}\} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 0$$

$$2(2 + \lambda) + (3 + \lambda) + 4 = 0$$

$$\lambda = \frac{-11}{3}$$

Putting $\lambda = -11/3$ in (i), we obtain the equation of the required plane as $\vec{r} \cdot (-5\hat{i} + 2\hat{j} + 12\hat{k}) = 47$

Example 3.53 **Find the equation of a plane containing the line of intersection of the planes $x + y + z - 6 = 0$ and $2x + 3y + 4z + 5 = 0$ and passing through (1, 1, 1).**

Sol. The equation of a plane passing through the line of intersection of the given planes is

$$(x + y + z - 6) + \lambda(2x + 3y + 4z + 5) = 0 \quad \text{(i)}$$

If it is passes through (1, 1, 1), $(1 + 1 + 1 - 6) + \lambda(2 + 3 + 4 + 5) = 0$

$$\Rightarrow \lambda = \frac{3}{14}$$

Putting $\lambda = 3/14$ in (i), we get

$$(x+y+z-6)+\frac{3}{14}(2x+3y+4z+5)=0$$

$$20x+23y+26z-69=0$$

Example 3.54 **The plane $ax+by=0$ is rotated through an angle α about its line of intersection with the plane $z=0$. Show that the equation to the plane in the new position is**

$$ax+by\pm z\sqrt{a^2+b^2}\tan\alpha=0.$$

Sol. Given planes are $ax+by=0$ (i)

and $z=0$ (ii)

Therefore, the equation of any plane passing through the line of intersection of planes (i) and (ii) may be taken as

$ax+by+kz=0$ (iii)

The direction cosines of a normal to the plane (iii) are

$$\frac{a}{\sqrt{a^2+b^2+k^2}},\ \frac{b}{\sqrt{a^2+b^2+k^2}} \text{ and } \frac{k}{\sqrt{a^2+b^2+k^2}}$$

The direction cosines of a normal to the plane (i) are

$$\frac{a}{\sqrt{a^2+b^2}}\ \frac{b}{\sqrt{a^2+b^2}} \text{ and } 0$$

Since the angle between the planes (i) and (iii) is α,

$$\cos\alpha=\frac{a\cdot a+b\cdot b+k\cdot 0}{\sqrt{a^2+b^2+k^2}\sqrt{a^2+b^2}}=\sqrt{\frac{a^2+b^2}{a^2+b^2+k^2}}$$

$$\Rightarrow k^2\cos^2\alpha=a^2(1-\cos^2\alpha)+b^2(1-\cos^2\alpha)$$

$$\Rightarrow k^2=\frac{(a^2+b^2)\sin^2\alpha}{\cos^2\alpha}\Rightarrow k=\pm\sqrt{a^2+b^2}\tan\alpha,$$

Putting this in (iii), we get the equation of the plane as $ax+by\pm z\sqrt{a^2+b^2}\tan\alpha=0$

DISTANCE OF A POINT FROM A PLANE

Vector Form

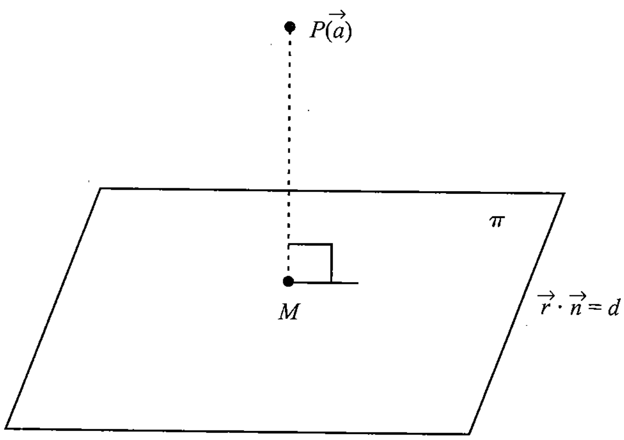


Fig. 3.25

Let $\pi\ (\vec{r}\cdot\vec{n}=d)$ be the given plane and $P\ (\vec{a})$ be the given point.

Let PM be the length of the perpendicular from P to the plane π.

Since line PM passes through $P(\vec{a})$ and is parallel to vector $\vec{n}$, which is normal to the plane π, the vector equation of line PM is: $\vec{r}=\vec{a}+\lambda\vec{n}$ (i)

Point M is the intersection of (i) and the given plane π. Therefore,

$$(\vec{a}+\lambda\vec{n})\cdot\vec{n}=d$$

$$\Rightarrow \vec{a}\cdot\vec{n}+\lambda\vec{n}\cdot\vec{n}=d$$

$$\Rightarrow \lambda=\frac{d-(\vec{a}\cdot\vec{n})}{|\vec{n}|^2}$$

Putting the value of λ in (i), we obtain the position vector of M given by $\vec{r}=\vec{a}+\left(\frac{d-\vec{a}\cdot\vec{n}}{|\vec{n}|^2}\right)\vec{n}$

$$\begin{aligned}\overrightarrow{PM} &= \text{P.V. of } M - \text{P.V. of } P\\ &= \vec{a}+\left(\frac{d-(\vec{a}\cdot\vec{n})}{|\vec{n}|^2}\right)\vec{n}-\vec{a}\\ &= \left(\frac{d-(\vec{a}\cdot\vec{n})}{|\vec{n}|^2}\right)\vec{n}\end{aligned}$$

$$\Rightarrow PM=|\overrightarrow{PM}|=\left|\frac{(d-\vec{a}\cdot\vec{n})\,\vec{n}}{|\vec{n}|^2}\right|=\frac{|d-(\vec{a}\cdot\vec{n})|\,|\vec{n}|}{|\vec{n}|^2}=\frac{|d-(\vec{a}\cdot\vec{n})|}{|\vec{n}|},$$ which is the required length.

Cartesian Form

Let PM be the length of the perpendicular from a point $P\ (x_1, y_1, z_1)$ to the plane $ax+by+cz+d=0$.

Then the equation of PM is $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=r$ (let) (i)

The coordinates of any point on this line are (x_1+ar, y_1+br, z_1+cr).

Thus the point coincides with M if and only if it lies on the plane.

i.e., $a(x_1+ar)+b(y_1+br)+c(z_1+cr)+d=0$

$$r=-\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2} \qquad \text{(ii)}$$

$$\begin{aligned}\text{Now, } PM &= \sqrt{(x_1+ar-x_1)^2+(y_1+br-y_1)^2+(z_1+cr-z_1)^2}\\ &= \sqrt{(a^2+b^2+c^2)r^2}\\ &= \sqrt{a^2+b^2+c^2}\ |r|\end{aligned}$$

$$= \sqrt{a^2+b^2+c^2}\left|\frac{-(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}\right| \qquad \text{from (ii)}$$

$$= \frac{|(ax_1+by_1+cz_1+d)|}{\sqrt{a^2+b^2+c^2}}$$

Also, if coordinates of M are (x_2, y_2, z_2), then

$$\frac{x_2-x_1}{a} = \frac{y_2-y_1}{b} = \frac{z_2-z_1}{c} = -\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2} \qquad \text{(iii)}$$

Image of a Point in a Plane

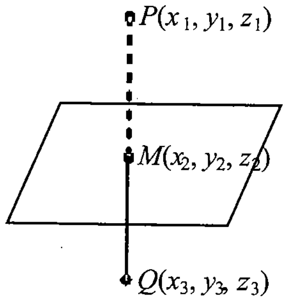


Fig. 3.26

Here Q is the image of P in the plane.

Therefore M is the midpoint of PQ.

Therefore from (iii)

$$\frac{\frac{x_3+x_1}{2}-x_1}{a} = \frac{\frac{y_3-y_1}{2}-y_1}{b} = \frac{\frac{z_3-z_1}{2}-z_1}{c}$$

$$= -\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$$

or

$$\frac{x_3-x_1}{a} = \frac{y_3-y_1}{b} = \frac{z_3-z_1}{c} = \frac{-2(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$$

DISTANCE BETWEEN PARALLEL PLANES

The distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is given by

$$d = \left|\frac{(d_2-d_1)}{\sqrt{a^2+b^2+c^2}}\right|$$

Proof:

Let $P(x_1, y_1, z_1)$ be point on plane $ax + by + cz + d_1 = 0$

then distance of this point from plane $ax + by + cz + d_2 = 0$ is

$$d = \frac{|ax_1 + by_1 + cz_1 + d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

also $ax_1 + by_1 + cz_1 + d_1 = 0$

$$\Rightarrow d = \frac{|d_2 - d_1|}{\sqrt{a^2 + b^2 + c^2}}$$

Example 3.55 **Find the length and the foot of the perpendicular from the point (7, 14, 5) to the plane $2x + 4y - z = 2$.**

Sol. The required length $= \dfrac{2(7) + 4(14) - (5) - 2}{\sqrt{2^2 + 4^2 + 1^2}} = \dfrac{14 + 56 - 5 - 2}{\sqrt{4 + 16 + 1}} = \dfrac{63}{\sqrt{21}}$

Let the coordinates of the foot of the perpendicular from the point P (7, 14, 5) be $M(\alpha, \beta, \gamma)$.
Then the direction ratios of PM are $\alpha - 7$, $\beta - 14$ and $\gamma - 5$.
Therefore, the direction ratios of the normal to the plane are $\alpha - 7$, $\beta - 14$ and $\gamma - 5$.

But the direction ratios of normal to the given plane $2x + 4y - z = 2$ are 2, 4 and -1.

Hence, $\dfrac{\alpha - 7}{2} = \dfrac{\beta - 14}{4} = \dfrac{\gamma - 5}{-1} = k$

$\therefore\ \alpha = 2k + 7, \beta = 4k + 14$ and $\gamma = -k + 5$. (i)

Since α, β and γ lie on the plane $2x + 4y - z = 2$, $2\alpha + 4\beta - \gamma = 2$

$\Rightarrow\ 2(7 + 2k) + 4(14 + 4k) - (5 - k) = 2$

$\Rightarrow 14 + 4k + 56 + 16k - 5 + k = 2$

$\Rightarrow 21k = -63$

$\Rightarrow k = -3$

Now, putting $k = -3$ in (i), we get

$\alpha = 1, \beta = 2, \gamma = 8$

Hence the foot of the perpendicular is (1, 2, 8)

Example 3.56 **Find the distance between the parallel planes $x + 2y - 2z + 1 = 0$ and $2x + 4y - 4z + 5 = 0$.**

Sol. We know that the distance between parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is

$$\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

Therefore, the distance between $x + 2y - 2z + 1 = 0$ and $x + 2y - 2z + \frac{5}{2} = 0$ is

$$\frac{|(5/2) - 1|}{\sqrt{1+4+4}} = \frac{1}{2}$$

Example 3.57 **Find the image of the line $\frac{x-1}{9} = \frac{y-2}{-1} = \frac{z+3}{-3}$ in the plane $3x - 3y + 10z - 26 = 0$.**

Sol.

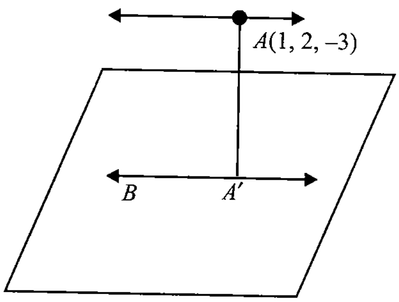


Fig. 3.27

$$\frac{x-1}{9} = \frac{y-2}{-1} = \frac{z+3}{-3} \qquad \text{(i)}$$

$$3x - 3y + 10z - 26 = 0 \qquad \text{(ii)}$$

The direction ratios of the line are 9, –1 and –3 and direction ratios of the normal to the given plane are 3, –3 and 10.

Since $9 \cdot 3 + (-1)(-3) + (-3)\,10 = 0$ and the point (1, 2, –3) of line (i) does not lie in plane (ii) for $3 \cdot 1 - 3 \cdot 2 + 10 \cdot (-3) - 26 \neq 0$, line (i) is parallel to plane (ii). Let A' be the image of point A (1, 2, –3) in plane (ii). Then the image of the line (i) in the plane (ii) is the line through A' and parallel to the line (i).

Let point A' be (p, q, r). Then

$$\frac{p-1}{3} = \frac{q-2}{-3} = \frac{r+3}{10} = -\frac{(3(1) - 3(2) + 10(-3) - 26)}{9+9+100} = \frac{1}{2}$$

The point is $A'(5/2, 1/2, 2)$

The equation of line BA' is $\frac{x-(5/2)}{9} = \frac{y-(1/2)}{-1} = \frac{z-2}{-3}$

EQUATION OF A PLANE BISECTING THE ANGLE BETWEEN TWO PLANES

Given planes are

$$a_1x + b_1y + c_1z + d_1 = 0 \qquad \text{(i)}$$

$$a_2x + b_2y + c_2z + d_2 = 0 \qquad \text{(ii)}$$

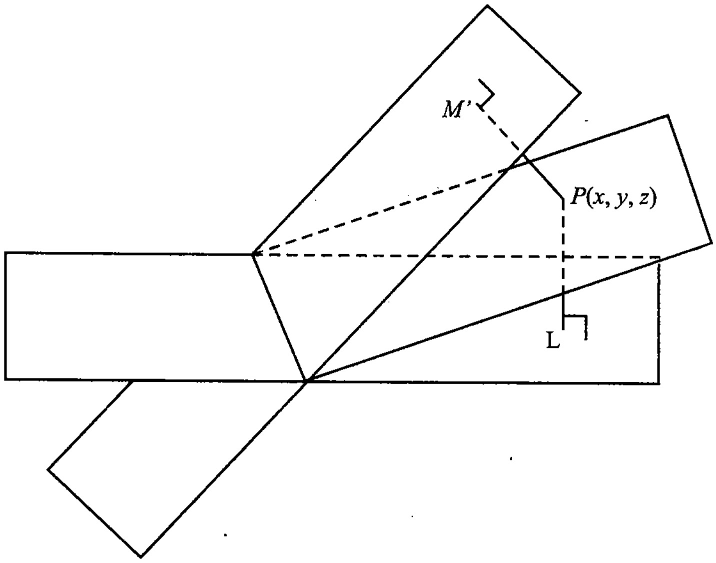


Fig. 3.28

Let $P(x, y, z)$ be a point on the plane bisecting the angle between (i) and (ii).

Let PL and PM be the length of the perpendiculars from P to planes (i) and (ii). Therefore,

$PL = PM$

$$\Rightarrow \left|\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}\right| = \left|\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

$$\Rightarrow \frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

This is the equation of the plane bisecting the angles between planes (i) and (ii).

Vector form

The equation of the plane bisecting the angle between planes $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$ is

$$\left|\frac{\vec{r}\cdot\vec{n}_1 - d_1}{\vec{n}_1}\right| = \left|\frac{\vec{r}\cdot\vec{n}_2 - d_2}{\vec{n}_2}\right|$$

Bisector of the Angle Between the Two Planes Containing the Origin

Let the equation of the two planes be

$a_1x + b_1y + c_1z + d_1 = 0$ and (i)

$a_2x + b_2y + c_2z + d_2 = 0$ (ii)

where d_1 and d_2 are positive.

The equation of the bisector of the angle between the planes (i) and (ii) containing the origin is

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

Bisector of the Acute and Obtuse Angles Between Two Planes

Let the two planes be

$$a_1x+b_1y+c_1z+d_1=0 \quad \text{(i)}$$

$$a_2x+b_2y+c_2z+d_2=0 \quad \text{(ii)}$$

where $d_1, d_2 > 0$

i. If $a_1a_2+b_1b_2+c_1c_2>0$, the origin lies in the obtuse angle between the two planes and the equation of the bisector of the obtuse angle is $\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=-\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$.

ii. If $a_1a_2+b_1b_2+c_1c_2<0$, the origin lies in the acute angle between the two planes and the equation of the bisector of the acute angle between the two planes is $\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=-\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$.

Example 3.58 **Find the equations of the bisectors of the angles between the planes $2x-y+2z+3=0$ and $3x-2y+6z+8=0$ and specify the plane which bisects the acute angle and the plane which bisects the obtuse angle.**

Sol. The given planes are $2x-y+2z+3=0$ and $3x-2y+6z+8=0$, where $d_1, d_2>0$ and $a_1a_2+b_1b_2+c_1c_2=6+2+12>0$.

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=-\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \quad \text{(obtuse angle bisector)}$$

$$\frac{a_1x+b_1y+c_1z+d}{\sqrt{a_2^2+b_2^2+c_2^2}}=\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \quad \text{(acute angle bisector)}$$

i.e., $\frac{2x-y+2z+3}{\sqrt{4+1+4}}=\pm\frac{3x-2y+6z+8}{\sqrt{9+4+36}}$

$\Rightarrow (14x-7y+14z+21)=\pm(9x-6y+18z+24)$

Taking the positive sign on the right hand side, we get

$5x-y-4z-3=0$ (obtuse angle bisector)

Taking the negative sign on the right hand side, we get

$23x-13y+32z+45=0$ (acute angle bisector)

TWO SIDES OF A PLANE

Let $ax + by + cz + d = 0$ be the plane. Then the points (x_1, y_1, z_1) and (x_2, y_2, z_2) lie on the same side or the opposite sides as $\dfrac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d} > 0$ or < 0, respectively.

Proof:

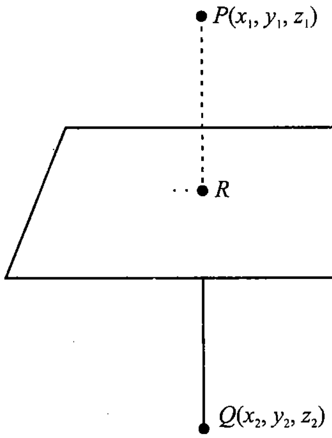


Fig. 3.29

Here the equation of the plane is $ax + by + cz + d = 0$. (i)

Let (i) divide the line segment joining P and Q at R internally in the ratio $m : n$.

Then $R\left(\dfrac{mx_2 + nx_1}{m+n}, \dfrac{my_2 + ny_1}{m+n}, \dfrac{mz_2 + nz_1}{m+n}\right)$

R lies on plane (i). Therefore,

$$a\left(\frac{mx_2 + nx_1}{m+n}\right) + b\left(\frac{my_2 + ny_1}{m+n}\right) + c\left(\frac{mz_2 + nz_1}{m+n}\right) + d = 0$$

$$a\,(mx_2 + nx_1) + b\,(my_2 + ny_1) + c\,(mz_2 + nz_1) + d\,(m + n) = 0$$

$$m\,(ax_2 + by_2 + cz_2 + d) + n\,(ax_1 + by_1 + cz_1 + d) = 0$$

$$\frac{m}{n} = -\frac{(ax_1 + by_1 + cz_1 + d)}{(ax_2 + by_2 + cz_2 + d)} \quad \text{(ii)}$$

Now, if $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$

are of same sign $\dfrac{m}{n} < 0$ (external division)

are of opposite signs $\dfrac{m}{n} > 0$ (internal division)

If $\dfrac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d} > 0$ (same side)

$\dfrac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d} < 0$ (opposite side)

Concept Application Exercise 3.3

1. Find the angle between the line $\dfrac{x+1}{3} = \dfrac{y-1}{2} = \dfrac{z-1}{4}$ and the plane $2x + y - 3z + 4 = 0$.
2. Find the distance between the line $\dfrac{x+1}{-3} = \dfrac{y-3}{2} = \dfrac{z-2}{1}$ and the plane $x + y + z + 3 = 0$.
3. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\dfrac{x-2}{3} = \dfrac{y+1}{4} = \dfrac{z-2}{12}$ and plane $x - y + z = 5$.
4. Find the equation of a plane which passes through the point (1, 2, 0) and which is perpendicular to the planes $x - y + z - 3 = 0$ and $2x + y - z + 4 = 0$.
5. Find the equation of the plane passing through the points $(1, 0, -1)$ and (3, 2, 2) and parallel to the line $x - 1 = \dfrac{1-y}{2} = \dfrac{z-2}{3}$.
6. Find the equation of a plane containing the lines $\dfrac{x-5}{4} = \dfrac{y-7}{4} = \dfrac{z+3}{-5}$ and $\dfrac{x-8}{7} = \dfrac{y-4}{1} = \dfrac{z-5}{3}$.
7. Find the equation of the plane passing through the straight line $\dfrac{x-1}{2} = \dfrac{y+2}{-3} = \dfrac{z}{5}$ and perpendicular to the plane $x - y + z + 2 = 0$.
8. Find the equation of the plane perpendicular to the line $\dfrac{x-1}{2} = \dfrac{y-3}{-1} = \dfrac{z-4}{2}$ and passing through the origin.
9. Find the equation of the plane passing through the line $\dfrac{x-1}{5} = \dfrac{y+2}{6} = \dfrac{z-3}{4}$ and point (4, 3, 7).
10. Find the angle between the line $\vec{r} = (\vec{i} + 2\vec{j} - \vec{k}) + \lambda(\vec{i} - \vec{j} + \vec{k})$ and the normal to the plane $\vec{r} \cdot (2\vec{i} - \vec{j} + \vec{k}) = 4$.
11. Find the equation of a plane which passes through the point (1, 2, 3) and which is at the maximum distance from the point $(-1, 0, 2)$.
12. Find the direction ratios of orthogonal projection of line $\dfrac{x-1}{1} = \dfrac{y+1}{-2} = \dfrac{z-2}{3}$ in the plane $x - y + 2z - 3 = 0$. Also find the direction ratios of the image of the line in the plane.
13. Find the equation of a plane which is parallel to the plane $x - 2y + 2z = 5$ and whose distance from the point (1, 2, 3) is 1.
14. Find the equation of a plane which passes through the point (1, 2, 3) and which is equally inclined to the planes $x - 2y + 2z - 3 = 0$ and $8x - 4y + z - 7 = 0$.
15. Find the equation of the image of the plane $x - 2y + 2z - 3 = 0$ in the plane $x + y + z - 1 = 0$.

SPHERES

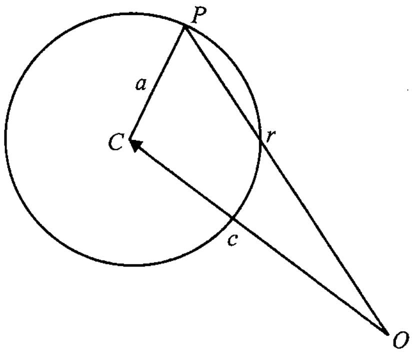


Fig. 3.30

A sphere is the locus of a point which moves in space in such a way that its distance from a fixed point always remains constant. The fixed point is called the centre and the constant distance is called the radius of the sphere.

Equation of a Sphere

Let $\vec{c}$ be the position vector of the centre C of the sphere and a be the radius of the sphere.

Let $\vec{r}$ be the position vector of any point P on the sphere.

Then $|\overrightarrow{CP}| = a$

But $\overrightarrow{CP} = \overrightarrow{OP} - \overrightarrow{OC} = \vec{r} - \vec{c}$

Thus, $|\vec{r} - \vec{c}| = a$

$\Rightarrow |\vec{r} - \vec{c}|^2 = a^2$

$\Rightarrow (\vec{r} - \vec{c})\cdot(\vec{r} - \vec{c}) = a^2$

Cartesian form

If $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then $\vec{r} - \vec{c} = (x - c_1)\hat{i} + (y - c_2)\hat{j} + (z - c_3)\hat{k}$

Now, $|\vec{r} - \vec{c}| = \sqrt{(x - c_1)^2 + (y - c_2)^2 + (z - c_3)^2}$

Therefore, the equation is $(x - c_1)^2 + (y - c_2)^2 + (z - c_3)^2 = a^2$

$\Rightarrow \; x^2 + y^2 + z^2 - 2c_1x - 2c_2y - 2c_3z + c_1^2 + c_2^2 + c_3^2 - a^2 = 0$

We usually write this equation as $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ (i)

Adding $u^2 + v^2 + w^2$ on both the sides of (i), we can write $(x + u)^2 + (y + v)^2 + (w + z)^2 = u^2 + v^2 + w^2 - d$.

This equation represents a sphere with centre at $(-u, -v, -w)$ and radius $\sqrt{u^2 + v^2 + w^2 - d}$. Note that we must have $u^2 + v^2 + w^2 - d \geq 0$.

Thus, (i) represents a sphere with centre at $(-u, -v, -w)$ and radius equal to $\sqrt{u^2 + v^2 + w^2 - d}$.

In particular, the equation of a sphere with centre at the origin is $|\vec{r}| = a$ or $x^2 + y^2 + z^2 = a^2$.

For a fixed sphere in space, we require four non-coplanar points which form a tetrahedron, or we can say that every tetrahedron has a unique circumscribed sphere.

Example 3.59 **Find the equation of a sphere whose centre is (3, 1, 2) and radius is 5.**

Sol. The equation of the sphere whose centre is (3, 1, 2) and radius is 5 is

$(x-3)^2 + (y-1)^2 + (z-2)^2 = 5^2$

$x^2 + y^2 + z^2 - 6x - 2y - 4z - 11 = 0$

Example 3.60 **Find the equation of the sphere passing through (0, 0, 0), (1, 0, 0), (0, 1, 0) and (0, 0, 1).**

Sol. Let the equation of the sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ (i)

As (i) passes through (0, 0, 0), (1, 0, 0), (0, 1, 0) and (0, 0, 1), we must have $d = 0$, $1 + 2u + d = 0$ $1 + 2v + d = 0$ and $1 + 2w + d = 0$

Since $d = 0$, we get $2u = 2v = 2w = -1$

Thus, the equation of the required sphere is $x^2 + y^2 + z^2 - x - y - z = 0$

Example 3.61 **Find the equation of the sphere which passes through (1, 0, 0), (0,1, 0) and (0, 0, 1), and whose centre lies on the plane 3x − y + z = 2.**

Sol. Let the equation of the required sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

As the sphere passes through (1, 0, 0), (0, 1, 0) and (0, 0, 1), we get

$1 + 2u + d = 0$, $1 + 2v + d = 0$ and $1 + 2w + d = 0$

$\Rightarrow u = v = w = -\dfrac{d+1}{2}$

Since the centre $(-u, -v, -w)$ lies on the plane $3x - y + z = 2$, we get $-3u + v - w = 2$

$\Rightarrow \dfrac{3}{2}(d+1) = 2$ or $d + 1 = \dfrac{4}{3}$ or $d = \dfrac{1}{3}$

Thus, $u = v = w = -2/3$

Therefore, the equation of the required sphere is $x^2 + y^2 + z^2 - \left(\dfrac{2}{3}\right)x - \left(\dfrac{2}{3}\right)y - \left(\dfrac{2}{3}\right)z + \dfrac{1}{3} = 0$

or $3(x^2 + y^2 + z^2) - 2(x + y + z) + 1 = 0$

Example 3.62 **Find the equation of a sphere which passes through (1, 0, 0) (0, 1, 0) and (0, 0, 1), and has radius as small as possible.**

Sol. Let the equation of the required sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ (i)

As the sphere passes through (1, 0, 0), (0, 1, 0) and (0, 0, 1), we get

$1 + 2u + d = 0$, $1 + 2v + d = 0$ and $1 + 2w + d = 0$

$\Rightarrow u = v = w = -\frac{1}{2}(d+1)$

If R is the radius of the sphere, then $R^2 = u^2 + v^2 + w^2 - d$

$$\Rightarrow R^2 = \frac{3}{4}(d+1)^2 - d$$

$$= \frac{3}{4}\left[d^2 + 2d + 1 - \frac{4}{3}d\right]$$

$$= \frac{3}{4}\left[d^2 + \frac{2}{3}d + 1\right]$$

$$= \frac{3}{4}\left[\left(d + \frac{1}{3}\right)^2 + 1 - \frac{1}{9}\right]$$

$$= \frac{3}{4}\left[\left(d + \frac{1}{3}\right)^2 + \frac{8}{9}\right]$$

The last equation shows that R^2 (and thus R) will be the least if and only if $d = -1/3$.

Therefore, $u = v = w = -\frac{1}{2}\left(1 - \frac{1}{3}\right) = -\frac{1}{3}$

Hence, the equation of the required sphere is $x^2 + y^2 + z^2 - \frac{2}{3}(x + y + z) - \frac{1}{3} = 0$

or $3(x^2 + y^2 + z^2) - 2(x + y + z) - 1 = 0$

Example 3.63 **Find the locus of a point which moves such that the sum of the squares of its distance from the points A (1, 2, 3), B (2, −3, 5) and C (0, 7, 4) is 120.**

Sol. Let $P(x, y, z)$ be any point on the locus. Then $PA^2 + PB^2 + PC^2 = 120$

$\Rightarrow (x-1)^2 + (y-2)^2 + (z-3)^2 + (x-2)^2 + (y+3)^2 + (z-5)^2 + (x-0)^2 + (y-7)^2 + (z-4)^2 = 120$

$3x^2 + 3y^2 + 3z^2 - 6x - 12y - 24z + 117 = 120$

$x^2 + y^2 + z^2 - 2x - 4y - 8z - 1 = 0$

This represents a sphere with centre at (1, 2, 4) and radius equal to $\sqrt{1^2 + 2^2 + 4^2 + 1} = \sqrt{22}$

Diameter Form of the Equation of a Sphere

Let AB be the diameter of a sphere whose centre is C. Let the vectors of the extremeties A and B of the diameter be $\vec{a}$ and $\vec{b}$, respectively. Let P be any point on this sphere. Suppose the position vector of P is $\vec{r}$. We know that the angle in a hemisphere is a right angle.

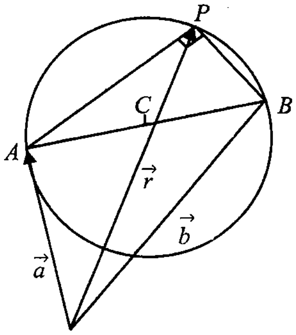


Fig. 3.31

Thus, $\angle APB = \pi/2$

$\overrightarrow{AP} \cdot \overrightarrow{BP} = 0$ (i)

But $\overrightarrow{AP} = \vec{r} - \vec{a}$ and $\overrightarrow{BP} = \vec{r} - \vec{b}$

Thus, (i) can be written as $\left(\vec{r} - \vec{a}\right) \cdot \left(\vec{r} - \vec{b}\right) = 0$

This is the required equation of the sphere.

Cartesian form

$\vec{a} = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}, \vec{b} = x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Then $\vec{r} - \vec{a} = (x - x_1)\,\hat{i} + (y - y_1)\,\hat{j} + (z - z_1)\,\hat{k}$

$\vec{r} - \vec{b} = (x - x_2)\,\hat{i} + (y - y_2)\,\hat{j} + (z - z_2)\,\hat{k}$

Thus, $(\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{b}) = 0$ gives

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$

Example 3.64 **Find the equation of the sphere described on the joint of points *A* and *B* having position vectors $2\hat{i} + 6\hat{j} - 7\hat{k}$ and $-2\hat{i} + 4\hat{j} - 3\hat{k}$, respectively, as the diameter. Find the centre and the radius of the sphere.**

Sol. If point *P* with positive vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is any point on the sphere, then

$\overrightarrow{AP} \cdot \overrightarrow{BP} = 0$

$(x - 2)(x + 2) + (y - 6)(y - 4) + (z + 7)(z + 3) = 0$

$\Rightarrow (x^2 - 4) + (y^2 - 10y + 24) + (z^2 + 10z + 21) = 0$

$\Rightarrow x^2 + y^2 + z^2 - 10y + 10z + 41 = 0$

The centre of this sphere is (0, 5, −5) and its radius is $\sqrt{5^2 + (-5)^2 - 41} = \sqrt{9} = 3$

Example 3.65 **Find the radius of the circular section in which the sphere $|\vec{r}| = 5$ is cut by the plane $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k}) = 3\sqrt{3}$.**

Sol. Let A be the foot of the perpendicular from the centre O to the plane

$\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k}) - 3\sqrt{3} = 0$

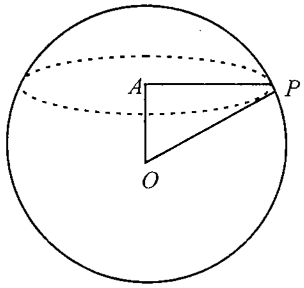


Fig. 3.32

Then $|OA| = \left|\dfrac{0\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)-3\sqrt{3}}{|\hat{i}+\hat{j}+\hat{k}|}\right| = \dfrac{3\sqrt{3}}{\sqrt{3}} = 3$ (Perpendicular distance of a point from the plane)

If P is any point on the circle, then P lies on the plane as well as on the sphere. Therefore,

OP = radius of the sphere = 5

Now $AP^2 = OP^2 - OA^2 = 5^2 - 3^2 = 16 \Rightarrow AP = 4$

Example 3.66 **Show that the plane $2x - 2y + z + 12 = 0$ touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$.**

Sol. The given plane will touch the given sphere if the perpendicular distance from the centre of the sphere to the plane is equal to the radius of the sphere. The centre of the given sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ is $(1, 2, -1)$ and its radius is $\sqrt{1^2 + 2^2 + (-1)^2 - (-3)} = 3$.

Length of the perpendicular from $(1, 2, -1)$ to the plane $2x - 2y + z + 12 = 0$ is

$$\left|\frac{2(1) - 2(2) + (-1) + 12}{\sqrt{2^2 + (-2)^2 + 1^2}}\right| = \frac{9}{3} = 3$$

Thus, the given plane touches the given sphere.

Example 3.67 **A variable plane passes through a fixed point (a, b, c) and cuts the coordinate axes at points A, B and C. Show that the locus of the centre of the sphere $OABC$ is $\dfrac{a}{x} + \dfrac{b}{y} + \dfrac{c}{z} = 2$.**

Sol. Let (α, β, γ) be any point on the locus. Then according to the given condition, (α, β, γ) is the centre of the sphere through the origin. Therefore, its equation is given by

$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = (0-\alpha)^2 + (0-\beta)^2 + (0-\gamma)^2$

$x^2 + y^2 + z^2 - 2\alpha x - 2\beta y - 2\gamma z = 0$

To obtain its point of intersection with the x-axis, we put $y = 0$ and $z = 0$, so that

$x^2 - 2\alpha x = 0 \Rightarrow x(x - 2\alpha) = 0 \Rightarrow x = 0$ or $x = 2\alpha$

Thus the plane meets x-axis at O (0, 0, 0) and A $(2\alpha, 0, 0)$. Similarly, it meets y-axis at O (0, 0, 0) and B $(0, 2\beta, 0)$, and z-axis at O (0, 0, 0) and C $(0, 0, 2\gamma)$.

The equation of the plane through A, B and C is

$$\frac{x}{2\alpha} + \frac{y}{2\beta} + \frac{z}{2\gamma} = 1 \quad \text{(intercept form)}$$

Since it passes through (a, b, c), we get

$$\frac{a}{2\alpha} + \frac{b}{2\beta} + \frac{c}{2\gamma} = 1 \text{ or } \frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 2$$

Hence, locus of (α, β, γ) is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$

Example 3.68 **A sphere of constant radius k passes through the origin and meets the axes at A, B and C. Prove that the centroid of triangle ABC lies on the sphere $9(x^2 + y^2 + z^2) = 4k^2$.**

Sol. Let the equation of any sphere passing through the origin and having radius k be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0 \quad \text{(i)}$$

As the radius of the sphere is k, we get

$$u^2 + v^2 + w^2 = k^2 \quad \text{(ii)}$$

Note that (i) meets the x-axis at O (0, 0, 0) and A $(-2u, 0, 0)$; y-axis at O (0, 0, 0) and B $(0, -2v, 0)$; and z-axis at O (0, 0, 0) and C $(0, 0, -2w)$.

Let the centroid of the triangle ABC be (α, β, γ). Then

$$\alpha = -\frac{2u}{3}, \beta = -\frac{2v}{3}, \gamma = -\frac{2w}{3} \quad \Rightarrow \quad u = -\frac{3\alpha}{2}, v = -\frac{3\beta}{2}, w = -\frac{3v}{2}$$

Putting this in (ii), we get

$$\left(\frac{-3}{2}\alpha\right)^2 + \left(\frac{-3}{2}\beta\right)^2 + \left(\frac{-3}{2}\gamma\right)^2 = k^2$$

$$\Rightarrow \quad \alpha^2 + \beta^2 + \gamma^2 = \frac{4}{9}k^2$$

This shows that the centroid of triangle ABC lies on $x^2 + y^2 + z^2 = \frac{4}{9}k^2$

Concept Application Exercise 3.4

1. Find the plane of the intersection of $x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0$ and $4x^2 + 4y^2 + 4z^2 + 4x + 4y + 4z - 1 = 0$.
2. Find the radius of the circular section of the sphere $|\vec{r}| = 5$ by the plane $\vec{r}\cdot(\vec{i} + 2\vec{j} - \vec{k}) = 4\sqrt{3}$.
3. A point $P(x, y, z)$ is such that $3PA = 2PB$, where A and B are the points $(1, 3, 4)$ and $(1, -2, -1)$, respectively. Find the equation to the locus of the point P and verify that the locus is a sphere.
4. The extremities of a diameter of a sphere lie on the positive y- and positive z-axes at distances 2 and 4, respectively. Show that the sphere passes through the origin and find the radius of the sphere.
5. A plane passes through a fixed point (a, b, c). Show that the locus of the foot of the perpendicular to it from the origin is the sphere $x^2 + y^2 + z^2 - ax - by - cz = 0$.

Exercises

Subjective Type

Solutions on page 3.79

1. If variable lines in two adjacent positions have direction cosines l, m and n and $(l + \delta l)$, $(m + \delta m)$, $(n + \delta n)$, show that the small angle $\delta\theta$ between the two positions is given by $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$.
2. Find the equation of the plane containing the line $\frac{y}{b} + \frac{z}{c} = 1, x = 0$, and parallel to the line $\frac{x}{a} - \frac{z}{c} = 1$, $y = 0$.
3. A variable plane passes through a fixed point (α, β, γ) and meets the axes at A, B and C. Show that the locus of the point of intersection of the planes through A, B and C parallel to the coordinate planes is $\alpha x^{-1} + \beta y^{-1} + \gamma z^{-1} = 1$.
4. Show that the straight lines whose direction cosines are given by the equations $al + bm + cn = 0$ and $ul^2 + vm^2 + wn^2 = 0$ are parallel or perpendicular as
$\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$ or $a^2(v + w) + b^2(w + u) + c^2(u + v) = 0$.
5. Find the perpendicular distance of a corner of a cube of unit side length from a diagonal not passing through it.
6. A point P moves on a plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. A plane through P and perpendicular to OP meets the coordinate axes at A, B and C. If the planes through A, B and C parallel to the planes $x = 0$, $y = 0$ and $z = 0$, respectively, intersect at Q, find the locus of Q.
7. If the planes $x - cy - bz = 0$, $cx - y + az = 0$ and $bx + ay - z = 0$ pass through a straight line, then find the value of $a^2 + b^2 + c^2 + 2abc$.
8. P is a point and PM and PN are the perpendiculars from P to z-x and x-y planes. If OP makes angles θ, α, β and γ with the plane OMN and the x-y, y-z and z-x planes, respectively, then prove that $\operatorname{cosec}^2\theta = \operatorname{cosec}^2\alpha + \operatorname{cosec}^2\beta + \operatorname{cosec}^2\gamma$.

9. A variable plane $lx + my + nz = p$ (where l, m, n are direction cosines of normal) intersects the coordinate axes at points A, B and C, respectively. Show that the foot of the normal on the plane from the origin is the orthocentre of triangle ABC and hence find the coordinates of the circumcentre of triangle ABC.

10. Let $x - y\sin\alpha - z\sin\beta = 0$, $x\sin\alpha + z\sin\gamma - y = 0$ and $x\sin\beta + y\sin\gamma - z = 0$ be the equations of the planes such that $\alpha + \beta + \gamma = \pi/2$ (where α, β and $\gamma \neq 0$). Then show that there is a common line of intersection of the three given planes.

11. Let a plane $ax + by + cz + 1 = 0$, where a, b and c are parameters, make an angle 60° with the line $x = y = z$, 45° with the line $x = y - z = 0$ and θ with the plane $x = 0$. The distance of the plane from point (2, 1, 1) is 3 units. Find the value of θ and the equation of the plane.

12. Prove that for all values of λ and μ, the planes $\frac{2x}{a}+\frac{y}{b}+\frac{2z}{c}-1+\lambda\left(\frac{x}{a}-\frac{2y}{b}-\frac{z}{c}-2\right)=0$ and $\frac{4x}{a}-\frac{3y}{b}-5+\mu\left(\frac{5y}{b}+\frac{4z}{c}+3\right)=0$ intersect on the same line.

13. OA, OB and OC, with O as the origin, are three mutually perpendicular lines whose direction cosines and l_r, m_r and n_r (r = 1, 2 and 3). If the projections of OA and OB on the plane $z = 0$ make angles ϕ_1 and ϕ_2, respectively, with the x-axis, prove that $\tan(\phi_1 - \phi_2) = \pm n_3/n_1n_2$.

14. O is the origin and lines OA, OB and OC have direction cosines l_r, m_r and n_r (r = 1, 2 and 3). If lines OA', OB' and OC' bisect angles BOC, COA and AOB, respectively, prove that planes AOA', BOB' and COC' pass through the line $\frac{x}{l_1+l_2+l_3}=\frac{y}{m_1+m_2+m_3}=\frac{z}{n_1+n_2+n_3}$.

15. If P be any point on the plane $lx + my + nz = p$ and Q be a point on the line OP such that $OP \cdot OQ = p^2$, then find the locus of the point Q.

16. If a variable plane forms a tetrahedron of constant volume $64k^3$ with the coordinate planes, find the locus of the centroid of the tetrahedron.

Objective Type

Solutions on page 3.89

Each question has four choices *a*, *b*, *c* and *d*, out of which *only one* answer is correct. Find the correct answer.

1. In a three-dimensional xyz space, the equation $x^2 - 5x + 6 = 0$ represents
   a. points **b.** planes
   c. curves **d.** pair of straight lines

2. The line $\frac{x-2}{3}=\frac{y+1}{2}=\frac{z-1}{-1}$ intersects the curve $xy = c^2$, $z = 0$ if c is equal to
   a. ± 1 **b.** $\pm 1/3$ **c.** $\pm\sqrt{5}$ **d.** none of these

3. Let the equations of a line and a plane be $\frac{x+3}{2}=\frac{y-4}{3}=\frac{z+5}{2}$ and $4x-2y-z=1$, respectively, then

a. the line is parallel to the plane
b. the line is perpendicular to the plane
c. the line lies in the plane
d. none of these

4. The length of the perpendicular from the origin to the plane passing through the point $\vec{a}$ and containing the line $\vec{r}=\vec{b}+\lambda\vec{c}$ is

a. $\frac{[\vec{a}\ \vec{b}\ \vec{c}]}{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}$
b. $\frac{[\vec{a}\ \vec{b}\ \vec{c}]}{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}|}$
c. $\frac{[\vec{a}\ \vec{b}\ \vec{c}]}{|\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}$
d. $\frac{[\vec{a}\ \vec{b}\ \vec{c}]}{|\vec{c}\times\vec{a}+\vec{a}\times\vec{b}|}$

5. The distance of point A (–2, 3, 1) from the line PQ through P (–3, 5, 2), which makes equal angles with the axes is

a. $2/\sqrt{3}$ **b.** $\sqrt{14/3}$ **c.** $16/\sqrt{3}$ **d.** $5/\sqrt{3}$

6. The Cartesian equation of the plane $\vec{r}=(1+\lambda-\mu)\hat{i}+(2-\lambda)\hat{j}+(3-2\lambda+2\mu)\hat{k}$ is

a. $2x+y=5$ **b.** $2x-y=5$ **c.** $2x+z=5$ **d.** $2x-z=5$

7. A unit vector parallel to the intersection of the planes $\vec{r}\cdot\left(\hat{i}-\hat{j}+\hat{k}\right)=5$ and $\vec{r}\cdot\left(2\hat{i}+\hat{j}-3\hat{k}\right)=4$ is

a. $\frac{2\hat{i}+5\hat{j}-3\hat{k}}{\sqrt{38}}$ **b.** $\frac{2\hat{i}-5\hat{j}+3\hat{k}}{\sqrt{38}}$ **c.** $\frac{-2\hat{i}-5\hat{j}-3\hat{k}}{\sqrt{38}}$ **d.** $\frac{-2\hat{i}+5\hat{j}-3\hat{k}}{\sqrt{38}}$

8. Let L_1 be the line $\vec{r}_1=2\hat{i}+\hat{j}-\hat{k}+\lambda(\hat{i}+2\hat{k})$ and let L_2 be the line $\vec{r}_2=3\hat{i}+\hat{j}+\mu(\hat{i}+\hat{j}-\hat{k})$. Let π be the plane which contains the line L_1 and is parallel to L_2. The distance of the plane π from the origin is

a. $\sqrt{2/7}$ **b.** 1/7 **c.** $\sqrt{6}$ **d.** none

9. For the line $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$, which one of the following is incorrect?

a. it lies in the plane $x-2y+z=0$
b. it is same as line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$
c. it passes through (2, 3, 5)
d. it is parallel to the plane $x-2y+z-6=0$

10. The value of m for which straight line $3x-2y+z+3=0=4x-3y+4z+1$ is parallel to the plane $2x-y+mz-2=0$ is

a. –2 **b.** 8 **c.** – 18 **d.** 11

11. The intercept made by the plane $\vec{r}\cdot\vec{n}=q$ on the x-axis is

a. $\frac{q}{\hat{i}\cdot\vec{n}}$ **b.** $\frac{\hat{i}\cdot\vec{n}}{q}$ **c.** $\frac{\hat{i}\cdot\vec{n}}{q}$ **d.** $\frac{q}{|\vec{n}|}$

12. Equation of a line in the plane $\pi \equiv 2x - y + z - 4 = 0$ which is perpendicular to the line l whose equation is $\frac{x-2}{1} = \frac{y-2}{-1} = \frac{z-3}{-2}$ and which passes through the point of intersection of l and π is

a. $\frac{x-2}{1} = \frac{y-1}{5} = \frac{z-1}{-1}$

b. $\frac{x-1}{3} = \frac{y-3}{5} = \frac{z-5}{-1}$

c. $\frac{x+2}{2} = \frac{y+1}{-1} = \frac{z+1}{1}$

d. $\frac{x-2}{2} = \frac{y-1}{-1} = \frac{z-1}{1}$

13. If the foot of the perpendicular from the origin to a plane is $P(a, b, c)$, the equation of the plane is

a. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$

b. $ax + by + cz = 3$

c. $ax + by + cz = a^2 + b^2 + c^2$

d. $ax + by + cz = a + b + c$

14. The equation of a plane which passes through the point of intersection of lines $\frac{x-1}{3} = \frac{y-2}{1} = \frac{z-3}{2}$, and $\frac{x-3}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ and at greatest distance from point (0, 0, 0) is

a. $4x + 3y + 5z = 25$ b. $4x + 3y = 5z = 50$ c. $3x + 4y + 5z = 49$ d. $x + 7y - 5z = 2$

15. Let $A(\vec{a})$ and $B(\vec{b})$ be points on two skew lines $\vec{r} = \vec{a} + \lambda \vec{p}$ and $\vec{r} = \vec{b} + u\vec{q}$ and the shortest distance between the skew lines is 1, where $\vec{p}$ and $\vec{q}$ are unit vectors forming adjacent sides of a parallelogram enclosing an area of $\frac{1}{2}$ units. If an angle between AB and the line of shortest distance is 60°, then $AB =$

a. $\frac{1}{2}$ b. 2 c. 1 d. $\lambda \in R - \{0\}$

16. Let $A(1, 1, 1)$, $B(2, 3, 5)$ and $C(-1, 0, 2)$ be three points, then equation of a plane parallel to the plane ABC which is at distance 2 is

a. $2x - 3y + z + 2\sqrt{14} = 0$

b. $2x - 3y + z - \sqrt{14} = 0$

c. $2x - 3y + z + 2 = 0$

d. $2x - 3y + z - 2 = 0$

17. The point on the line $\frac{x-2}{1} = \frac{y+3}{-2} = \frac{z+5}{-2}$ at a distance of 6 from the point $(2, -3, -5)$ is

a. $(3, -5, -3)$ b. $(4, -7, -9)$ c. $(0, 2, -1)$ d. $(-3, 5, 3)$

18. The coordinates of the foot of the perpendicular drawn from the origin to the line joining the points $(-9, 4, 5)$ and $(10, 0, -1)$ will be

a. $(-3, 2, 1)$ b. $(1, 2, 2)$ c. $(4, 5, 3)$ d. none of these

19. If $P_1: \vec{r} \cdot \vec{n_1} - d_1 = 0$, $P_2: \vec{r} \cdot \vec{n_2} - d_2 = 0$ and $P_3: \vec{r} \cdot \vec{n_3} - d_3 = 0$ are three planes and $\vec{n_1}, \vec{n_2}$ and $\vec{n_3}$ are three non-coplanar vectors, then three lines $P_1 = 0, P_2 = 0$; $P_2 = 0, P_3 = 0$ and $P_3 = 0, P_1 = 0$ are

a. parallel lines b. coplanar lines c. coincident lines d. concurrent lines

20. The length of projection of the line segment joining the points (1, 0, – 1) and (–1, 2, 2) on the plane $x + 3y - 5z = 6$, is equal to

a. 2 b. $\sqrt{\frac{271}{53}}$ c. $\sqrt{\frac{472}{31}}$ d. $\sqrt{\frac{474}{35}}$

21. The number of planes that are equidistant from four non-coplanar points is

a. 3 b. 4 c. 7 d. 9

22. In a three dimensional co-ordinate system, P, Q and R are images of a point A (a, b, c) in the x-y, y-z and z-x planes, respectively. If G is the centroid of triangle PQR, then area of triangle AOG is (O is the origin)

a. 0 b. $a^2 + b^2 + c^2$ c. $\frac{2}{3}(a^2 + b^2 + c^2)$ d. none of these

23. A plane passing through (1, 1, 1) cuts positive direction of co-ordinate axes at A, B and C, then the volume of tetrahedron $OABC$ satisfies

a. $V \le \frac{9}{2}$ b. $V \ge \frac{9}{2}$ c. $V = \frac{9}{2}$ d. none of these

24. If lines $x = y = z$ and $x = \frac{y}{2} = \frac{z}{3}$, and third line passing through (1, 1, 1) form a triangle of area $\sqrt{6}$ units, then point of intersection of third line with second line will be

a. $(1, 2, 3)$ b. $(2, 4, 6)$ c. $\left(\frac{4}{3}, \frac{8}{3}, \frac{12}{3}\right)$ d. none of these

25. The point of intersection of the line passing through (0, 0, 1) and intersecting the lines $x + 2y + z = 1$, $-x + y - 2z = 2$ and $x + y = 2$, $x + z = 2$ with xy plane is

a. $\left(\frac{5}{3}, -\frac{1}{3}, 0\right)$ b. $(1, 1, 0)$ c. $\left(\frac{2}{3}, -\frac{1}{3}, 0\right)$ d. $\left(-\frac{5}{3}, \frac{1}{3}, 0\right)$

26. Shortest distance between the lines $\frac{x-1}{1} = \frac{y-1}{1} = \frac{z-1}{1}$ and $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{1}$ is equal to

a. $\sqrt{14}$ b. $\sqrt{7}$ c. $\sqrt{2}$ d. none of these

27. Distance of point P $(\vec{p})$ from the plane $\vec{r} \cdot \vec{n} = 0$ is

a. $|\vec{p} \cdot \vec{n}|$ b. $\frac{|\vec{p} \times \vec{n}|}{|\vec{n}|}$ c. $\frac{|\vec{p} \cdot \vec{n}|}{|\vec{n}|}$ d. none of these

28. The reflection of the point $\vec{a}$ in the plane $\vec{r} \cdot \vec{n} = q$ is

a. $\vec{a} + \frac{(\vec{q} - \vec{a} \cdot \vec{n})}{|\vec{n}|}$

b. $\vec{a} + 2\left(\frac{(\vec{q} - \vec{a} \cdot \vec{n})}{|\vec{n}|^2}\right)\vec{n}$

c. $\vec{a} + \frac{2(\vec{q} + \vec{a} \cdot \vec{n})}{|\vec{n}|}\vec{n}$

d. none of these

29. Line $\vec{r} = \vec{a} + \lambda\vec{b}$ will not meet the plane $\vec{r} \cdot \vec{n} = q$, if

a. $\vec{b} \cdot \vec{n} = 0, \vec{a} \cdot \vec{n} = q$

b. $\vec{b} \cdot \vec{n} \ne 0, \vec{a} \cdot \vec{n} \ne q$

c. $\vec{b} \cdot \vec{n} = 0, \vec{a} \cdot \vec{n} \ne q$

d. $\vec{b} \cdot \vec{n} \ne 0, \vec{a} \cdot \vec{n} = q$

30. If a line makes an angle of $\frac{\pi}{4}$ with the positive direction of each of x-axis and y-axis, then the angle that the line makes with the positive direction of the z-axis is

a. $\frac{\pi}{3}$ b. $\frac{\pi}{4}$ c. $\frac{\pi}{2}$ d. $\frac{\pi}{6}$

31. The ratio in which the plane $\vec{r}\cdot(\vec{i}-2\vec{j}+3\vec{k})=17$ divides the line joining the points $-2\vec{i}+4\vec{j}+7\vec{k}$ and $3\vec{i}-5\vec{j}+8\vec{k}$ is

a. 1 : 5 b. 1 : 10 c. 3 : 5 d. 3 :10

32. The image of the point $(-1, 3, 4)$ in the plane $x-2y=0$ is

a. $\left(-\frac{17}{3},-\frac{19}{3},4\right)$ b. $(15,11,4)$ c. $\left(-\frac{17}{3},-\frac{19}{3},1\right)$ d. $\left(\frac{9}{5},-\frac{13}{5},4\right)$

33. The distance between the line: $\vec{r}=2\hat{i}-2\hat{j}+3\hat{k}+\lambda(\hat{i}-\hat{j}+4\hat{k})$ and the plane $\vec{r}\cdot(\hat{i}+5\hat{j}+\hat{k})=5$ is

a. $\frac{10}{3\sqrt{3}}$ b. $\frac{10}{9}$ c. $\frac{10}{3}$ d. $\frac{3}{10}$

34. Let L be the line of intersection of the planes $2x+3y+z=1$ and $x+3y+2z=2$. If L makes an angle α with the positive x-axis, then $\cos\alpha$ equals

a. $\frac{1}{2}$ b. 1 c. $\frac{1}{\sqrt{2}}$ d. $\frac{1}{\sqrt{3}}$

35. The length of the perpendicular drawn from $(1, 2, 3)$ to the line $\frac{x-6}{3}=\frac{y-7}{2}=\frac{z-7}{-2}$ is

a. 4 b. 5 c. 6 d. 7

36. If angle θ between the line $\frac{x+1}{1}=\frac{y-1}{2}=\frac{z-2}{2}$ and the plane $2x-y+\sqrt{\lambda}\,z+4=0$ is such that $\sin\theta=\frac{1}{3}$, the value of λ is

a. $\frac{-3}{5}$ b. $\frac{5}{3}$ c. $\frac{-4}{3}$ d. $\frac{3}{4}$

37. The intersection of the spheres $x^2+y^2+z^2+7x-2y-z=13$ and $x^2+y^2+z^2-3x+3y+4z=8$ is the same as the intersection of one of the spheres and the plane

a. $x-y-z=1$ b. $x-2y-z=1$ c. $x-y-2z=1$ d. $2x-y-z=1$

38. A plane makes intercepts OA, OB and OC whose measurements are b and c on the OX, OY and OZ axes. The area of triangle ABC is

a. $\frac{1}{2}(ab+bc+ca)$ b. $\frac{1}{2}abc\,(a+b+c)$

c. $\frac{1}{2}(a^2b^2+b^2c^2+c^2a^2)^{1/2}$ d. $\frac{1}{2}(a+b+c)^2$

39. A line makes an angle θ with each of the x- and z-axes. If the angle β, which it makes with y-axis, is such that $\sin^2 \beta = 3 \sin^2 \theta$, then $\cos^2 \theta$ equals

a. $\frac{2}{3}$ **b.** $\frac{1}{5}$ **c.** $\frac{3}{5}$ **d.** $\frac{2}{5}$

40. The shortest distance from the plane $12x + y + 3z = 327$ to the sphere $x^2 + y^2 + z^2 + 4x - 2y - 6z = 155$ is

a. 39 **b.** 26 **c.** $41\frac{4}{13}$ **d.** 13

41. A tetrahedron has vertices $O(0, 0, 0)$, $A(1, 2, 1)$, $B(2, 1, 3)$ and $C(-1, 1, 2)$, then angle between faces OAB and ABC will be:

a. $\cos^{-1}\left(\frac{17}{31}\right)$ **b.** 30° **c.** 90° **d.** $\cos^{-1}\left(\frac{19}{35}\right)$

42. The radius of the circle in which the sphere $x^2 + y^2 + z^2 + 2z - 2y - 4z - 19 = 0$ is cut by the plane $x + 2y + 2z + 7 = 0$ is

a. 2 **b.** 3 **c.** 4 **d.** 1

43. The lines: $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar if:

a. $k = 1$ or -1 **b.** $k = 0$ or -3 **c.** $k = 3$ or -3 **d.** $k = 0$ or -1

44. The point of intersection of the lines $\frac{x-5}{3} = \frac{y-7}{-1} = \frac{z+2}{1}$ and $\frac{x+3}{-36} = \frac{y-3}{2} = \frac{z-6}{4}$ is

a. $\left(21, \frac{5}{3}, \frac{10}{3}\right)$ **b.** $(2, 10, 4)$ **c.** $(-3, 3, 6)$ **d.** $(5, 7, -2)$

45. Two systems of rectangular axes have the same origin. If a plane cuts them at distance a, b, c and a', b', c' from the origin, then:

a. $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2} = 0$ **b.** $\frac{1}{a^2} - \frac{1}{b^2} - \frac{1}{c^2} + \frac{1}{a'^2} - \frac{1}{b'^2} - \frac{1}{c'^2} = 0$

c. $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - \frac{1}{a'^2} - \frac{1}{b'^2} - \frac{1}{c'^2} = 0$ **d.** $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2} = 0$

46. The plane, which passes through the point (3, 2, 0) and the line $\frac{x-3}{1} = \frac{y-6}{5} = \frac{z-4}{4}$ is:

a. $x - y + z = 1$ **b.** $x + y + z = 5$ **c.** $x + 2y - z = 1$ **d.** $2x - y + z = 5$

47. The direction ratios of a normal to the plane through (1, 0, 0) and (0, 1, 0), which makes an angle of $\frac{\pi}{4}$ with the plane $x + y = 3$ are

a. $< 1, \sqrt{2}, 1 >$ **b.** $< 1, 1, \sqrt{2} >$ **c.** $< 1, 1, 2 >$ **d.** $< \sqrt{2}, 1, 1 >$

48. The centre of the circle given by: $\vec{r}\cdot(\hat{i}+2\hat{j}+2\hat{k})=15$ and $|\vec{r}-(\hat{j}+2\hat{k})|=4$ is

a. $(0,1,2)$ b. $(1,3,4)$ c. $(-1,3,4)$ d. none of these

49. The lines which intersect the skew lines $y=mx, z=c$; $y=-mx, z=-c$ and the x-axis lie on the surface

a. $cz=mxy$ b. $xy=cmz$ c. $cy=mxz$ d. none of these

50. Distance of the point $P(\vec{p})$ from the line $\vec{r}=\vec{a}+\lambda\vec{b}$ is

a. $\left|(\vec{a}-\vec{p})+\frac{((\vec{p}-\vec{a})\cdot\vec{b})\vec{b}}{|\vec{b}|^2}\right|$ b. $\left|(\vec{b}-\vec{p})+\frac{((\vec{p}-\vec{a})\cdot\vec{b})\vec{b}}{|\vec{b}|^2}\right|$

c. $\left|(\vec{a}-\vec{p})+\frac{((\vec{p}-\vec{b})\cdot\vec{b})\vec{b}}{|\vec{b}|^2}\right|$ d. none of these

51. From the point $P(a, b, c)$, let perpendiculars PL and PM be drawn to YOZ and ZOX planes, respectively. Then the equation of the plane OLM is

a. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ b. $\frac{x}{a}+\frac{y}{b}-\frac{z}{c}=0$

c. $\frac{x}{a}-\frac{y}{b}-\frac{z}{c}=0$ d. $\frac{x}{a}-\frac{y}{b}+\frac{z}{c}=0$

52. The plane $\vec{r}\cdot\vec{n}=q$ will contain the line $\vec{r}=\vec{a}+\lambda\vec{b}$, if

a. $\vec{b}\cdot\vec{n}\neq 0, \vec{a}\cdot\vec{n}\neq q$ b. $\vec{b}\cdot\vec{n}=0, \vec{a}\cdot\vec{n}\neq q$

c. $\vec{b}\cdot\vec{n}=0, \vec{a}\cdot\vec{n}=q$ d. $\vec{b}\cdot\vec{n}\neq 0, \vec{a}\cdot\vec{n}=q$

53. The projection of point $P(\vec{p})$ on the plane $\vec{r}\cdot\vec{n}=q$ is $(\vec{s})$, then

a. $\vec{s}=\frac{(q-\vec{p}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$ b. $\vec{s}=\vec{p}+\frac{(q-\vec{p}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$

c. $\vec{s}=\vec{p}-\frac{(\vec{p}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$ d. $\vec{s}=\vec{p}-\frac{(q-\vec{p}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$

54. The angle between $\hat{i}$ line of the intersection of the plane $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})=0$ and $\vec{r}\cdot(3\hat{i}+3\hat{j}+\hat{k})=0$, is

a. $\cos^{-1}\left(\frac{1}{3}\right)$ b. $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ c. $\cos^{-1}\left(\frac{2}{\sqrt{3}}\right)$ d. none of these

55. The line $\frac{x+6}{5}=\frac{y+10}{3}=\frac{z+14}{8}$ is the hypotenuse of an isosceles right angled triangle whose opposite vertex is (7, 2, 4). Then which of the following is not the side of the triangle?

a. $\frac{x-7}{2}=\frac{y-2}{-3}=\frac{z-4}{6}$ b. $\frac{x-7}{3}=\frac{y-2}{6}=\frac{z-4}{2}$

c. $\frac{x-7}{3}=\frac{y-2}{5}=\frac{z-4}{-1}$ d. none of these

56. The equation of the plane which passes through the line of intersection of planes $\vec{r}\cdot\vec{n_1}=q_1$, $\vec{r}\cdot\vec{n_2}=q_2$ and is parallel to the line of intersection of planes $\vec{r}\cdot\vec{n_3}=q_3$ and $\vec{r}\cdot\vec{n_4}=q_4$, is

a. $[\vec{n_2}\,\vec{n_3}\,\vec{n_4}](\vec{r}\cdot\vec{n_1}-q_1)=[\vec{n_1}\,\vec{n_3}\,\vec{n_4}](\vec{r}\cdot\vec{n_2}-q_2)$

b. $[\vec{n_1}\,\vec{n_2}\,\vec{n_3}](\vec{r}\cdot\vec{n_4}-q_4)=[\vec{n_4}\,\vec{n_3}\,\vec{n_1}](\vec{r}\cdot\vec{n_2}-q_2)$

c. $[\vec{n_4}\,\vec{n_3}\,\vec{n_1}](\vec{r}\cdot\vec{n_4}-q_4)=[\vec{n_1}\,\vec{n_2}\,\vec{n_3}](\vec{r}\cdot\vec{n_2}-q_2)$

d. none of these

57. Consider triangle AOB in the x-y plane, where $A\equiv(1,0,0)$; $B\equiv(0,2,0)$; and $O\equiv(0,0,0)$. The new position of O, when triangle is rotated about side AB by 90° can be

a. $\left(\frac{4}{5},\frac{3}{5},\frac{2}{\sqrt{5}}\right)$ b. $\left(\frac{-3}{5},\frac{\sqrt{2}}{5},\frac{2}{\sqrt{5}}\right)$ c. $\left(\frac{4}{5},\frac{2}{5},\frac{2}{\sqrt{5}}\right)$ d. $\left(\frac{4}{5},\frac{2}{5},\frac{1}{\sqrt{5}}\right)$

58. Let $\vec{a}=\hat{i}+\hat{j}$ and $\vec{b}=2\hat{i}-\hat{k}$, then the point of intersection of the lines $\vec{r}\times\vec{a}=\vec{b}\times\vec{a}$ and $\vec{r}\times\vec{b}=\vec{a}\times\vec{b}$ is

a. (3, – 1, 1) b. (3, 1, – 1) c. (–3, 1, 1) d. (–3, – 1, – 1)

59. The coordinates of the point P on the line $\vec{r}=(\hat{i}+\hat{j}+\hat{k})+\lambda(-\hat{i}+\hat{j}-\hat{k})$ which is nearest to the origin is

a. $\left(\frac{2}{3},\frac{4}{3},\frac{2}{3}\right)$ b. $\left(-\frac{2}{3},-\frac{4}{3},\frac{2}{3}\right)$ c. $\left(\frac{2}{3},\frac{4}{3},-\frac{2}{3}\right)$ d. None of these

60. The ratio in which the line segment joining the points whose position vectors are $2\hat{i}-4\hat{j}-7\hat{k}$ and $-3\hat{i}+5\hat{j}-8\hat{k}$ is divided by the plane whose equation is $\hat{r}\cdot(\hat{i}-2\hat{j}+3\hat{k})=13$, is

a. 13 : 12 internally b. 12 : 25 externally c. 13 : 25 internally d. 37 : 25 internally

61. Which of the following are equations for the plane passing through the points $P(1,1,-1)$, $Q(3,0,2)$ and $R(-2,1,0)$?

a. $(2\hat{i}-3\hat{j}+3\hat{k})\cdot((x+2)\hat{i}+(y-1)\hat{j}+z\hat{k})=0$

b. $x=3-t$, $y=-11t$, $z=2-3t$

c. $(x+2)+11(y-1)=3z$

d. $(2\hat{i}-\hat{j}+3\hat{k})\times(-3\hat{i}+\hat{k})\cdot((x+2)\hat{i}+(y-1)\hat{j}+z\hat{k})=0$

62. Given $\vec{\alpha}=3\hat{i}+\hat{j}+2\hat{k}$ and $\vec{\beta}=\hat{i}-2\hat{j}-4\hat{k}$ are the position vectors of the points A and B. Then the distance of the point $-\hat{i}+\hat{j}+\hat{k}$ from the plane passing through B and perpendicular to AB is

a. 5 b. 10 c. 15 d. 20

63. L_1 and L_2 are two lines whose vector equations are

$L_1 : \vec{r} = \lambda((\cos\theta + \sqrt{3})\hat{i} + (\sqrt{2}\sin\theta)\hat{j} + (\cos\theta - \sqrt{3})\hat{k})$

$L_2 : \vec{r} = \mu\left(a\hat{i} + b\hat{j} + c\hat{k}\right)$, where λ and μ are scalars and α is the acute angle between L_1 and L_2. If the angle 'α' is independent of θ, then the value of 'α' is

a. $\frac{\pi}{6}$ **b.** $\frac{\pi}{4}$ **c.** $\frac{\pi}{3}$ **d.** $\frac{\pi}{2}$

64. The shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ is

a. $\sqrt{30}$ **b.** $2\sqrt{30}$ **c.** $5\sqrt{30}$ **d.** $3\sqrt{30}$

65. The line through $\hat{i} + 3\hat{j} + 2\hat{k}$ and $\perp$ to the line $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(2\hat{i} + \hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} + 6\hat{j} + \hat{k}) + \mu(\hat{i} + 2\hat{j} + 3\hat{k})$ is

a. $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(-\hat{i} + 5\hat{j} - 3\hat{k})$ **b.** $\vec{r} = \hat{i} + 3\hat{j} + 2\hat{k} + \lambda(\hat{i} - 5\hat{j} + 3\hat{k})$

c. $\vec{r} = \hat{i} + 3\hat{j} + 2\hat{k} + \lambda(\hat{i} + 5\hat{j} + 3\hat{k})$ **d.** $\vec{r} = \hat{i} + 3\hat{j} + 2\hat{k} + \lambda(-\hat{i} - 5\hat{j} - 3\hat{k})$

66. The equation of the plane passing through the lines $\frac{x-4}{1} = \frac{y-3}{1} = \frac{z-2}{2}$ and $\frac{x-3}{1} = \frac{y-2}{-4} = \frac{z}{5}$ is

a. $11x - y - 3z = 35$ **b.** $11x + y - 3z = 35$ **c.** $11x - y + 3z = 35$ **d.** none of these

67. The three planes $4y + 6z = 5$; $2x + 3y + 5z = 5$ and $6x + 5y + 9z = 10$

a. meet in a point **b.** have a line in common

c. form a triangular prism **d.** none of these

68. The equation of the plane through the line of intersection of the planes $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$ and parallel to the line $y = 0$ and $z = 0$ is

a. $(ab' - a'b)x + (bc' - b'c)y + (ad' - a'd) = 0$

b. $(ab' - a'b)x + (bc' - b'c)y + (ad' - a'd)z = 0$

c. $(ab' - a'b)y + (ac' - a'c)z + (ad' - a'd) = 0$

d. none of these

69. Equation of the plane passing through the points (2, 2, 1) and (9, 3, 6), and $\perp$ to the plane $2x + 6y + 6z - 1 = 0$ is

a. $3x + 4y + 5z = 9$ **b.** $3x + 4y - 5z = 9$

c. $3x + 4y - 5z = 9$ **d.** none of the above

70. Value of λ such that the line $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z-1}{\lambda}$ is $\perp$ to normal to the plane $\vec{r}\cdot(2\vec{i} + 3\vec{j} + 4\vec{k}) = 0$ is

a. $-\frac{13}{4}$ **b.** $-\frac{17}{4}$ **c.** 4 **d.** none of these

71. The equation of the plane through the intersection of the planes $x + 2y + 3z - 4 = 0$ and $4x + 3y + 2z + 1 = 0$ and passing through the origin is

a. $17x + 14y + 11z = 0$ b. $7x + 4y + z = 0$

c. $x + 14y + 11z = 0$ d. $17x + y + z = 0$

72. The plane $4x + 7y + 4z + 81 = 0$ is rotated through a right angle about its line of intersection with the plane $5x + 3y + 10z = 25$. The equation of the plane in its new position is

a. $x - 4y + 6z = 106$ b. $x - 8y + 13z = 103$

c. $x - 4y + 6z = 110$ d. $x - 8y + 13z = 105$

73. The vector equation of the plane passing through the origin and the line of intersection of the planes $\vec{r}\cdot\vec{a} = \lambda$ and $\vec{r}\cdot\vec{b} = \mu$ is

a. $\vec{r}\cdot(\lambda\vec{a} - \mu\vec{b}) = 0$ b. $\vec{r}\cdot(\lambda\vec{b} - \mu\vec{a}) = 0$ c. $\vec{r}\cdot(\lambda\vec{a} + \mu\vec{b}) = 0$ d. $\vec{r}\cdot(\lambda\vec{b} + \mu\vec{a}) = 0$

74. The lines $\vec{r} = \vec{a} + \lambda(\vec{b}\times\vec{c})$ and $\vec{r} = \vec{b} + \mu(\vec{c}\times\vec{a})$ will intersect if

a. $\vec{a}\times\vec{c} = \vec{b}\times\vec{c}$ b. $\vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c}$ c. $\vec{b}\times\vec{a} = \vec{c}\times\vec{a}$ d. none of these

75. The projection of the line $\frac{x+1}{-1} = \frac{y}{2} = \frac{z-1}{3}$ on the plane $x - 2y + z = 6$ is the line of intersection of this plane with the plane

a. $2x + y + 2 = 0$ b. $3x + y - z = 2$ c. $2x - 3y + 8z = 3$ d. none of these

76. The direction cosines of a line satisfy the relations $\lambda(l + m) = n$ and $mn + nl + lm = 0$. The value of λ, for which the two lines are perpendicular to each other, is

a. 1 b. 2 c. 1/2 d. none of these

77. The intercepts made on the axes by the plane which bisects the line joining the points (1, 2, 3) and (–3, 4, 5) at right angles are

a. $\left(-\frac{9}{2}, 9, 9\right)$ b. $\left(\frac{9}{2}, 9, 9\right)$ c. $\left(9, -\frac{9}{2}, 9\right)$ d. $\left(9, \frac{9}{2}, 9\right)$

78. The pair of lines whose direction cosines are given by the equations $3l + m + 5n = 0$ and $6mn - 2nl + 5lm = 0$, are

a. parallel b. perpendicular c. inclined at $\cos^{-1}\left(\frac{1}{6}\right)$ d. none of these

79. A sphere of constant radius $2k$ passes through the origin and meets the axes in A, B and C. The locus of a centroid of the tetrahedron $OABC$ is

a. $x^2 + y^2 + z^2 = 4k^2$ b. $x^2 + y^2 + z^2 = k^2$

c. $2(k^2 + y^2 + z)^2 = k^2$ d. none of these

80. A plane passes through a fixed point (a, b, c). The locus of the foot of the perpendicular to it from the origin is a sphere of radius

a. $\frac{1}{2}\sqrt{a^2 + b^2 + c^2}$ b. $\sqrt{a^2 + b^2 + c^2}$

c. $a^2 + b^2 + c^2$ d. $\frac{1}{2}(a^2 + b^2 + c^2)$

81. What is the nature of the intersection of the set of planes $x + ay + (b + c)z + d = 0$, $x + by + (c + a)z + d = 0$ and $x + cy + (a + b)z + d = 0$?

a. They meet at a point
b. They form a triangular prism
c. They pass through a line
d. They are at equal distance from the origin

82. Find the equation of a straight line in the plane $\vec{r} \cdot \vec{n} = d$ which is parallel to $\vec{r} = \vec{a} + \lambda \vec{b}$ and passes through the foot of the perpendicular drawn from point $P(\vec{a})$ to $\vec{r} \cdot \vec{n} = d$ (where $\vec{n} \cdot \vec{b} = 0$).

a. $\vec{r} = \vec{a} + \left(\dfrac{d - \vec{a}\cdot\vec{n}}{n^2}\right)\vec{n} + \lambda \vec{b}$
b. $\vec{r} = \vec{a} + \left(\dfrac{d - \vec{a}\cdot\vec{n}}{n}\right)\vec{n} + \lambda \vec{b}$
c. $\vec{r} = \vec{a} + \left(\dfrac{\vec{a}\cdot\vec{n} - d}{n^2}\right)\vec{n} + \lambda \vec{b}$
d. $\vec{r} = \vec{a} + \left(\dfrac{\vec{a}\cdot\vec{n} - d}{n}\right)\vec{n} + \lambda \vec{b}$

83. What is the equation of the plane which passes through the z-axis and is perpendicular to the line $\dfrac{x-a}{\cos\theta} = \dfrac{y+2}{\sin\theta} = \dfrac{z-3}{0}$?

a. $x + y\tan\theta = 0$
b. $y + x\tan\theta = 0$
c. $x\cos\theta - y\sin\theta = 0$
d. $x\sin\theta - y\cos\theta = 0$

84. A straight line L on the xy-plane bisects the angle between OX and OY. What are the direction cosines of L?

a. $<(1/\sqrt{2}), (1/\sqrt{2}), 0>$
b. $<(1/2), (\sqrt{3}/2), 0>$
c. $<0, 0, 1>$
d. $<(2/3), (2/3), (1/3)>$

85. For what value(s) of a, will the two points $(1, a, 1)$ and $(-3, 0, a)$ lie on opposite sides of the plane $3x + 4y - 12z + 13 = 0$?

a. $a < -1$ or $a > 1/3$
b. $a = 0$ only
c. $0 < a < 1$
d. $-1 < a < 1$

86. If the plane $\dfrac{x}{2} + \dfrac{y}{3} + \dfrac{z}{6} = 1$ cuts the axes of coordinates at points A, B and C, then find the area of the triangle ABC.

a. 18 sq unit
b. 36 sq unit
c. $3\sqrt{14}$ sq unit
d. $2\sqrt{14}$ sq unit

Multiple Correct Answers Type

Solutions on page 3.111

Each question has four choices *a*, *b*, *c* and *d*, out of which *one or more* are correct.

1. Let PM be the perpendicular from the point $P(1, 2, 3)$ to the x-y plane. If $\overrightarrow{OP}$ makes an angle θ with the positive direction of the z-axis and $\overrightarrow{OM}$ makes an angle ϕ with the positive direction of x-axis, where O is the origin and θ and ϕ are acute angles, then

   a. $\cos\theta\cos\phi = 1/\sqrt{14}$ **b.** $\sin\theta\sin\phi = 2/\sqrt{14}$ **c.** $\tan\phi = 2$ **d.** $\tan\theta = \sqrt{5}/3$

2. Let P_1 denote the equation of a plane to which the vector $(\hat{i} + \hat{j})$ is normal and which contains the line whose equation is $\vec{r} = \hat{i} + \hat{j} + \hat{k} + \lambda(\hat{i} - \hat{j} - \hat{k})$ and P_2 denote the equation of the plane containing the line L and a point with position vector $\hat{j}$. Which of the following holds good?

   a. The equation of P_1 is $x + y = 2$.
   b. The equation of P_2 is $\vec{r}\cdot(\hat{i} - 2\hat{j} + \hat{k}) = 2$.
   c. The acute angle between P_1 and P_2 is $\cot^{-1}(\sqrt{3})$.
   d. The angle between the plane P_2 and the line L is $\tan^{-1}\sqrt{3}$.

3. If the planes $\vec{r}\cdot(\hat{i} + \hat{j} + \hat{k}) = q_1$, $\vec{r}\cdot(\hat{i} + 2a\hat{j} + \hat{k}) = q_2$ and $\vec{r}\cdot(a\hat{i} + a^2\hat{j} + \hat{k}) = q_3$ intersect in a line, then the value of a is

   a. 1 **b.** 1/2 **c.** 2 **d.** 0

4. A line with direction cosines proportional to 1, –5 and –2 meets lines $x = y + 5 = z + 1$ and $x + 5 = 3y = 2z$. The coordinates of each of the points of the intersection are given by

   a. $(2, -3, 1)$ **b.** $(1, 2, 3)$ **c.** $(0, 5/3, 5/2)$ **d.** $(3, -2, 2)$

5. Let $P = 0$ be the equation of a plane passing through the line of intersection of the planes $2x - y = 0$ and $3z - y = 0$ and perpendicular to the plane $4x + 5y - 3z = 8$. Then the points which lie on the plane $P = 0$ is/are

   a. $(0, 9, 17)$ **b.** $(1/7, 2, 1/9)$ **c.** $(1, 3, -4)$ **d.** $(1/2, 1, 1/3)$

6. The equation of the lines $x + y + z - 1 = 0$ and $4x + y - 2z + 2 = 0$ written in the symmetrical form is

   a. $\dfrac{x-1}{2} = \dfrac{y+2}{-1} = \dfrac{z-2}{2}$ **b.** $\dfrac{x+(1/2)}{1} = \dfrac{y-1}{-2} = \dfrac{z-(1/2)}{1}$

   c. $\dfrac{x}{1} = \dfrac{y}{-2} = \dfrac{z-1}{1}$ **d.** $\dfrac{x+1}{1} = \dfrac{y-2}{-2} = \dfrac{z-0}{1}$

7. Consider the planes $3x - 6y + 2z + 5 = 0$ and $4x - 12y + 3z = 3$. The plane $67x - 162y + 47z + 44 = 0$ bisects the angle between the given planes which

   a. contains the origin **b.** is acute **c.** is obtuse **d.** none of these

8. If the lines $\dfrac{x-2}{1} = \dfrac{y-3}{1} = \dfrac{z-4}{\lambda}$ and $\dfrac{x-1}{\lambda} = \dfrac{y-4}{2} = \dfrac{z-5}{1}$ intersect, then

   a. $\lambda = -1$ **b.** $\lambda = 2$ **c.** $\lambda = -3$ **d.** $\lambda = 0$

9. The equations of the plane which passes through (0, 0, 0) and which is equally inclined to the planes $x - y + z - 3 = 0$ and $x + y + z + 4 = 0$ is/are

a. $y = 0$ **b.** $x = 0$ **c.** $x + y = 0$ **d.** $x + z = 0$

10. The x-y plane is rotated about its line of intersection with the y-z plane by 45°, then the equation of the new plane is/are

a. $z + x = 0$ **b.** $z - y = 0$ **c.** $x + y + z = 0$ **d.** $z - x = 0$

11. The equation of the plane which is equally inclined to the lines $\frac{x-1}{2} = \frac{y}{-2} = \frac{z+2}{-1}$ and $\frac{x+3}{8} = \frac{y-4}{1} = \frac{z}{-4}$ and passing through the origin is/are

a. $14x - 5y - 7z = 0$ **b.** $2x + 7y - z = 0$ **c.** $3x - 4y - z = 0$ **d.** $x + 2y - 5z = 0$

12. Which of the following lines lie on the plane $x + 2y - z + 4 = 0$?

a. $\frac{x-1}{1} = \frac{y}{-1} = \frac{z-5}{-1}$

b. $x - y + z = 2x + y - z = 0$

c. $\vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda(3\hat{i} + \hat{j} + 5\hat{k})$

d. none of these

13. If the volume of tetrahedron $ABCD$ is 1 cubic units, where $A(0, 1, 2)$, $B(-1, 2, 1)$ and $C(1, 2, 1)$, then the locus of point D is

a. $x + y - z = 3$ **b.** $y + z = 6$ **c.** $y + z = 0$ **d.** $y + z = -3$

14. A rod of length 2 units whose one end is (1, 0, –1) and other end touches the plane $x - 2y + 2z + 4 = 0$, then

a. The rod sweeps the figure whose volume is π cubic units.

b. The area of the region which the rod traces on the plane is 2π.

c. The length of projection of the rod on the plane is $\sqrt{3}$ units.

d. The centre of the region which the rod traces on the plane is $\left(\frac{2}{3}, \frac{2}{3}, \frac{-5}{3}\right)$.

15. Consider a set of points R in the space which is at a distance of 2 units from the line $\frac{x}{1} = \frac{y-1}{-1} = \frac{z+2}{2}$ between the planes $x - y + 2z + 3 = 0$ and $x - y + 2z - 2 = 0$.

a. The volume of the bounded figure by points R and the planes is $(10/3\sqrt{3})\pi$ cube units.

b. The area of the curved surface formed by the set of points R is $(20\pi/\sqrt{6})$ sq. units.

c. The volume of the bounded figure by the set of points R and the planes is $(20\pi/\sqrt{6})$ cubic units.

d. The area of the curved surface formed by the set of points R is $(10/\sqrt{3})\pi$ sq. units.

16. The equation of a line passing through the point $\vec{a}$ parallel to the plane $\vec{r}.\vec{n} = q$ and perpendicular to the line $\vec{r} = \vec{b} + t\vec{c}$ is

a. $\vec{r} = \vec{a} + \lambda(\vec{n} \times \vec{c})$

b. $(\vec{r} - \vec{a}) \times (\vec{n} \times \vec{c}) = 0$

c. $\vec{r} = \vec{b} + \lambda(\vec{n} \times \vec{c})$

d. none of these

Reasoning Type

Solutions on page 3.116

Each question has four choices *a, b, c* and *d*, out of which *only one* is correct. Each question contains Statement 1 and Statement 2.

a. Both the statements are true, and Statement 2 is the correct explanation for Statement 1.
b. Both the statements are true, but Statement 2 is not the correct explanation for Statement 1.
c. Statement 1 is true and Statement 2 is false.
d. Statment 1 is false and Statement 2 is true.

1. **Statement 1:** Lines $\vec{r} = \hat{i} - \hat{j} + \lambda(\hat{i} + \hat{j} - \hat{k})$ and $\vec{r} = 2\hat{i} - \hat{j} + \mu(\hat{i} + \hat{j} - \hat{k})$ do not intersect.

   Statement 2: Skew lines never intersect.

2. **Statement 1:** Lines $\vec{r} = \hat{i} + \hat{j} - \hat{k} + \lambda(3\hat{i} - \hat{j})$ and $\vec{r} = 4\hat{i} - \hat{k} + \mu(2\hat{i} + 3\hat{k})$ intersect.

   Statement 2: If $\vec{b} \times \vec{d} = \vec{0}$, then lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{c} + \lambda\vec{d}$ do not intersect.

3. The equation of two straight lines are $\frac{x-1}{2} = \frac{y+3}{1} = \frac{z-2}{-3}$ and $\frac{x-2}{1} = \frac{y-1}{-3} = \frac{z+3}{2}$.

   Statement 1: The given lines are coplanar.
   Statement 2: The equations $2x_1 - y_1 = 1, x_1 + 3y_1 = 4$ and $3x_1 + 2y_1 = 5$ are consistent.

4. **Statement 1:** A plane passes through the point $A(2, 1, -3)$. If distance of this plane from origin is maximum, then its equation is $2x + y - 3z = 14$.

   Statement 2: If the plane passing through the point $A(\vec{a})$ is at maximum distance from origin, then normal to the plane is vector $\vec{a}$.

5. **Statement 1:** Line $\frac{x-1}{1} = \frac{y-0}{2} = \frac{z+2}{-1}$ lies in the plane $2x - 3y - 4z - 10 = 0$.

   Statement 2: If line $\vec{r} = \vec{a} + \lambda\vec{b}$ lies in the plane $\vec{r} \cdot \vec{c} = n$ (where n is scalar), then $\vec{b} \cdot \vec{c} = 0$.

6. **Statement 1:** Let θ be the angle between the line $\frac{x-2}{2} = \frac{y-1}{-3} = \frac{z+2}{-2}$ and the plane $x + y - z = 5$. Then $\theta = \sin^{-1}(1/\sqrt{51})$.

   Statement 2: The angle between a straight line and a plane is the complement of the angle between the line and the normal to the plane.

7. **Statement 1:** Let $A(\vec{i} + \vec{j} + \vec{k})$ and $B(\vec{i} - \vec{j} + \vec{k})$ be two points. Then point $P(2\vec{i} + 3\vec{j} + \vec{k})$ lies exterior to the sphere with AB as its diameter.
   Statement 2: If A and B are any two points and P is a point in space such that $\overrightarrow{PA} \cdot \overrightarrow{PB} > 0$, then point P lies exterior to the sphere with AB as its diameter.

8. **Statement 1:** There exists a unique sphere which passes through the three non-collinear points and which has the least radius.
   Statement 2: The centre of such a sphere lies on the plane determined by the given three points.

9. **Statement 1:** There exist two points on the line $\frac{x-1}{1} = \frac{y}{-1} = \frac{z+2}{2}$ which are at a distance of 2 units from point $(1, 2, -4)$.

   Statement 2: Perpendicular distance of point $(1, 2, -4)$ from the line $\frac{x-1}{1} = \frac{y}{-1} = \frac{z+2}{2}$ is 1 unit.

10. Statement 1: The shortest distance between the lines $\frac{x}{-3}=\frac{y-1}{1}=\frac{z+1}{-1}$ and $\frac{x-2}{1}=\frac{y-3}{2}=\left(\frac{z+(13/7)}{-1}\right)$ is zero.

Statement 2: The given lines are perpendicular.

Linked Comprehension Type

Solutions on page 3.117

Based on each paragraph, three multiple-choice questions have to be answered. Each question has four choices *a*, *b*, *c* and *d*, out of which *only one* is correct.

For Problems 1–3

Given four points $A(2, 1, 0)$, $B(1, 0, 1)$, $C(3, 0, 1)$ and $D(0, 0, 2)$. Point D lies on a line L orthogonal to the plane determined by the points A, B and C.

1. The equation of the plane ABC is

   a. $x+y+z-3=0$ **b.** $y+z-1=0$ **c.** $x+z-1=0$ **d.** $2y+z-1=0$

2. The equation of the line L is

   a. $\vec{r}=2\hat{k}+\lambda(\hat{i}+\hat{k})$ **b.** $\vec{r}=2\hat{k}+\lambda(2\hat{j}+\hat{k})$ **c.** $\vec{r}=2\hat{k}+\lambda(\hat{j}+\hat{k})$ **d.** none

3. The perpendicular distance of D from the plane ABC is

   a. $\sqrt{2}$ **b.** 1/2 **c.** 2 **d.** $1/\sqrt{2}$

For Problems 4–6

A ray of light comes along the line $L=0$ and strikes the plane mirror kept along the plane $P=0$ at B. $A(2, 1, 6)$ is a point on the line $L=0$ whose image about $P=0$ is A'. It is given that $L=0$ is $\frac{x-2}{3}=\frac{y-1}{4}=\frac{z-6}{5}$ and $P=0$ is $x+y-2z=3$.

4. The coordinates of A' are

   a. $(6,5,2)$ **b.** $(6,5,-2)$ **c.** $(6,-5,2)$ **d.** none of these

5. The coordinates of B are

   a. $(5,10,6)$ **b.** $(10,15,11)$ **c.** $(-10,-15,-14)$ **d.** none of these

6. If $L_1=0$ is the reflected ray, then its equation is

   a. $\frac{x+10}{4}=\frac{y-5}{4}=\frac{z+2}{3}$ **b.** $\frac{x+10}{3}=\frac{y+15}{5}=\frac{z+14}{5}$

   c. $\frac{x+10}{4}=\frac{y+15}{5}=\frac{z+14}{3}$ **d.** none of these

For Problems 7–9

Consider three planes $2x + py + 6z = 8, x + 2y + qz = 5$ and $x + y + 3z = 4$.

7. Three planes intersect at a point if

a. $p = 2, q \neq 3$ **b.** $p \neq 2, q \neq 3$ **c.** $p \neq 2, q = 3$ **d.** $p = 2, q = 3$

8. Three planes do not have any common point of intersection if

a. $p = 2, q \neq 3$ **b.** $p \neq 2, q \neq 3$ **c.** $p \neq 2, q = 3$ **d.** $p = 2, q = 3$

9. The planes have infinite points common among them if

a. $p = 2, q \in 3$ **b.** $p \in 2, q \in 3$ **c.** $p \neq 2, q = 3$ **d.** $p = 2, q = 3$

For Problems 10–12

Consider a plane $x + y - z = 1$ and point $A(1, 2, -3)$. A line L has the equation $x = 1 + 3r$, $y = 2 - r$ and $z = 3 + 4r$.

10. The coordinate of a point B of line L such that AB is parallel to the plane is

a. (10, –1, 15) **b.** (–5, 4, –5) **c.** (4, 1, 7) **d.** (–8, 5, –9)

11. The equation of the plane containing line L and point A has the equation

a. $x - 3y + 5 = 0$ **b.** $x + 3y - 7 = 0$ **c.** $3x - y - 1 = 0$ **d.** $3x + y - 5 = 0$

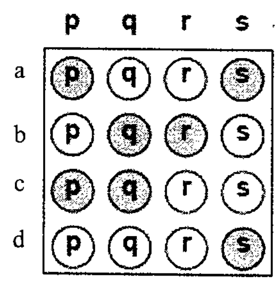
12. The distance between the points on the line which are at a distance of $4/\sqrt{3}$ from the plane is

a. $4\sqrt{26}$ **b.** 20 **c.** $10\sqrt{13}$ **d.** none of these

Matrix-Match Type

Solutions on page 3.120

Each question contains statements given in two columns which have to be matched. Statements (a, b, c, d) in Column I have to be matched with statements (p, q, r, s) in Column II. If the correct matches are a → p, s; b → q, r; c → p, q and d → s, then the correctly bubbled 4 × 4 matrix should be as follows:



1.

	Column I	Column II
a.	A vector perpendicular to the line $x = 2t + 1, y = t + 2$ and $z = -t - 3$	p. $7\hat{i} + 3\hat{j} + 5\hat{k}$
b.	A vector parallel to the planes $x + y + z - 3 = 0$ and $2x - y + 3z = 0$	q. $4\hat{i} - \hat{j} - 3\hat{k}$
c.	A vector along which the distance between the lines $\frac{x}{2} = \frac{y}{-3} = \frac{z}{-1}$ and $\vec{r} = (3\hat{i} - \hat{j} + \hat{k}) + t(\hat{i} + \hat{j} - 2\hat{k})$ is the shortest	r. $-11\hat{i} + 7\hat{j} + 5\hat{k}$
d.	A vector normal to the plane $\vec{r} = -\hat{i} + 4\hat{j} - 6\hat{k} + \lambda(\hat{i} + 3\hat{j} - 2\hat{k}) + \mu(-\hat{i} + 2\hat{j} - 5\hat{k})$	s. $\hat{i} + 3\hat{j} + \hat{k}$

2.

Column I	Column II
a. Lines $\frac{x-1}{-2} = \frac{y+2}{3} = \frac{z}{-1}$ and $\vec{r} = (3\hat{i} - \hat{j} + \hat{k}) + t(\hat{i} + \hat{j} + \hat{k})$ are	p. intersecting
b. Lines $\frac{x+5}{1} = \frac{y-3}{7} = \frac{z+3}{3}$ and $x - y + 2z - 4 = 0 = 2x + y - 3z + 5 = 0$ are	q. perpendicular
c. Lines $(x = t - 3, y = -2t + 1, z = -3t - 2)$ and $\vec{r} = (t+1)\hat{i} + (2t+3)\hat{j} + (-t-9)\hat{k}$ are	r. parallel
d. Lines $\vec{r} = (\hat{i} + 3\hat{j} - \hat{k}) + t(2\hat{i} - \hat{j} - \hat{k})$ and $\vec{r} = (-\hat{i} - 2\hat{j} + 5\hat{k}) + s(\hat{i} - 2\hat{j} + \frac{3}{4}\hat{k})$ are	s. skew

3.

	Column I	Column II
a.	The coordinates of a point on the line $x = 4y + 5, z = 3y - 6$ at a distance 3 from the point $(5, 3, -6)$ is/are	p. $(-1, -2, 0)$
b.	The plane containing the lines $\frac{x-2}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and parallel to $\hat{i} + 4\hat{j} + 7\hat{k}$ has the point	q. $(5, 0, -6)$
c.	A line passes through two points $A(2, -3, -1)$ and $B(8, -1, 2)$. The coordinates of a point on this line nearer to the origin and at a distance of 14 units from A is/are	r. $(2, 5, 7)$
d.	The coordinates of the foot of the perpendicular from the point $(3, -1, 11)$ on the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ is/are	s. $(14, 1, 5)$

4.

	Column I	Column II
a.	The distance between the line $\vec{r} = (2\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - \hat{j} + 4\hat{k})$ and plane $\vec{r} \cdot (\hat{i} + 5\hat{j} + \hat{k}) = 5$	**p.** $\frac{25}{3\sqrt{14}}$
b.	Distance between parallel planes $\vec{r} \cdot (2\hat{i} - \hat{j} + 3\hat{k}) = 4$ and $\vec{r} \cdot (6\hat{i} - 3\hat{j} + 9\hat{k}) + 13 = 0$ is	**q.** 13/7
c.	The distance of a point (2, 5, – 3) from the plane $\vec{r} \cdot (6\hat{i} - 3\hat{j} + 2\hat{k}) = 4$ is	**r.** $\frac{10}{3\sqrt{3}}$
d.	The distance of the point (1, 0, – 3) from the plane $x - y - z - 9 = 0$ measured parallel to line $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$	**s.** 7

5.

	Column I	Column II
a.	Image of the point (3, 5, 7) in the plane $2x + y + z = -18$ is	**p.** $(-1, -1, -1)$
b.	The point of intersection of the line $\frac{x-2}{-3} = \frac{y-1}{-2} = \frac{z-3}{2}$ and the plane $2x + y - z = 3$ is	**q.** $(-21, -7, -5)$
c.	The foot of the perpendicular from the point (1, 1, 2) to the plane $2x - 2y + 4z + 5 = 0$ is	**r.** $\left(\frac{5}{2}, \frac{2}{3}, \frac{8}{3}\right)$
d.	The intersection point of the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ is	**s.** $\left(-\frac{1}{12}, \frac{25}{12}, \frac{-2}{12}\right)$

Integer Answer Type

Solutions on page 3.124

1. Find the number of spheres of radius r touching the coordinate axes.
2. Find the distance of the z-axis from the image of the point M (2, –3, 3) in the plane $x - 2y - z + 1 = 0$.
3. If the length of the projection of the line segment with points (1, 0, – 1) and (–1, 2, 2) to the plane $x + 3y - 5z = 6$ is d, then find the value of $[d/2]$ where $[\cdot]$ represent greatest integer function.
4. If the angle between the plane $x - 3y + 2z = 1$ and the line $\frac{x-1}{2} = \frac{y-1}{1} = \frac{z-1}{-3}$ is θ, then find the value of cosec θ.
5. Let A_1, A_2, A_3, A_4 be the areas of the triangular faces of a tetrahedron, and h_1, h_2, h_3, h_4 be the corresponding altitudes of the tetrahedron. If volume of tetrahedron is 1/6 cubic units, then find the minimum value of $(A_1 + A_2 + A_3 + A_4)(h_1 + h_2 + h_3 + h_4)$ (in cubic units).

6. Let the equation of the plane containing line $x-y-z-4=0=x+y+2z-4$ and parallel to the line of intersection of the planes $2x+3y+z=1$ and $x+3y+2z=2$ be $x+Ay+Bz+C=0$. Then find the value of $|A+B+C-4|$.
7. Let $P(a, b, c)$ be any point on the plane $3x+2y+z=7$, then find the least value of $2(a^2+b^2+c^2)$.
8. The plane denoted by $P_1 : 4x + 7y + 4z + 81 = 0$ is rotated through a right angle about its line of intersection with the plane $P_2 : 5x + 3y + 10z = 25$. If the plane in its new position be denoted by P, and the distance of this plane from the origin is d, then find the value of $[k/2]$ (where $[\cdot]$ represents greatest integer less than or equal to k).
9. The distance of the point $P(-2, 3, -4)$ from the line $\frac{x+2}{3}=\frac{2y+3}{4}=\frac{3z+4}{5}$ measured parallel to the plane $4x+12y-3z+1=0$ is d, then find the value of $(2d-8)$.
10. The position vectors of the four angular points of a tetrahedron $OABC$ are (0, 0, 0), (0, 0, 2), (0, 4, 0) and (6, 0, 0), respectively. A point P inside the tetrahedron is at the same distance 'r' from the four plane faces of the tetrahedron. Find the value of $9r$.

Archives

Solutions on page 3.127

Subjective Type

1. (i) Find the equation of the plane passing through the points (2, 1, 0), (5, 0, 1) and (4, 1, 1).

   (ii) If P is the point (2, 1, 6), then find the point Q such that PQ is perpendicular to the plane in (i) and the midpoint of PQ lies on it. **(IIT-JEE, 2003)**
2. Find the equation of a plane passing through (1, 1, 1) and parallel to the lines L_1 and L_2 having direction ratios (1, 0, –1) and (1, – 1, 0), respectively. Find the volume of tetrahedron formed by origin and the points where this plane intersects the coordinate axes. **(IIT-JEE, 2004)**
3. A parallelepiped S has base points A, B, C and D and upper face points A', B', C' and D'. The parallelepiped is compressed by upper face $A'B'C'D'$ to form a new parallelepiped T having upper face points A'', B'', C'' and D''. The volume of parallelepiped T is 90 percent of the volume of parallelepiped S. Prove that the locus of A'' is a plane. **(IIT-JEE, 2004)**
4. Find the equation of the plane containing the lines $2x-y+z-3=0$ and $3x+y=z=5$ and at a distance of $1/\sqrt{6}$ from the point $(2, 1, -1)$. **(IIT-JEE, 2005)**
5. A line with positive direction cosines passes through the point P (2, – 1, 2) and makes equal angles with the coordinate axes. The line meets the plane $2x+y+z=9$ at point Q. Find the length of the line segment PQ. **(IIT-JEE, 2009)**

Objective Type

Multiple choice questions with one correct answer

1. The value of k such that $\frac{x-4}{1}=\frac{y-2}{1}=\frac{z-k}{2}$ lies in the plane $2x-4y+z=7$, is

   a. 7 **b.** – 7 **c.** no real value **d.** 4

   (IIT-JEE, 2003)

2. If the lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect, then the value of k is

a. 3/2 b. 9/2 c. −2/9 d. −3/2

(IIT-JEE, 2004)

3. A variable plane at a distance of 1 unit from the origin cuts the coordinate axes at A, B and C. If the centroid $D(x, y, z)$ of triangle ABC satisfies the relation $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=k$, then the value of k is

a. 3 b. 1 c. 1/3 d. 9

(IIT-JEE, 2005)

4. A plane which is perpendicular to two planes $2x - 2y + z = 0$ and $x - y + 2z = 4$ passes through $(1, -2, 1)$. The distance of the plane from the point $(1, 2, 2)$ is

a. 0 b. 1 c. $\sqrt{2}$ d. $2\sqrt{2}$

(IIT-JEE, 2006)

5. Let $P(3, 2, 6)$ be a point in space and Q be a point on line $\vec{r}=(\hat{i}-\hat{j}+2\hat{k})+\mu(-3\hat{i}+\hat{j}+5\hat{k})$. Then the value of μ for which the vector $\overrightarrow{PQ}$ is parallel to the plane $x-4y+3z=1$ is

a. 1/4 b. −1/4 c. 1/8 d. −1/8

(IIT-JEE, 2009)

6. Equation of the plane containing the straight line $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3}=\frac{y}{4}=\frac{z}{2}$ and $\frac{x}{4}=\frac{y}{2}=\frac{z}{3}$ is

a. $x+2y-2z=0$ b. $3x+2y-2z=0$ c. $x-2y+z=0$ d. $5x+2y-4z=0$

(IIT-JEE, 2010)

7. If the distance of the point $P(1, -2, 1)$ from the plane $x + 2y - 2z = \alpha$, where $\alpha > 0$, is 5, then the foot of the perpendicular from P to the place is

a. $\left(\frac{8}{3},\frac{4}{3},-\frac{7}{3}\right)$ b. $\left(\frac{4}{3},-\frac{4}{3},\frac{1}{3}\right)$ c. $\left(\frac{1}{3},\frac{2}{3},\frac{10}{3}\right)$ d. $\left(\frac{2}{3},-\frac{1}{3},\frac{5}{2}\right)$

(IIT-JEE, 2010)

Assertion and reasoning type

Each question has four choices *a*, *b*, *c* and *d*, out of which *only one* is correct. Each question contains Statement 1 and Statement 2.

a. Both the statements are true, and Statement 2 is the correct explanation for Statement 1.
b. Both the statements are true, but Statement 2 is not the correct explanation for Statement 1.
c. Statement 1 is true and Statement 2 is false.
d. Statment 1 is false and Statement 2 is true.

1. Consider the planes $3x-6y-2z=15$ and $2x+y-2z=5$.

   Statement 1: The parametric equations of the line of intersection of the given planes are $x = 3 + 14t$, $y = 2t$, $z = 15t$.

   Statement 2: The vector $14\hat{i} + 2\hat{j} + 15\hat{k}$ is parallel to the line of intersection of the given planes.

2. Consider three planes $P_1: x-y+z=1$, $P_2: x+y-z=-1$ and $P_3: x-3y+3z=2$.
   Let L_1, L_2 and L_3 be the lines of intersection of the planes P_2 and P_3, P_3 and P_1, and P_1 and P_2, respectively.

   Statement 1: At least two of the lines L_1, L_2 and L_3 are non-parallel.

   Statement 2: The three planes do not have a common point.

(IIT-JEE, 2009)

Comprehension type

For Problems 1–3

Consider the lines $L_1: \dfrac{x+1}{3}=\dfrac{y+2}{1}=\dfrac{z+1}{2}$, $L_2: \dfrac{x-2}{1}=\dfrac{y+2}{2}=\dfrac{z-3}{3}$ **(IIT-JEE, 2008)**

1. The unit vector perpendicular to both L_1 and L_2 is

   a. $\dfrac{-\hat{i}+7\hat{j}+7\hat{k}}{\sqrt{99}}$ **b.** $\dfrac{-\hat{i}-7\hat{j}+5\hat{k}}{5\sqrt{3}}$ **c.** $\dfrac{-\hat{i}+7\hat{j}+5\hat{k}}{5\sqrt{3}}$ **d.** $\dfrac{7\hat{i}-7\hat{j}-\hat{k}}{\sqrt{99}}$

2. The shortest distance between L_1 and L_2 is

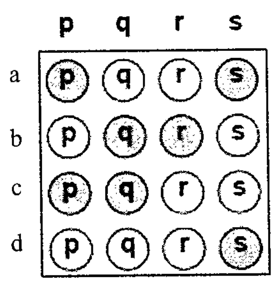
   a. 0 **b.** $\dfrac{17}{\sqrt{3}}$ **c.** $\dfrac{41}{5\sqrt{3}}$ **d.** $\dfrac{17}{5\sqrt{3}}$

3. The distance of the point (1, 1, 1) from the plane passing through the point (–1, –2, –1) and whose normal is perpendicular to both the lines L_1 and L_2 is

   a. $\dfrac{12}{\sqrt{65}}$ **b.** $\dfrac{14}{\sqrt{75}}$ **c.** $\dfrac{13}{\sqrt{75}}$ **d.** $\dfrac{13}{\sqrt{65}}$

Matrix-match type

Each question contains statements given in two columns which have to be matched. Statements (a, b, c, d) in Column I have to be matched with statements (p, q, r, s) in Column II. If the correct matches are a → p, s; b → q, r; c → p, q and d → s, then the correctly bubbled 4 × 4 matrix should be as follows:



1. Consider the linear equations $ax + by + cz = 0$, $bx + cy + az = 0$ and $cx + ay + bz = 0$.
Match the conditions/expressions in Column I with statements in Column II. **(IIT-JEE, 2007)**

Column I	Column II
a. $a + b + c \neq 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$	**p.** The equations represent planes meeting only at a single point.
b. $a + b + c = 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$	**q.** The equations represent the line $x = y = z$.
c. $a + b + c \neq 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$	**r.** The equations represent identical planes.
d. $a + b + c = 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$	**s.** The equations represent the whole of the three-dimensional space.

Integer Answer Type

1. If the distance between the plane $Ax - 2y + z = d$ and the plane containing the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ is $\sqrt{6}$, then find the value of $|d|$.

(IIT-JEE, 2010)

ANSWERS AND SOLUTIONS

Subjective Type

1. Since l, m and n, and $(l + \delta m)$, $(m + \delta m)$, $(n + \delta n)$ are the direction cosines, we have

$l^2 + m^2 + n^2 = 1$ (i)

$(l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1$

$\Rightarrow l^2 + m^2 + n^2 + 2l\,\delta l + 2m\,\delta m + 2n\,\delta n + (\delta l)^2 + (\delta m)^2 + (\delta n)^2 = 1$

$\Rightarrow 2(l\,\delta l + m\,\delta m + n\,\delta n) = -(\{\delta l)^2 + (\delta m)^2 + (\delta n)^2\}$ (ii)

Now it is given that $\delta\theta$ is the angle between two adjacent positions of the line. Therefore

$\cos\delta\theta = l(l + \delta l) + m(m + \delta m) + n(n + \delta n)$ (iii)

Now $\cos\delta\theta = 1 - \frac{(\delta\theta)^2}{2!} + \frac{(\delta\theta)^2}{4!} - \cdots$

If $\delta\theta$ is small, then $\cos\delta\theta = 1 - \frac{(\delta\theta)^2}{2}$

Then from (iii), we have $1 - \frac{(\delta\theta)^2}{2} = (l^2 + m^2 + n^2) + (l\,\delta l + m\,\delta m + n\,\delta n)$

$\Rightarrow\quad 1 - \frac{(\delta\theta)^2}{2} = 1 - \frac{1}{2}\{(\delta l)^2 + (\delta m)^2 + (\delta n)^2\}$ (using (i) and (ii))

$\Rightarrow\quad (\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$

2. The equation of the first line may be written as $\frac{y}{b}-\frac{1}{2}=\frac{1}{2}-\frac{z}{c}, x=0$

or $\frac{x}{0}=\frac{y-\frac{1}{2}b}{b}=\frac{z-\frac{1}{2}c}{-c}$ (i)

Similarly, the equation of the second line may be written as

$\frac{x-\frac{1}{2}a}{a}=\frac{y}{0}=\frac{z+\frac{1}{2}c}{c}$ (ii)

The equation of any plane passing through line (i) is

$A(x)+B\left(y-\frac{1}{2}b\right)+C\left(z-\frac{1}{2}c\right)=0,$ (iii)

where $A\cdot 0+B\cdot b-C\cdot c=0$ (iv)

Now plane (iii) will be parallel to line (ii) if

$A\cdot a+B\cdot 0-C\cdot c=0$ (v)

Solving (iv) and (v), we have $\frac{A}{bc}=\frac{B}{-ca}=\frac{C}{-ab}$

Putting these values of A, B and C in (iii), the equation of the required plane is

$$bcx-ca\left(y-\frac{1}{2}b\right)-ab\left(z-\frac{1}{2}c\right)=0 \text{ or } \frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$$

3. Let the equation of the variable plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$ (i)

where a, b and c are the parameters.

Plane (i) passes through the point (α, β, γ). Therefore,

$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=1$ (ii)

Plane (i) meets the coordinate axes at points A, B and C. The equations of the planes passing through A, B and C and parallel to the coordinate planes are, respectively,

$x=a, y=b, z=c$ (iii)

The locus of the point of intersection of these planes is obtained by eliminating the parameters a, b and c between (ii) and (iii). Putting the values of a, b and c from (iii) in (ii), the required locus is given by $\frac{\alpha}{x}+\frac{\beta}{y}+\frac{\gamma}{z}=1$ or $\alpha x^{-1}+\beta y^{-1}+\gamma z^{-1}=1$

4. Here, $l=-\frac{(bm+cn)}{a}$ and $ul^2+m^2v+wn^2=0$.

Eliminating l, we get

$$\frac{u(bm+cn)^2}{a^2}+vm^2+wn^2=0$$

$$u(b^2m^2+2bcmn+c^2n^2)+va^2m^2+wa^2n^2=0$$

$$(b^2u+a^2v)m^2+(2bcu)mn+(c^2u+a^2w)n^2=0$$

$\Rightarrow (b^2u+a^2v)\left(\frac{m}{n}\right)^2+(2bcu)\left(\frac{m}{n}\right)+(c^2u+a^2w)=0$, which is quadratic in (m/n) having roots m_1/n_1 and m_2/n_2

a. If the straight lines are parallel, the quadratic in m/n has equal roots, i.e., discriminant = 0

$$\Rightarrow (2bcu)^2 - 4(b^2u + a^2v)(c^2u + a^2w) = 0$$

$$\Rightarrow b^2c^2u^2 = (b^2u + a^2v)(c^2u + a^2w)$$

$$\Rightarrow a^2vw + b^2uw + c^2uv = 0$$

$$\Rightarrow \frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}=0$$

b. If the straight lines are perpendicular,

$$\Rightarrow \frac{m_1}{n_1}\cdot\frac{m_2}{n_2}=\frac{c^2u+a^2w}{b^2u+a^2v} \quad \text{(product of roots)}$$

$$\Rightarrow \frac{m_1m_2}{c^2u+a^2w}=\frac{n_1n_2}{b^2u+a^2v} \quad \text{(i)}$$

Similarly, by eliminating n, we get

$$\frac{l_1l_2}{b^2w+c^2v}=\frac{m_1m_2}{c^2u+a^2w} \quad \text{(ii)}$$

From (i) and (ii)

$$\frac{l_1l_2}{b^2w+c^2v}=\frac{m_1m_2}{c^2u+a^2w}=\frac{n_1n_2}{b^2u+a^2v}=\lambda$$

Since they are perpendicular, $l_1l_2+m_1m_2+n_1n_2=0$

$$\Rightarrow \lambda(b^2w+c^2v)+\lambda(c^2u+a^2w)+\lambda(b^2u+a^2v)=0$$

$$\Rightarrow a^2(v+w)+b^2(w+u)+c^2(u+v)=0$$

5.

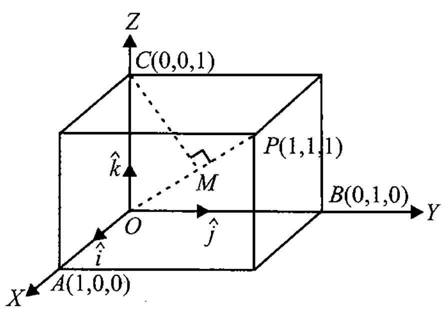


Fig. 3.33

Let the edges OA, OB and OC of the unit cube be along OX, OY and OZ, respectively.

Since $OA = OB = OC = 1$ unit, $\overrightarrow{OA}=\hat{i}$, $\overrightarrow{OB}=\hat{j}$ and $\overrightarrow{OC}=\hat{k}$

Let CM be perpendicular from the corner C on the diagonal OP. The vector equation of OP is

$\vec{r} = \lambda(\hat{i} + \hat{j} + \hat{k})$

OM = projection of $\overrightarrow{OC}$ on $\overrightarrow{OP} = \dfrac{\overrightarrow{OC}\cdot\overrightarrow{OP}}{|\overrightarrow{OP}|} = \hat{k}\cdot\dfrac{(\hat{i}+\hat{j}+\hat{k})}{\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

Now $OC^2 = OM^2 + CM^2$

$\Rightarrow CM^2 = OC^2 - OM^2 = 1 - \dfrac{1}{3} = \dfrac{2}{3} \Rightarrow CM = \sqrt{\dfrac{2}{3}}$

6. The given plane is $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1.$ (i)

Let $P(h, k, l)$ be the point on the plane

$\dfrac{h}{a} + \dfrac{k}{b} + \dfrac{l}{c} = 1$ (ii)

$\Rightarrow OP = \sqrt{h^2 + k^2 + l^2}$

Direction cosines of OP are $\dfrac{h}{\sqrt{h^2+k^2+l^2}}, \dfrac{k}{\sqrt{h^2+k^2+l^2}}$ and $\dfrac{l}{\sqrt{h^2+k^2+l^2}}$.

The equation of the plane through P and normal to OP is

$$\frac{hx}{\sqrt{h^2+k^2+l^2}} + \frac{ky}{\sqrt{h^2+k^2+l^2}} + \frac{lz}{\sqrt{h^2+k^2+l^2}} = \sqrt{h^2+k^2+l^2}$$

or $hx + ky + lz = h^2 + k^2 + l^2$

Therefore, $A \equiv \left(\dfrac{h^2+k^2+l^2}{h}, 0, 0\right), B \equiv \left(0, \dfrac{h^2+k^2+l^2}{k}, 0\right)$ and $C \equiv \left(0, 0, \dfrac{h^2+k^2+l^2}{l}\right)$

If $Q(\alpha, \beta, \gamma)$, then $\alpha = \dfrac{h^2+k^2+l^2}{h}$, $\beta = \dfrac{h^2+k^2+l^2}{k}$ and $\gamma = \dfrac{h^2+k^2+l^2}{l}$ (iii)

Now, $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2} = \dfrac{h^2+k^2+l^2}{(h^2+k^2+l^2)^2} = \dfrac{1}{h^2+k^2+l^2}$ (iv)

From (iii), $h = \dfrac{h^2+k^2+l^2}{\alpha} \Rightarrow \dfrac{h}{a} = \dfrac{h^2+k^2+l^2}{a\alpha}$

Similarly, $\dfrac{k}{b} = \dfrac{h^2+k^2+l^2}{b\beta}$ and $\dfrac{l}{c} = \dfrac{h^2+k^2+l^2}{c\gamma}$

$\dfrac{h^2+k^2+l^2}{a\alpha} + \dfrac{h^2+k^2+l^2}{b\beta} + \dfrac{h^2+k^2+l^2}{c\gamma} = \dfrac{h}{a} + \dfrac{k}{b} + \dfrac{l}{c} = 1$ (from (ii)) or $\quad \frac{1}{a\alpha}+\frac{1}{b\beta}+\frac{1}{c\gamma}=\frac{1}{h^2+k^2+l^2}=\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$ (from (iv))

The required equation of locus is $\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}=\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}$

7. The given planes are

$x - cy - bz = 0$ (i)

$cx - y + az = 0$ (ii)

$bx + ay - z = 0$ (iii)

The equation of the planes passing through the line of intersection of planes (i) and (ii) may be taken as $(x - cy - bz) + \lambda(cx - y + az) = 0$

or $x(1 + \lambda c) - y(c + \lambda) + z(-b + a\lambda) = 0$ (iv)

If planes (iii) and (iv) are the same, then Eqs. (iii) and (iv) will be identical.

$$\frac{1+c\lambda}{b}=\frac{-(c+\lambda)}{a}=\frac{-b+a\lambda}{-1}$$

$$\lambda = -\frac{(a+bc)}{(ac+b)} \text{ and } \lambda=-\frac{(ab+c)}{(1-a^2)}$$

$$\therefore \frac{-(a+bc)}{(ac+b)}=-\frac{(ab+c)}{(1-a^2)}$$

$a - a^3 + bc - a^2bc = a^2bc + ac^2 + ab^2 + bc$

$\Rightarrow 2a^2bc + ac^2 + ab^2 + a^3 - a = 0$

$\Rightarrow a(2abc + c^2 + b^2 + a^2 - 1) = 0$

$\Rightarrow a^2 + b^2 + c^2 + 2abc = 1$

Alternative method:

Since the planes pass through origin, the given planes have a common line of intersection if given system of equations has a non-trivial solution

$$\Rightarrow \begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

$\Rightarrow a^2 + b^2 + c^2 + 2abc = 1$

8. Let P be (x_1, y_1, z_1). Point M is $(x_1, 0, z_1)$ and N is $(x_1, y_1, 0)$.

So normal to plane OMN is $\overrightarrow{OM}\times\overrightarrow{ON}=\vec{x}$ (say). Therefore,

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & 0 & z_1 \\ x_1 & y_1 & 0 \end{vmatrix} = \hat{i}(-y_1 z_1)-\hat{j}(-x_1 z_1)+\hat{k}(x_1 y_1)$$

$$\sin\theta = \frac{-x_1y_1z + x_1y_1z + x_1y_1z_1}{\sqrt{x_1^2+y_1^2+z_1^2}\sqrt{\Sigma x_1^2y_1^2}} \left(\text{because } \sin\theta = \frac{\vec{n}\times\overrightarrow{OP}}{|\vec{n}||\overrightarrow{OP}|}\right)$$

$$\Rightarrow \operatorname{cosec}^2\theta = \frac{\Sigma x_1^2 \Sigma x_1^2 y_1^2}{(x_1y_1z_1)^2} = \frac{\Sigma x_1^2}{x_1^2} + \frac{\Sigma x_1^2}{y_1^2} + \frac{\Sigma x_1^2}{z_1^2}$$

Now, $\sin\alpha = \dfrac{\overrightarrow{OP}\cdot\hat{k}}{|\overrightarrow{OP}|} = \dfrac{z_1}{\sqrt{\Sigma x_1^2}}$, $\sin\beta = \dfrac{x_1}{\sqrt{\Sigma x_1^2}}$ and $\sin\gamma = \dfrac{y_1}{\sqrt{\Sigma x_1^2}}$

Now, $\operatorname{cosec}^2\alpha + \operatorname{cosec}^2\beta + \operatorname{cosec}^2\gamma = \dfrac{\Sigma x_1^2}{x_1^2} + \dfrac{\Sigma x_1^2}{y_1^2} + \dfrac{\Sigma x_1^2}{z_1^2} = \operatorname{cosec}^2\theta$

Hence proved.

9. $\dfrac{x}{p/l} + \dfrac{y}{p/m} + \dfrac{z}{p/n} = 1$

The foot of normal on plane has coordinates $H(lp, mp, np)$.

Direction ratios of AH are $lp - (p/l)$, mp and np and direction ratios of BC are 0, $-p/m$, and p/n.

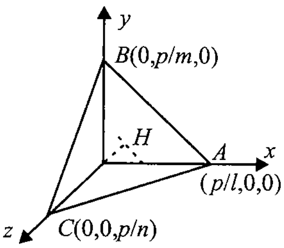


Fig. 3.34

$$\Rightarrow \left(lp - \frac{p}{l}\right)\cdot 0 + (mp)\left(-\frac{p}{m}\right) + (np)\left(\frac{p}{n}\right) = 0$$

Hence, AH is perpendicular to BC.

Similarly, BH is perpendicular to AC and CH is perpendicular to AB.

Hence, H is the orthocenter.

Moreover, in any triangle, G (centroid) divides OH in the ratio 1 : 2.

Hence,

$$G \equiv \left(\frac{p}{3l}, \frac{p}{3m}, \frac{p}{3n}\right)$$

$$H \equiv (lp, mp, np)$$

$$\Rightarrow O \equiv \left(\frac{p - l^2p}{2l}, \frac{p - m^2p}{2m}, \frac{p - n^2p}{2n}\right).$$

10. $x - y \sin \alpha - z \sin \beta = 0$ (i)

$x \sin \alpha + z \sin \gamma - y = 0$ (ii)

$x \sin \beta + y \sin \gamma - z = 0$ (iii)

These planes pass through origin. Let l, m and n be the direction cosines of the line of intersection of planes (i) and (ii). Then

$l \cdot 1 - m \sin \alpha - n \sin \beta = 0$

$l \sin \alpha - m \cdot 1 + n \sin \gamma = 0$

$$\Rightarrow \frac{l}{-\sin\gamma \sin\alpha - \sin\beta} = \frac{m}{-\sin\beta \sin\alpha - \sin\gamma} = \frac{n}{-1 + \sin^2 \alpha} \quad \text{(iv)}$$

If $\alpha + \beta + \gamma = \frac{\pi}{2} \Rightarrow \beta = \frac{\pi}{2} - (\alpha + \gamma)$

$$\sin \beta = \sin\left(\frac{\pi}{2} - (\alpha + \gamma)\right) = \cos(\alpha + \gamma)$$

$\sin \beta = \cos \alpha \cos \gamma - \sin \alpha \sin \gamma$

$\sin \beta + \sin \alpha \sin \gamma = \cos \alpha \cos \gamma$

Similarly, $\sin \gamma + \sin \beta \sin \alpha = \cos \alpha \cos \beta$

From equation (iv), we get $\dfrac{l}{\cos\alpha\cos\gamma} = \dfrac{m}{\cos\alpha\cos\beta} = \dfrac{n}{\cos^2 \alpha}$

$$\frac{l}{\cos\gamma} = \frac{m}{\cos\beta} = \frac{n}{\cos\alpha} \quad \text{(v)}$$

The line of intersection of planes (i) and (ii) also passes through the origin. Then the equation of the line is

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n}$$

$$\Rightarrow \frac{x}{\cos\gamma} = \frac{y}{\cos\beta} = \frac{z}{\cos\alpha} \quad \text{(vi)}$$

If the line also lies on plane (iii), then the three planes will intersect on this straight line.

The angle between line and normal of plane (iii) should be $\pi/2$.

$\Rightarrow \cos \gamma \sin \beta + \cos \beta \sin \gamma + \cos \alpha(-1) = \sin (\beta + \gamma) - \cos \alpha$

$$= \sin\left(\frac{\pi}{2} - \alpha\right) - \cos\alpha = 0$$

Hence $\dfrac{x}{\cos\gamma} = \dfrac{y}{\cos\beta} = \dfrac{z}{\cos\alpha}$ is the common line of the intersection of the three given planes.

11. $ax + by + cz + 1 = 0$ (i)

It makes an angle 60° with the line $x = y = z$. So we get

$$\sin 60^\circ = \frac{a+b+c}{\sqrt{3}\,\Sigma a^2} \Rightarrow 3\sqrt{\Sigma a^2} = 2(a + b + c) \quad \text{(ii)}$$

Plane (i) makes an angle of 45° with the line $x = y - z = 0$ $\left(\text{or } \frac{x}{0} = \frac{y}{1} = \frac{z}{1}\right)$

$$\sin 45° = \frac{b+c}{\sqrt{2}\sqrt{\Sigma a^2}} \Rightarrow \sqrt{\Sigma a^2} = b+c \qquad \text{(iii)}$$

Plane (i) makes an angle θ with the plane $x = 0$. So we get

$$\cos\theta = \frac{a}{\sqrt{\Sigma a^2}} \qquad \text{(iv)}$$

From (ii) and (iii), we get

$$(\sqrt{\Sigma a^2}) = 2a$$

$$\Rightarrow \frac{a}{\sqrt{\Sigma a^2}} = \frac{1}{2}$$

From (iv), $\cos\theta = 1/2 \Rightarrow \theta = 60°$

Distance of plane (i) from the point (2, 1, 1) is 3 units.

$$\Rightarrow \frac{2a+b+c+1}{\sqrt{\Sigma a^2}} = \pm 3$$

$$\Rightarrow \pm 3\sqrt{\Sigma a^2} = 2a+b+c+1$$

Case I:

$$3\sqrt{\Sigma a^2} = 2a + b + c + 1 \qquad \text{(v)}$$

From (ii) and (v), we get

$$b + c - 1 = 0 \qquad \text{(vi)}$$

and from (iii) and (iv), we get

$$2a + b + c + 1 = 3(b + c) \qquad \text{(vii)}$$

From (vi) and (vii), we get

$$a = \frac{1}{2}, b = \frac{(2\mp\sqrt{2})}{4} \text{ and } c = \frac{2\pm\sqrt{2}}{4}.$$

Hence, the set of such planes is $2x + (2\pm\sqrt{2})y + (2\pm\sqrt{2})z + 4 = 0$.

Case II:

$$-3\sqrt{\Sigma a^2} = 2a+b+c+1$$

$$a = \frac{-1}{10}, b = \frac{-(2\pm\sqrt{2})}{20} \text{ and } c = \frac{-(2\mp\sqrt{2})}{20}$$

Hence, the other set of the planes is $2x + (2\pm\sqrt{2})y + (2\mp\sqrt{2})z - 20 = 0$.

12. Let the given planes intersect on the line with direction ratios l, m and n. In that case,

$$(2+\lambda)\frac{l}{a}+(1-2\lambda)\frac{m}{b}+(2-\lambda)\frac{n}{c}=0 \quad \text{(i)}$$

$$\text{and } \frac{4l}{a}-(3-5\mu)\frac{m}{b}+4\mu.\frac{n}{c}=0 \quad \text{(ii)}$$

$$\text{Hence, } \frac{l/a}{6-6\mu-3\lambda-3\lambda\mu}=\frac{m/b}{8-8\mu-4\lambda-4\lambda\mu}=\frac{n/c}{-10+10\mu+5\lambda+5\lambda\mu}$$

$$\text{or } \frac{l/a}{3(2-2\mu-\lambda-\lambda\mu)}=\frac{m/b}{4(2-2\mu-\lambda-\lambda\mu)}=\frac{n/c}{-5(2-2\mu-\lambda-\lambda\mu)}$$

$$\text{or } \frac{l/a}{3}=\frac{m/b}{4}=\frac{n/c}{-5} \quad (\text{provided } 2-2\mu-\lambda-\lambda\mu\neq 0)$$

which are independent of λ and μ. Hence a line with direction ratios $(3a, 4b, -5c)$ lies in both the planes.

For $2-2\mu-\lambda-\lambda\mu=0$ or $\lambda=\dfrac{2(1-\mu)}{1+\mu}$, planes (i) and (ii) coincide with each other. Hence, the two given families of planes intersect on the same line.

13. Let A_1 and B_1 be the projections of A and B on the plane $z=0$. Let OA, OB and OC be of the unit length each so that the coordinates of A, B and C are $A(l_1, m_1, n_1)$, $B(l_2, m_2, n_2)$ and $C(l_3, m_3, n_3)$. The coordinates of A_1 and B_1, therefore, are $A_1(l_1, m_1, 0)$ and $B_1(l_2, m_2, 0)$. Since OA_1 and OB_1 make angles ϕ_1 and ϕ_2, respectively, with the x-axis, the angle between OA_1 and OB_1 is $\phi_1 \sim \phi_2$. Hence

$$\cos(\phi_1-\phi_2)=\frac{l_1l_2+m_1m_2}{\sqrt{l_1^2+m_1^2}\sqrt{l_2^2+m_2^2}} \quad \text{(i)}$$

Also OA, OB and OC are mutually perpendicular so that

$l_1l_2+m_1m_2+n_1n_2=0$

and $l_1^2+m_1^2+n_1^2=1$

Eq. (i), therefore, yields

$$\cos(\phi_1-\phi_2)=\frac{-n_1n_2}{\sqrt{1-n_1^2}\sqrt{1-n_2^2}}$$

$$\Rightarrow \sec^2(\phi_1-\phi_2)=\frac{1-n_1^2-n_2^2+n_1^2n_2^2}{n_1^2n_2^2}=1+\frac{1-n_1^2-n_2^2}{n_1^2n_2^2}=1+\frac{n_3^2}{n_1^2n_2^2}$$

$$\Rightarrow \tan^2(\phi_1-\phi_2)=\frac{n_3^2}{n_1^2n_2^2}$$

$$\Rightarrow \tan(\phi_1-\phi_2)=\pm\frac{n_3}{n_1n_2}$$

14. If θ is the angle BCO, then the direction cosines of OA' (bisector of $\angle BOC$) are $\dfrac{l_2+l_3}{2\cos(\theta/2)}$, $\dfrac{m_2+m_3}{2\cos(\theta/2)}$ and $\dfrac{n_2+n_3}{2\cos(\theta/2)}$ or the direction ratios of OA' are l_2+l_3, m_2+m_3 and n_2+n_3.

Also, the direction cosines of OA are l_1, m_1 and n_1. Hence the equation of plane AOA' is

$$\begin{vmatrix} x & y & z \\ l_2+l_3 & m_2+m_3 & n_2+n_3 \\ l_1 & m_1 & n_1 \end{vmatrix} = 0$$

Applying $R_2 \to R_2 + R_3$, we get the equation of plane AOA' as

$$\begin{vmatrix} x & y & z \\ l_1+l_2+l_3 & m_1+m_2+m_3 & n_1+n_2+n_3 \\ l_1 & m_1 & n_1 \end{vmatrix} = 0$$

$\Rightarrow$ For all values of r, the point $((l_1+l_2+l_3)r, (m_1+m_2+m_3)r$ and $(n_1+n_2+n_3)r)$ lies on plane AOA'. Hence, the line $\dfrac{x}{l_1+l_2+l_3} = \dfrac{y}{m_1+m_2+m_3} = \dfrac{z}{n_1+n_2+n_3} = r$ lies on plane AOA'. Similarly, this line lies on planes BOB' and COC' also. Hence, all the three planes, AOA', BOB' and COC', pass through the line.

15. Let $P(\alpha, \beta, \gamma)$ and $Q(x_1, y_1, z_1)$ be the given points.

Direction ratios of OP are α, β and γ and those of OQ are x_1, y_1 and z_1.

Since O, Q and P are collinear, $\dfrac{\alpha}{x_1} = \dfrac{\beta}{y_1} = \dfrac{\gamma}{z_1} = k$ (say) (i)

As $P(\alpha, \beta, \gamma)$ lies on the plane $lx + my + nz = p$,

$l\alpha + m\beta + n\gamma = p$, or

$klx_1 + kmy_1 + knz_1 = p$ (using (i)) (ii)

Since $OP \cdot OQ = p^2$,

$$\sqrt{\alpha^2+\beta^2+\gamma^2} \cdot \sqrt{x_1^2+y_1^2+z_1^2} = p^2$$

$$\Rightarrow \sqrt{k^2x_1^2+k^2y_1^2+k^2z_1^2} \cdot \sqrt{x_1^2+y_1^2+z_1^2} = p^2$$

$\Rightarrow k(x_1^2+y_1^2+z_1^2) = p^2$ (iii)

From (ii) and (iii), $\dfrac{lx_1+my_1+nz_1}{x_1^2+y_1^2+z_1^2} = \dfrac{1}{p}$ or $p(lx_1+my_1+nz_1) = (x_1^2+y_1^2+z_1^2)$

Hence, the locus of Q is $p(lx+my+nz) = (x^2+y^2+z^2)$

16. Let the variable plane intersect the coordinate axes at $A(a, 0, b)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Then the equation of the plane will be $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ (i)

Let $P(\alpha, \beta, \gamma)$ be the centroid of tetrahedron $OABC$. Then,

$\alpha = \dfrac{a}{4}$, $\beta = \dfrac{b}{4}$ and $\gamma = \dfrac{c}{4}$, or $a = 4\alpha$, $b = 4\beta$ and $c = 4\gamma$

$\Rightarrow$ Volume of tetrahedron = (Area of ΔAOB) OC

$$\Rightarrow 64k^3 = \frac{1}{3}\left(\frac{1}{2}ab\right)c = \frac{abc}{6} \Rightarrow 64\,k^3 = \frac{(4\alpha)(4\beta)(4\gamma)}{6} \Rightarrow k^3 = \frac{\alpha\beta\gamma}{6}$$

Therefore, the required locus of $P(\alpha, \beta, \gamma)$ is $xyz = 6k^3$

Objective Type

1. **b.** $x^2 - 5x + 6 = 0$
   $\Rightarrow x - 2 = 0, x - 3 = 0$
   which represents planes.
2. **c.** We have $z = 0$ for the point, where the line intersects the curve.
   Therefore, $\frac{x-2}{3} = \frac{y+1}{2} = \frac{0-1}{-1}$
   $\Rightarrow \frac{x-2}{3} = 1$ and $\frac{y+1}{2} = 1$
   $\Rightarrow x = 5$ and $y = 1$
   Putting these values in $xy = c^2$, we get
   $5 = c^2 \Rightarrow c = \pm\sqrt{5}$
3. **a.** $4(2) - 2(3) - 1(2) = 0$
   Also, point $(-3, 4, -5)$ does not lie on the plane.
   Therefore, the line is parallel to the plane.
4. **c.** The given plane passes through $\vec{a}$ and is parallel to the vectors $\vec{b} - \vec{a}$ and $\vec{c}$. So it is normal to $(\vec{b} - \vec{a}) \times \vec{c}$. Hence, its equation is
   $(\vec{r} - \vec{a}) \cdot ((\vec{b} - \vec{a}) \times \vec{c}) = 0$
   or $\vec{r} \cdot (\vec{b} \times \vec{c} + \vec{c} \times \vec{a}) = [\vec{a}\,\vec{b}\,\vec{c}]$
   The length of the perpendicular from the origin to this plane is
   $$\frac{[\vec{a}\,\vec{b}\,\vec{c}]}{|\vec{b} \times \vec{c} + \vec{c} \times \vec{a}|}$$
5. **b.** Here, $\alpha = \beta = \gamma$
   $\because \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
   $\therefore \cos\alpha = \frac{1}{\sqrt{3}}$
   DC's of PQ are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

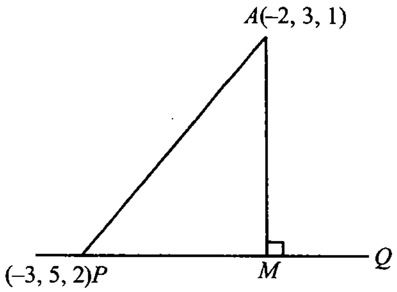


Fig. 3.35

PM = Projection of AP on PQ

$$= \left|(-2+3)\frac{1}{\sqrt{3}}+(3-5)\frac{1}{\sqrt{3}}+(1-2)\frac{1}{\sqrt{3}}\right| = \frac{2}{\sqrt{3}}$$

and $AP = \sqrt{(-2+3)^2+(3-5)^2+(1-2)^2} = \sqrt{6}$

$$AM = \sqrt{(AP)^2-(PM)^2} = \sqrt{6-\frac{4}{3}} = \sqrt{\frac{14}{3}}$$

6. **c.** Given plane is $\vec{r} = (1+\lambda-\mu)\hat{i}+(2-\lambda)\hat{j}+(3-2\lambda+2\mu)\hat{k}$

$\Rightarrow \quad \vec{r} = (\hat{i}+2\hat{j}+3\hat{k})+\lambda(\hat{i}-\hat{j}-2\hat{k})+\mu(-\hat{i}+2\hat{k})$

which is a plane passing through $\vec{a} = \hat{i}+2\hat{j}+3\hat{k}$ and parallel to the vectors $\vec{b} = \hat{i}-\hat{j}-2\hat{k}$ and $\vec{c} = -\hat{i}+2\hat{k}$

Therefore, it is perpendicular to the vector $\vec{n} = \vec{b}\times\vec{c} = -2\hat{i}-\hat{k}$

Hence, equation of plane is $-2(x-1)+(0)(y-2)-(z-3)=0$ or $2x+z=5$

7. **c.** $\hat{a} = \pm\frac{\vec{n}_1\times\vec{n}_2}{|\vec{n}_1\times\vec{n}_2|} = \pm\frac{2\hat{i}+5\hat{j}+3\hat{k}}{\sqrt{38}}$ (where $\vec{n}_1$ and $\vec{n}_2$ are normal to the planes)

8. **a.** Equation of the plane containing L_1, $A(x-2)+B(y-1)+C(z+1)=0$

where $A+2C=0; A+B-C=0$

$\Rightarrow A=-2C,\ B=3C,\ C=C$

$\Rightarrow$ Plane is $-2(x-2)+3(y-1)+z+1=0$ or $2x-3y-z-2=0$

Hence, $p = \left|\frac{-2}{\sqrt{14}}\right| = \sqrt{\frac{2}{7}}$

9. **c.** (1, 2, 3) satisfies the plane $x-2y+z=0$ and also $(\hat{i}+2\hat{j}+3\hat{k})\cdot(\hat{i}-2\hat{j}+\hat{k})=0$

Since the lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ both satisfy (0, 0, 0) and (1, 2, 3), both are same. Given line is obviously parallel to the plane $x-2y+z=6$.

10. **a.** Vector $\left((3\hat{i}-2\hat{j}+\hat{k})\times(4\hat{i}-3\hat{j}+4\hat{k})\right)$ is perpendicular to $2\hat{i}-\hat{j}+m\hat{k}$

$$\Rightarrow \begin{vmatrix} 3 & -2 & 1 \\ 4 & -3 & 4 \\ 2 & -1 & m \end{vmatrix} = 0 \qquad \Rightarrow m=-2$$

11. **a.** x intercept is say x_1

$\Rightarrow$ Plane passes through it

$\therefore\ x_1\,\hat{i}\cdot\vec{n}=q \Rightarrow x_1=\dfrac{q}{\hat{i}\cdot\vec{n}}$

12. **b.** Let direction ratios of the line be (a, b, c), then

$2a-b+c=0$ and $a-b-2c=0$, i.e., $\frac{a}{3}=\frac{b}{5}=\frac{c}{-1}$

Therefore, direction ratios of the line are $(3, 5, -1)$.

Any point on the given line is $(2+\lambda, 2-\lambda, 3-2\lambda)$, it lies on the given plane π if

$2(2+\lambda)-(2-\lambda)+(3-2\lambda)=4$

$\Rightarrow 4+2\lambda-2+\lambda+3-2\lambda=4 \Rightarrow \lambda=-1$

Therefore, the point of intersection of the line and the plane is (1, 3, 5).

Therefore, equation of the required line is

$$\frac{x-1}{3}=\frac{y-3}{5}=\frac{z-5}{-1}$$

13. **c.** Direction ratios of OP are (a, b, c)

Therefore, equation of the plane is

$a(x-a)+b(y-b)+c(z-c)=0$

i.e. $xa+yb+zc=a^2+b^2+c^2$

14. **b.** Let a point $(3\lambda+1, \lambda+2, 2\lambda+3)$ of the first line also lies on the second line

Then $\frac{3\lambda+1-3}{1}=\frac{\lambda+2-1}{2}=\frac{2\lambda+3-2}{3} \Rightarrow \lambda=1$

Hence, the point of intersection P of the two lines is (4, 3, 5).

Equation of plane perpendicular to OP, where O is (0, 0, 0) and passing through P is

$4x+3y+5z=50$.

15. **b.** $1=\left|(\vec{b}-\vec{a})\cdot\dfrac{(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$

$\Rightarrow |\vec{b}-\vec{a}|\cos 60^\circ=1 \Rightarrow AB=2$

16. **a.** $A(1, 1, 1), B(2, 3, 5), C(-1, 0, 2)$ direction ratios of AB are $< 1, 2, 4 >$.
Direction ratios of AC are $<-2, -1, 1>$.
Therefore, direction ratios of normal to plane ABC are $< 2, -3, 1 >$
As a result, equation of the plane ABC is $2x - 3y + z = 0$.
Let the equation of the required plane is $2x - 3y + z = k$, then $\left|\frac{k}{\sqrt{4+9+1}}\right| = 2$

$$k = \pm 2\sqrt{14}$$

Hence, equation of the required plane is $2x - 3y + z + 2\sqrt{14} = 0$

17. **b.** Direction cosines of the given line are $\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}$
Hence, the equation of line can be point in the form $\frac{x-2}{1/3} = \frac{y+3}{-2/3} = \frac{z+5}{-2/3} = r$

Therefore, any point on the line is $\left(2+\frac{r}{3}, -3-\frac{2r}{3}, -5-\frac{2r}{3}\right)$, where $r = \pm 6$.
Points are $(4, -7, -9)$ and $(0, 1, -1)$

18. **d.** Let AD be the perpendicular and D be the foot of the perpendicular which divides BC in the ratio $\lambda : 1$, then

$$D\left(\frac{10\lambda-9}{\lambda+1}, \frac{4}{\lambda+1}, \frac{-\lambda+5}{\lambda+1}\right). \quad \text{(i)}$$

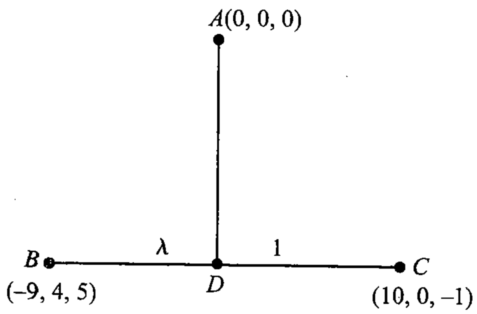


Fig. 3.36

The direction ratios of AD are $\frac{10\lambda-9}{\lambda+1}, \frac{4}{\lambda+1}$ and $\frac{-\lambda+5}{\lambda+1}$ and direction ratios of BC are $19, -4$ and -6.
Since $AD \perp BC$, we get

$$19\left(\frac{10\lambda-9}{\lambda+1}\right) - 4\left(\frac{4}{\lambda+1}\right) - 6\left(\frac{-\lambda+5}{\lambda+1}\right) = 0$$

$$\Rightarrow \lambda = \frac{31}{28}$$

Hence, on putting the value of λ in (i), we get required foot of the perpendicular, i.e., $\left(\frac{58}{59}, \frac{112}{59}, \frac{109}{59}\right)$.

19. **d.** $P_1 = P_2 = 0$, $P_2 = P_3 = 0$ and $P_3 = P_1 = 0$ are lines of intersection of the three planes P_1, P_2 and P_3. As $\vec{n_1}, \vec{n_2}$ and $\vec{n_3}$ are non-coplanar, planes P_1, P_2 and P_3 will intersect at unique point. So the given lines will pass through a fixed point.

20. **d.** Let $A(1, 0, -1), B(-1, 2, 2)$

Direction ratios of segment AB are $< 2, -2, -3 >$.

$$\cos\theta = \frac{|2\times 1 + 3(-2) - 5(-3)|}{\sqrt{1+9+25}\sqrt{4+4+9}} = \frac{11}{\sqrt{17}\sqrt{35}} = \frac{11}{\sqrt{595}}$$

Length of projection = $(AB)\sin\theta$

$$= \sqrt{(2)^2+(2)^2+(3)^2} \times \sqrt{1-\frac{121}{595}}$$

$$= \sqrt{17}\frac{\sqrt{474}}{\sqrt{17}\sqrt{35}} = \sqrt{\frac{474}{35}} \text{ units}$$

21. **c.** Let the point be A, B, C and D.

The number of planes which have three points on one side and the fourth point on the other side is 4. The number of planes which have two points on each side of the plane is 3.

$\Rightarrow$ Number of planes is 7.

22. **a.** Point A is $(a, b, c) \Rightarrow$ Points P, Q, R are $(a, b, -c)$, $(-a, b, c)$ and $(a, -b, c)$, respectively

$\Rightarrow$ Centroid of triangle PQR is $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right) \Rightarrow G \equiv \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$

$\Rightarrow A, O, G$ are collinear $\Rightarrow$ area of triangle AOG is zero.

23. **b.** Let the equation of the plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$\Rightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$

$\Rightarrow$ Volume of tetrahedron $OABC = V = \frac{1}{6}(abc)$

Now $(abc)^{1/3} \geq \dfrac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \geq 3$ (G.M. $\geq$ H.M.)

$\Rightarrow abc \geq 27 \Rightarrow V \geq \frac{9}{2}$

24.

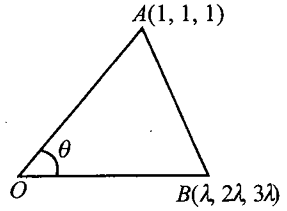


Fig. 3.37

b. Let any point on second line be $(\lambda, 2\lambda, 3\lambda)$

$$\cos\theta = \frac{6}{\sqrt{42}}, \ \sin\theta = \frac{\sqrt{6}}{\sqrt{42}}$$

$$\Delta_{OAB} = \frac{1}{2}(OA)\ OB \sin\theta = \frac{1}{2}\sqrt{3}\ \lambda\ \sqrt{14} \times \frac{\sqrt{6}}{\sqrt{42}} = \sqrt{6}$$

$\Rightarrow \lambda = 2$

So B is $(2, 4, 6)$

25. **a.** Equation of line $x + 2y + z - 1 + \lambda(-x + y - 2z - 2) = 0$ (i)

$x + y - 2 + \mu(x + z - 2) = 0$ (ii)

$(0, 0, 1)$ lies on it $\Rightarrow \lambda = 0, \mu = -2$

For point of intersection, $z = 0$ and solve (i) and (ii).

26. **c.** Since the given lines are parallel.

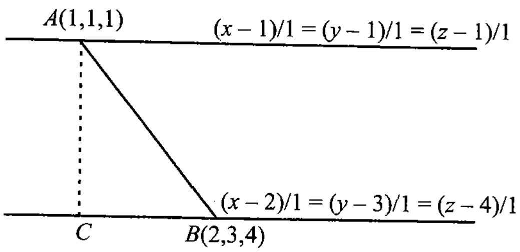


Fig. 3.38

From the figure, we get

$$BC = \frac{(2-1)1}{\sqrt{3}} + \frac{(3-1)1}{\sqrt{3}} + \frac{(4-1)1}{\sqrt{3}} = \frac{1+2+3}{\sqrt{3}} = 2\sqrt{3}$$

$$AB = \sqrt{1+4+9} = \sqrt{14}$$

Shortest distance $= AC = \sqrt{14-12} = \sqrt{2}$

27. **c.** Let $Q\ (\vec{q})$ be the foot of altitude drawn from 'P' to the plane $\vec{r} \cdot \vec{n} = 0$.

$\Rightarrow \vec{q} - \vec{p} = \lambda \vec{n} \quad \Rightarrow \vec{q} = \vec{p} + \lambda \vec{n}$

Also $\vec{q} \cdot \vec{n} = 0 \quad \Rightarrow (\vec{p} + \lambda \vec{n}) \cdot \vec{n} = 0$

$$\Rightarrow \lambda = -\frac{\vec{p} \cdot \vec{n}}{|\vec{n}|^2} \Rightarrow \vec{q} - \vec{p} = -\frac{(\vec{p} \cdot \vec{n})}{|\vec{n}|^2}\vec{n}$$

Thus, required distance $= |\vec{q} - \vec{p}| = \dfrac{|\vec{p} \cdot \vec{n}|}{|\vec{n}|} = |\vec{p} \cdot \hat{n}|$

28. **b.** Given plane is $\vec{r} \cdot \vec{n} = q$ (i)

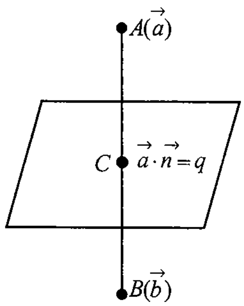


Fig. 3.39

Let the image of $A\ (\vec{a})$ in the plane be $B\ (\vec{b})$.

Equation of AC is $\vec{r} = \vec{a} + \lambda \vec{n}$ ($\because$ AC is normal to the plane) (ii)

Solving (i) and (ii), we get

$$(\vec{a} + \lambda \vec{n}) \cdot \vec{n} = q$$

$$\Rightarrow \lambda = \frac{q - \vec{a} \cdot \vec{n}}{|\vec{n}|^2}$$

$$\therefore\ \overrightarrow{OC} = \vec{a} + \frac{(q - \vec{a} \cdot \vec{n})}{|\vec{n}|^2} \cdot \vec{n}$$

But $\overrightarrow{OC} = \dfrac{\vec{a} + \vec{b}}{2}$

$$\therefore\ \vec{a} + \frac{(q - \vec{a} \cdot \vec{n})\vec{n}}{|\vec{n}|^2} = \frac{\vec{a} + \vec{b}}{2}$$

$$\Rightarrow \vec{b} = \vec{a} + 2\left(\frac{q - \vec{a} \cdot \vec{n}}{|\vec{n}|^2}\right)\vec{n}$$

29. **c.** We must have $\vec{b} \cdot \vec{n} = 0$ (because the line and the plane must be parallel) and $\vec{a} \cdot \vec{n} \neq q$ (as point $\vec{a}$ on the line should not lie on the plane).

30. **c.** Here $l = \cos \dfrac{\pi}{4}$, $m = \cos \dfrac{\pi}{4}$

Let the line make an angle 'γ' with z-axis

$$\therefore\ l^2 + m^2 + n^2 = 1$$

$$\Rightarrow \cos^2 \frac{\pi}{4} + \cos^2 \frac{\pi}{4} + \cos^2 \gamma = 1$$

$$\Rightarrow \frac{1}{2} + \frac{1}{2} + \cos^2 \gamma = 1$$

$$\Rightarrow 2\cos^2 \gamma = 0 \Rightarrow \cos \gamma = 0 \Rightarrow \gamma = \frac{\pi}{2}$$

31. **d.** Let the plane $\vec{r}\cdot(\vec{i}-2\vec{j}+3\vec{k})=17$ divide the line joining the points

$-2\vec{i}+4\vec{j}+7\vec{k}$ and $3\vec{i}-5\vec{j}+8\vec{k}$ in the ratio $t:1$ at point P.

Therefore, point P is

$$\frac{3t-2}{t+1}\vec{i}+\frac{-5t+4}{t+1}\vec{j}+\frac{8t+7}{t+1}\vec{k}$$

This lies on the given plane

$$\therefore\ \frac{3t-2}{t+1}\cdot(1)+\frac{-5t+4}{t+1}(-2)+\frac{8t+7}{t+1}(3)=17$$

Solving, we get

$$t=\frac{3}{10}$$

32. **d.** Let $P(\alpha,\beta,\gamma)$ be the image of the point $Q(-1,3,4)$.

Midpoint of PQ lies on $x-2y=0$. Then,

$$\frac{\alpha-1}{2}-2\left(\frac{\beta+3}{2}\right)=0$$

$$\Rightarrow \alpha-1-2\beta-6=0\Rightarrow \alpha-2\beta=7 \qquad \text{(i)}$$

Also PQ is perpendicular to the plane. Then,

$$\frac{\alpha+1}{1}=\frac{\beta-3}{-2}=\frac{\gamma-4}{0} \qquad \text{(ii)}$$

Solving (i) and (ii), we get

$$\alpha=\frac{9}{5},\beta=-\frac{13}{5},\gamma=4$$

Therefore, image is

$$\left(\frac{9}{5},-\frac{13}{5},4\right)$$

Alternative method:

For image,

$$\frac{\alpha-(-1)}{1}=\frac{\beta-3}{-2}=\frac{\gamma-4}{0}=\frac{-2(-1-2(3))}{(1)^2+(-2)^2}$$

$$\Rightarrow\quad \alpha=\frac{9}{5},\beta=-\frac{13}{5},\gamma=4$$

33. **a.** It is obvious that the given line and plane are parallel.

Given point on the line is $A(2,-2,3)$.

$B(0,0,5)$ is a point on the plane

$\therefore \overrightarrow{AB} = (2-0)\hat{i} + (-2-0)\hat{j} + (3-5)\hat{k}$

Then distance of B from the plane = projection of $\overrightarrow{AB}$ on vector $\hat{i} + 5\hat{j} + \hat{k}$

$$p = \left| \frac{\left(2\hat{i} - 2\hat{j} - 2\hat{k}\right) \cdot \left(\hat{i} + 5\hat{j} + \hat{k}\right)}{\sqrt{1+25+1}} \right|$$

$$= \left| \frac{2-10-2}{\sqrt{27}} \right| = \frac{10}{3\sqrt{3}}$$

34. **d.** Since line of intersection is perpendicular to both the planes, direction ratios of the line of intersection

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 1 \\ 1 & 3 & 2 \end{vmatrix} = 3\hat{i} - 3\hat{j} + 3\hat{k}.$$

Hence, $\cos\alpha = \dfrac{3}{\sqrt{9+9+9}} = \dfrac{1}{\sqrt{3}}$

35. **d.** Let P be the point (1, 2, 3) and PN be the length of the perpendicular from P on the given line.

Coordinates of point N are $(3\lambda + 6, 2\lambda + 7, -2\lambda + 7)$.

Now PN is perpendicular to the given line or vector $3\vec{i} + 2\vec{j} - 2\vec{k}$

$\Rightarrow 3(3\lambda+6-1) + 2(2\lambda+7-2) - 2(-2\lambda+7-3) = 0$

$\Rightarrow \lambda = -1$

Then, point N is (3, 5, 9)

$\Rightarrow PN = 7$

36. **b.** The line is $\dfrac{x+1}{1} = \dfrac{y-1}{2} = \dfrac{z-2}{2}$ and the plane is $2x - y + \sqrt{\lambda}\, z + 4 = 0$.

If θ be the angle between the line and the plane, then $90° - \theta$ is the angle between the line and normal to the plane

$$\Rightarrow \cos(90° - \theta) = \frac{(1)(2) + (2)(-1) + (2)\left(\sqrt{\lambda}\right)}{\sqrt{1+4+4}\sqrt{4+1+\lambda}}$$

$$\Rightarrow \sin\theta = \frac{2-2+2\sqrt{\lambda}}{3\sqrt{5+\lambda}} \Rightarrow \frac{1}{3} = \frac{2\sqrt{\lambda}}{3\sqrt{5+\lambda}}$$

$\Rightarrow \sqrt{5+\lambda} = 2\sqrt{\lambda}$

$\Rightarrow 5 + \lambda = 4\lambda$

$\Rightarrow 3\lambda = 5$

$\Rightarrow \lambda = \dfrac{5}{3}$

37. **d.** The given spheres are

$x^2 + y^2 + z^2 + 7x - 2y - z - 13 = 0$ (i)

and $x^2 + y^2 + z^2 - 3x + 3y + 4z - 8 = 0$ (ii)

Subtracting (ii) from (i), we get

$10x - 5y - 5z - 5 = 0$

$\Rightarrow 2x - y - z = 1$

38. **c.** Plane meets axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Then area of ΔABC,

$$= \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$= \frac{1}{2} |(-a\hat{i} + b\hat{j}) \times (-a\hat{i} + c\hat{k})|$$

$$= \frac{1}{2}\sqrt{(a^2b^2 + b^2c^2 + c^2a^2)}$$

39. **c.** Here $\sin^2 \beta = 3 \sin^2 \theta$ (i)

By the question, $\cos^2\theta + \cos^2\theta + \cos^2\beta = 1$ (ii)

$\Rightarrow \cos^2 \beta = 1 - 2\cos^2 \theta$ (iii)

Adding (i) and (iii), we get

$1 = 1 + 3 \sin^2\theta - 2 \cos^2 \theta$

$\Rightarrow 1 = 1 + 3(1 - \cos^2\theta) - 2\cos^2\theta$

$\Rightarrow 5 \cos^2 \theta = 3$

$\Rightarrow \cos^2 \theta = \frac{3}{5}$

40. **d.** The given sphere is

$$x^2 + y^2 + z^2 + 4x - 2y - 6z - 155 = 0$$

Its centre is $(-2, 1, 3)$ and radius $= \sqrt{4 + 1 + 9 + 155} = \sqrt{169} = 13$

Therefore, distance of centre $(-2, 1, 3)$ from the plane $12x + 4y + 3z = 327$

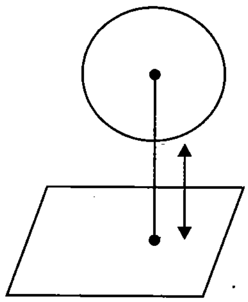


Fig. 3.40

$$= \frac{|12(-2) + 4(1) + 3(3) - 327|}{\sqrt{144 + 16 + 9}} = 26$$

Hence, the shortest distance is 13.

41. **d.** Vector perpendicular to the face OAB is

$$\overrightarrow{OA} \times \overrightarrow{OB} = (\hat{i} + 2\hat{j} + \hat{k}) \times (2\hat{i} + \hat{j} + 3\hat{k})$$
$$= 5\hat{i} - \hat{j} - 3\hat{k}$$

Vector perpendicular to face ABC is

$$\overrightarrow{AB} \times \overrightarrow{AC} = (\hat{i} - \hat{j} + 2\hat{k}) \times (-2\hat{i} - \hat{j} + \hat{k})$$

$$= \hat{i} - 5\hat{j} - 3\hat{k}$$

Since the angle between the face = angle between their normal, therefore

$$\cos\theta = \frac{5+5+9}{\sqrt{35}\sqrt{35}} = \frac{19}{35} \Rightarrow \theta = \cos^{-1}\left(\frac{19}{35}\right)$$

42. **b.** Center of the sphere is (–1, 1, 2) and its radius = $\sqrt{1+1+4+19} = 5$

CL, perpendicular distance of C from plane, is $\left|\frac{-1+2+4+7}{\sqrt{1+4+4}}\right| = 4$

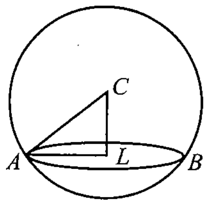


Fig. 3.41

Now $AL^2 = CA^2 - CL^2 = 25 - 16 = 9$

Hence, radius of the circle = $\sqrt{9} = 3$

43. **b.** The lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ (i)

and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ (ii)

are coplanar if $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_1 & n_2 \end{vmatrix} = 0$

or $\begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$

$\Rightarrow k^2 + 3k = 0$

$\Rightarrow k = 0$ or -3

44. **a.** Given lines are

$$\frac{x-5}{3}=\frac{y-7}{-1}=\frac{z+2}{1}=r_1 \quad \text{(say)}$$

and $\dfrac{x+3}{-36}=\dfrac{y-3}{2}=\dfrac{z-6}{4}=r_2$ (say)

$\therefore x=3r_1+5=-36r_2-3$,

$y=-r_1+7=3+2r_2$

and $z=r_1-2=4r_2+6$

On solving, we get

$$x=21, y=\frac{5}{3}, z=\frac{10}{3}$$

45. **c.** The planes are $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$ and $\dfrac{x}{a'}+\dfrac{y}{b'}+\dfrac{z}{c'}=1$

Since the perpendicular distance of the origin on the planes is same, therefore

$$\left|\frac{-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right|=\left|\frac{-1}{\sqrt{\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}}}\right|$$

$$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{a'^2}-\frac{1}{b'^2}-\frac{1}{c'^2}=0$$

46. **a.** The required plane is $\begin{vmatrix} x-3 & y-6 & z-4 \\ 3-3 & 2-6 & 0-4 \\ 1 & 5 & 4 \end{vmatrix}=0$

$$\Rightarrow \begin{vmatrix} x-3 & y-z-2 & z-4 \\ 0 & 0 & -4 \\ 1 & 1 & 4 \end{vmatrix}=0 \quad (\text{Operating } C_2 \to C_2-C_3)$$

$\Rightarrow 4(x-3-y+z+2)=0$

$\Rightarrow x-y+z=1$

47. **b.** Any plane through (1, 0, 0) is $a(x-1)+by+cz=0$. (i)

It passes through (0, 1, 0).

$\therefore \quad a(0-1)+b(1)+c(0)=0 \Rightarrow -a+b=0$ (ii)

(i) makes an angle of $\dfrac{\pi}{4}$ with $x+y=3$, therefore

$$\cos\frac{\pi}{4}=\frac{a(1)+b(1)+c(0)}{\sqrt{a^2+b^2+c^2}\sqrt{1+1+0}}$$

$\Rightarrow \dfrac{1}{\sqrt{2}} = \dfrac{a+b}{\sqrt{2}\sqrt{a^2+b^2+c^2}}$

$\Rightarrow a+b = \sqrt{a^2+b^2+c^2}$

Squaring, we get

$a^2+b^2+2ab = a^2+b^2+c^2$

$\Rightarrow 2ab = c^2 \Rightarrow 2a^2 = c^2$ (using (ii))

$\Rightarrow c = \sqrt{2}\,a$

Hence, $a : b : c = a : a : \sqrt{2}\,a$

$= 1 : 1 : \sqrt{2}$

48. **b.** The equation of the line through the centre $\hat{j}+2\hat{k}$ and normal to the given plane is

$\vec{r} = \hat{j}+2\hat{k}+\lambda(\hat{i}+2\hat{j}+2\hat{k})$ (i)

This meets the plane for which

$[\hat{j}+2\hat{k}+\lambda(\hat{i}+2\hat{j}+2\hat{k})]\cdot(\hat{i}+2\hat{j}+2\hat{k}) = 15$

$\Rightarrow 6+9\lambda = 15 \Rightarrow \lambda = 1$

Putting in (i), we get

$\vec{r} = \hat{j}+2\hat{k}+(\hat{i}+2\hat{j}+2\hat{k}) = \hat{i}+3\hat{j}+4\hat{k}$

Hence, centre is (1, 3, 4).

49. **c.** Equations of the planes through $y = mx$, $z = c$ and $y = -mx$, $z = -c$ are respectively,

$(y-mx)+\lambda_1(z-c) = 0$ (i)

and $(y+mx)+\lambda_2(z+c) = 0$ (ii)

It meets at x-axis, i.e., $y = 0 = z$.

$\therefore \lambda_2 = \lambda_1$

From (i) and (ii), $\dfrac{y-mx}{z-c} = \dfrac{y+mx}{z+c}$

$\therefore cy = mzx$

50. **c.** Let $Q\,(\vec{q})$ be the foot of altitude drawn from

$P\,(\vec{p})$ to the line $\vec{r} = \vec{a}+\lambda\vec{b}$,

$\Rightarrow (\vec{q}-\vec{p})\cdot\vec{b} = 0$ and $\vec{q} = \vec{a}+\lambda\vec{b}$

$\Rightarrow (\vec{a}+\lambda\vec{b}-\vec{p})\cdot\vec{b} = 0$

$\Rightarrow (\vec{a}-\vec{p})\cdot\vec{b}+\lambda|\vec{b}|^2 = 0$

$\Rightarrow \lambda = \dfrac{(\vec{p}-\vec{a})\cdot\vec{b}}{|\vec{b}|^2}$

$$\Rightarrow \vec{q} - \vec{p} = \vec{a} + \frac{((\vec{p} - \vec{a}) \cdot \vec{b})\vec{b}}{|\vec{b}|^2} - \vec{p}$$

$$\Rightarrow |\vec{q} - \vec{p}| = \left|(\vec{a} - \vec{p}) + \frac{((\vec{p} - \vec{a}) \cdot \vec{b})\vec{b}}{|\vec{b}|^2}\right|$$

51. **b.** Coordinates of L and M are $(0, b, c)$ and $(a, 0, c)$, respectively. Therefore, the equation of the plane passing through $(0, 0, 0)$, $(0, b, c)$ and $(a, 0, c)$ is

$$\begin{vmatrix} x-0 & y-0 & z-0 \\ 0 & b & c \\ a & 0 & c \end{vmatrix} = 0 \text{ or } \frac{x}{a} + \frac{y}{b} - \frac{z}{c} = 0$$

52. **c.** We must have $\vec{b} \cdot \vec{n} = 0$ and $\vec{a} \cdot \vec{n} = q$.

53. **b.** We have $\vec{s} - \vec{p} = \lambda \vec{n}$ and $\vec{s} \cdot \vec{n} = q$.

$$\Rightarrow (\lambda \vec{n} + \vec{p}) \cdot \vec{n} = q$$

$$\Rightarrow \lambda = \frac{q - \vec{p} \cdot \vec{n}}{|\vec{n}|^2}$$

$$\Rightarrow \vec{s} = \vec{p} + \frac{(q - \vec{p} \cdot \vec{n})\vec{n}}{|\vec{n}|^2}$$

54. **d.** Line of intersection of $\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) = 0$ and $\vec{r} \cdot (3\hat{i} + 3\hat{j} + \hat{k}) = 0$ will be parallel to $(3\hat{i} + 3\hat{j} + \hat{k}) \times (\hat{i} + 2\hat{j} + 3\hat{k})$, i.e., $7\hat{i} - 8\hat{j} + 3\hat{k}$.

If the required angle is θ, then

$$\cos \theta = \frac{7}{\sqrt{49 + 64 + 9}} = \frac{7}{\sqrt{122}}$$

55. **c.** Given one vertex $A(7, 2, 4)$ and line $\frac{x+6}{5} = \frac{y+10}{3} = \frac{z+14}{8}$

General point on above line $B \equiv (5\lambda - 6, 3\lambda - 10, 8\lambda - 14)$

Direction ratios of line AB are $<5\lambda - 13, 3\lambda - 12, 8\lambda - 18>$

Direction ratios of line BC are $<5, 3, 8>$

since angle between AB and BC is $\pi/4$

$$\cos \frac{\pi}{4} = \frac{(5\lambda - 3)5 + 3(3\lambda - 12) + 8(8\lambda - 18)}{\sqrt{5^2 + 3^2 + 8^2} \cdot \sqrt{(5\lambda - 13)^2 + (3\lambda - 12)^2 + (8\lambda - 18)^2}}$$

Squaring and solving, we have $\lambda = 3, 2$

Hence equation of lines are $\frac{x-7}{2} = \frac{y-2}{-3} = \frac{z-4}{6}$ and $\frac{x-7}{3} = \frac{y-2}{6} = \frac{z-4}{2}$

56. **a.** $\vec{r}\cdot\vec{n_1}+\lambda\vec{r}\cdot\vec{n_2}=q_1+\lambda q_2$ (i)

where λ is a parameter.

So, $\vec{n_1}+\lambda\vec{n_2}$ is normal to plane (i). Now, any plane parallel to the line of intersection of the planes $\vec{r}\cdot\vec{n_3}=q_3$ and $\vec{r}\cdot\vec{n_4}=q_4$ is of the form $\vec{r}\cdot(\vec{n_3}\times\vec{n_4})=d$. Hence we must have

$[\vec{n_1}+\lambda\vec{n_2}]\cdot[\vec{n_3}\times\vec{n_4}]=0$

$\Rightarrow [\vec{n_1}\,\vec{n_3}\,\vec{n_4}]+\lambda[\vec{n_2}\,\vec{n_3}\,\vec{n_4}]=0$

$\Rightarrow \lambda=\dfrac{-[\vec{n_1}\,\vec{n_3}\,\vec{n_4}]}{[\vec{n_2}\,\vec{n_3}\,\vec{n_4}]}$

$\Rightarrow$ On putting this value in Eq. (i), we have the equation of the required plane as

$\vec{r}\cdot\vec{n_1}-q_1=\dfrac{[\vec{n_1}\,\vec{n_3}\,\vec{n_4}]}{[\vec{n_2}\,\vec{n_3}\,\vec{n_4}]}(r\cdot\vec{n_2}-q_2)$

$\Rightarrow [\vec{n_2}\,\vec{n_3}\,\vec{n_4}](\vec{r}\cdot\vec{n_1}-q_1)=[\vec{n_1}\,\vec{n_3}\,\vec{n_4}](\vec{r}\cdot\vec{n_2}-q_2)$

57. **c.**

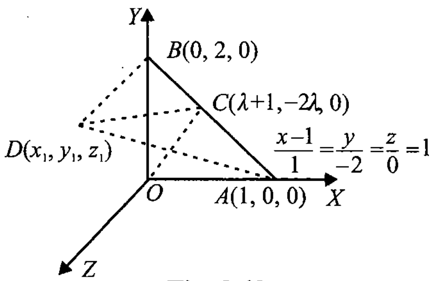


Fig. 3.42

Equation of line AB is $\dfrac{x-1}{1}=\dfrac{y}{-2}=\dfrac{z}{0}=\lambda$

Now $AB\perp OC\Rightarrow 1(\lambda+1)+(-2\lambda)(-2)=0\Rightarrow 5\lambda=-1\Rightarrow\lambda=-\dfrac{1}{5}$

$\Rightarrow C$ is $\left(\dfrac{4}{5},\dfrac{2}{5},0\right)$. Now

$x_1^2+(y_1-2)^2+z_1^2=4$ (i)

and $(x_1-1)^2+y_1^2+z_1^2=1$ (ii)

Now $OC\perp CD$

$\Rightarrow \left(x_1-\dfrac{4}{5}\right)\dfrac{4}{5}+\left(y_1-\dfrac{2}{5}\right)\dfrac{2}{5}+(z_1-0)\,0=0$ (iii)

From (i) and (ii), we get

$-4y_1 + 2x_1 = 0 \Rightarrow x_1 = 2y_1$

From (iii), putting $x_1 = 2y_1 \Rightarrow 2y_1 = \frac{4}{5} \Rightarrow y_1 = \frac{2}{5} \Rightarrow x_1 = \frac{4}{5}$. Putting this value of x_1 and y_1 in (i), we get

$z_1 = \pm\frac{2}{\sqrt{5}}$

58. **b.** Let $\vec{r} \times \vec{a} = \vec{b} \times \vec{a}$

$\Rightarrow (\vec{r} - \vec{b}) \times \vec{a} = \vec{0} \Rightarrow \vec{r} = \vec{b} + t\vec{a}$

Similarly, other line $\vec{r} = \vec{a} + k\vec{b}$, where t and k are scalars.

Now $\vec{a} + k\vec{b} = \vec{b} + t\vec{a}$

$\Rightarrow t = 1, k = 1$

(equating the coefficients of $\vec{a}$ and $\vec{b}$)

$\therefore \vec{r} = \vec{a} + \vec{b} = \hat{i} + \hat{j} + 2\hat{i} - \hat{k} = 3\hat{i} + \hat{j} - \hat{k}$

i.e., (3, 1, – 1)

59. **a.** Let the point P be (x, y, z), then the vector $x\hat{i} + y\hat{j} + z\hat{k}$ will lie on the line

$\Rightarrow (x-1)\hat{i} + (y-1)\hat{j} + (z-1)\hat{k} = -\lambda\hat{i} + \lambda\hat{j} - \lambda\hat{k}$

$\Rightarrow x = 1 - \lambda, y = 1 + \lambda$ and $z = 1 - \lambda$

Now point P is nearest to the origin $\Rightarrow D = (1 - \lambda)^2 + (1 + \lambda)^2 + (1 - \lambda)^2$

$\Rightarrow \frac{dD}{d\lambda} = -4(1 - \lambda) + 2(1 + \lambda) = 0 \Rightarrow \lambda = \frac{1}{3}$

$\Rightarrow$ the point is $\left(\frac{2}{3}, \frac{4}{3}, \frac{2}{3}\right)$

60. **b.** Let P be the point and it divides the line segment in the ratio $\lambda : 1$. Then,

$\overrightarrow{OP} = \vec{r} = \frac{-3\lambda + 2}{\lambda + 1}\hat{i} + \frac{5\lambda - 4}{\lambda + 1}\hat{j} + \frac{-8\lambda - 7}{\lambda + 1}\hat{k}$

It satisfies $\vec{r} \cdot (\hat{i} - 2\hat{j} + 3\hat{k}) = 13$. So,

$\frac{-3\lambda + 2}{\lambda + 1} - 2\frac{5\lambda - 4}{\lambda + 1} + 3\frac{-8\lambda - 7}{\lambda + 1} = 13$

or $-3\lambda + 2 - 2(5\lambda - 4) + 3(-8\lambda - 7) = 13(\lambda + 1)$

or $-37\lambda - 11 = 13\lambda + 13$ or $50\lambda = -24$ or $\lambda = -\frac{12}{25}$

61. **d.** $\vec{V_1}, \vec{V_2}, \overrightarrow{PS}$ are in the same plane

$\therefore (2\hat{i} - \hat{j} + 3\hat{k}) \times (-3\hat{i} + \hat{k}) \cdot ((x + 2)\hat{i} + (y - 1)\hat{j} + z\hat{k}) = 0$

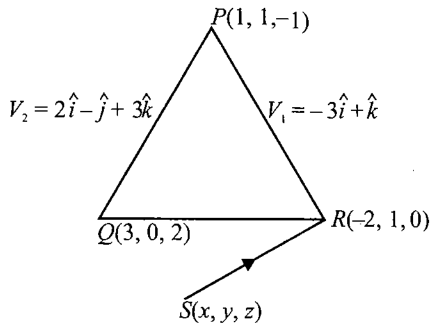


Fig. 3.43

62. **a.** $\overrightarrow{AB} = \vec{\beta} - \vec{\alpha} = -2\hat{i} - 3\hat{j} - 6\hat{k}$

Equation of the plane passing through B and perpendicular to AB is

$$(\vec{r} - \overrightarrow{OB}) \cdot \overrightarrow{AB} = 0$$

$$\vec{r} \cdot (2\hat{i} + 3\hat{j} + 6\hat{k}) + 28 = 0$$

Hence the required distance from $\vec{r} = -\hat{i} + \hat{j} + \hat{k}$

$$= \left| \frac{(-\hat{i} + \hat{j} + \hat{k}) \cdot (2\hat{i} + 3\hat{j} + 6\hat{k}) + 28}{|2\hat{i} + 3\hat{j} + 6\hat{k}|} \right| = \left| \frac{-2 + 3 + 6 + 28}{7} \right| = 5 \text{ units}$$

63. **a.** Both the lines pass through origin. Line L_1 is parallel to the vector $\vec{V_1}$

$$\vec{V_1} = (\cos\theta + \sqrt{3})\hat{i} + (\sqrt{2}\sin\theta)\hat{j} + (\cos\theta - \sqrt{3})\hat{k}$$

and L_2 is parallel to the vector $\vec{V_2}$

$$\vec{V_2} = a\hat{i} + b\hat{j} + c\hat{k}$$

$$\therefore \cos\alpha = \frac{\vec{V_1} \cdot \vec{V_2}}{|\vec{V_1}||\vec{V_2}|}$$

$$= \frac{a(\cos\theta + \sqrt{3}) + (b\sqrt{2})\sin\theta + c(\cos\theta - \sqrt{3})}{\sqrt{a^2 + b^2 + c^2}\sqrt{(\cos\theta + \sqrt{3})^2 + 2\sin^2\theta + (\cos\theta - \sqrt{3})^2}}$$

$$= \frac{(a + c)\cos\theta + b\sqrt{2}\sin\theta + (a - c)\sqrt{3}}{\sqrt{a^2 + b^2 + c^2}\sqrt{2 + 6}}$$

For $\cos\alpha$ to be independent of θ, we get
$a + c = 0$ and $b = 0$

$\therefore \cos\alpha = \dfrac{2a\sqrt{3}}{a\sqrt{2}\,2\sqrt{2}} = \dfrac{\sqrt{3}}{2}$

$\Rightarrow \alpha = \dfrac{\pi}{6}$.

64. **d.** Given lines are $\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + l(3\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = -3\hat{i} - 7\hat{j} + 6\hat{k} + m(-3\hat{i} + 2\hat{j} + 4\hat{k})$

Required shortest distance

$$= \frac{|(6\hat{i} + 15\hat{j} - 3\hat{k}) \cdot ((3\hat{i} - \hat{j} + \hat{k}) \times (-3\hat{i} + 2\hat{j} + 4\hat{k}))|}{|(3\hat{i} - \hat{j} + \hat{k}) \times (-3\hat{i} + 2\hat{j} + 4\hat{k})|}$$

$$= \frac{|(6\hat{i} + 15\hat{j} - 3\hat{k}) \cdot (-6\hat{i} - 15\hat{j} + 3\hat{k})|}{|-6\hat{i} - 15\hat{j} + 3\hat{k}|}$$

$$= \frac{36 + 225 + 9}{\sqrt{36 + 225 + 9}} = \frac{270}{\sqrt{270}} = \sqrt{270} = 3\sqrt{30}$$

65. **b.** The required line passes through the point $\hat{i} + 3\hat{j} + 2\hat{k}$ and is perpendicular to the lines $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(2\hat{i} + \hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} + 6\hat{j} + \hat{k}) + \mu(\hat{i} + 2\hat{j} + 3\hat{k})$; therefore it is parallel to the vector $\vec{b} = (2\hat{i} + \hat{j} + \hat{k}) \times (\hat{i} + 2\hat{j} + 3\hat{k}) = (\hat{i} - 5\hat{j} + 3\hat{k})$.

Hence, the equation of the required line is

$\vec{r} = (\hat{i} + 3\hat{j} + 2\hat{k}) + \lambda(\hat{i} - 5\hat{j} + 3\hat{k})$

66. **d.** Here, the required plane is

$a(x - 4) + b(y - 3) + c(z - 2) = 0$

Also $a + b + 2c = 0$ and $a - 4b + 5c = 0$

Solving, we have

$\dfrac{a}{5+8} = \dfrac{b}{2-5} = \dfrac{c}{-4-1} = k$

$\dfrac{a}{13} = \dfrac{b}{-3} = \dfrac{c}{-5} = k$

Therefore, the required equation of plane is $-13x + 3y + 5z + 33 = 0$

67. **b.** Plane passing through the line of intersection of planes $4y + 6z = 5$ and $2x + 3y + 5z = 5$ is

$(4y + 6z - 5) + \lambda(2x + 3y + 5z - 5) = 0$, or

$2\lambda x + (3\lambda + 4)y + (5\lambda + 6)z - 5\lambda - 5 = 0$

Clearly, for $\lambda = -3$, we get the plane $6x + 5y + 9z = 10$.

Hence, the given three planes have common line of intersection.

68. **c.** The equation of a plane through the line of intersection of the planes $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$ is

$(ax + by + cz + d) + \lambda(a'x + b'y + c'z + d') = 0$

or $x(a + \lambda a') + y(b + \lambda b') + z(c + \lambda c') + d + \lambda d' = 0$ (i)

This is parallel to x-axis, i.e., $y = 0, z = 0$. Therefore,

$1(a + \lambda a') + 0(b + \lambda b') + 0(c + \lambda c') = 0$

$\Rightarrow \lambda = -\frac{a}{a'}$

Putting the value of λ in (i), the required plane is $y(a'b - ab') + z(a'c - ac') + a'd - ad' = 0$

or $(ab' - a'b)y + (ac' - a'c)z + ad' - a'd = 0$

69. **b.** Any plane through (2, 2, 1) is

$a(x - 2) + b(y - 2) + c(z - 1) = 0$ (i)

It passes through (9, 3, 6) if $7a + b + 5c = 0$. (ii)

Also (i) is perpendicualr to $2x + 6y + 6z - 1 = 0$, we have

$2a + 6b + 6c = 0$

$\therefore a + 3b + 3c = 0$ (iii)

$\therefore \frac{a}{-12} = \frac{b}{-16} = \frac{c}{20}$ or $\frac{a}{3} = \frac{b}{4} = \frac{c}{-5}$ (from (ii) and (iii))

Therefore, the required plane is $3(x - 2) + 4(y - 2) - 5(z - 1) = 0$ or $3x + 4y - 5z - 9 = 0$.

70. **a.** Since line is parallel to the plane vector, $2\vec{i} + 3\vec{j} + \lambda\vec{k}$ is perpendicular to the normal to the plane $2\vec{i} + 3\vec{j} + 4\vec{k}$

$\Rightarrow 2 \times 2 + 3 \times 3 + 4\lambda = 0$

$\Rightarrow \lambda = -\frac{13}{4}$

71. **a.** Any plane through the given planes is $x + 2y + 3z - 4 + \lambda(4x + 3y + 2z + 1) = 0$

It passes through (0, 0, 0). Therefore,

$-4 + \lambda = 0$

$\therefore \lambda = 4$

Therefore, the required plane is $x + 2y + 3z + 4(4x + 3y + 2z) = 0$ or $17x + 14y + 11z = 0$.

72. **a.** The equation of the plane through the line of intersection of the planes $4x + 7y + 4z + 81 = 0$ and $5x + 3y + 10z = 25$ is $(4x + 7y + 4z + 81) + \lambda(5x + 3y + 10z - 25) = 0$

$\Rightarrow (4 + 5\lambda)x + (7 + 3\lambda)y + (4 + 10\lambda)z + 81 - 25\lambda = 0$ (i)

which is perpendicular to $4x + 7y + 4z + 81 = 0$

$\Rightarrow 4(4 + 5\lambda) + 7(7 + 3\lambda) + 4(4 + 10\lambda) = 0$

$\Rightarrow 81\lambda + 81 = 0$

$\Rightarrow \lambda = -1$

Hence the plane is $x - 4y + 6z = 106$

73. **b.** The equation of a plane through the line of intersection of the planes $\vec{r}\cdot\vec{a}=\lambda$ and $\vec{r}\cdot\vec{b}=\mu$ is

$(\vec{r}\cdot\vec{a}-\lambda)+k(\vec{r}\cdot\vec{b}-\mu)=0$ or $\vec{r}\cdot(\vec{a}+k\vec{b})=\lambda+k\mu$ (i)

This passes through the origin, therefore

$$\vec{0}(\vec{a}+k\vec{b})=\lambda+\mu k \Rightarrow k=\frac{-\lambda}{\mu}$$

Putting the value of k in (i), we get the equation of the required plane as

$$\vec{r}\cdot(\mu\vec{a}-\lambda\vec{b})=0 \Rightarrow \vec{r}\cdot(\lambda\vec{b}-\mu\vec{a})=0$$

74. **b.** The lines $\vec{r}=\vec{a}+\lambda(\vec{b}\times\vec{c})$ and $\vec{r}=\vec{b}+\mu(\vec{c}\times\vec{a})$ pass through points $\vec{a}$ and $\vec{b}$, respectively, and are parallel to the vectors $\vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$, respectively. Therefore, they intersect if $\vec{a}-\vec{b}, \vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$ are coplanar and so

$$(\vec{a}-\vec{b})\cdot\{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\}=0$$

$$\Rightarrow (\vec{a}-\vec{b})\cdot([\vec{b}\ \vec{c}\ \vec{a}]\vec{c}-[\vec{b}\ \vec{c}\ \vec{c}]\vec{a})=0$$

$$\Rightarrow ((\vec{a}-\vec{b})\cdot\vec{c})[\vec{b}\ \vec{c}\ \vec{a}]=0$$

$$\Rightarrow \vec{a}\cdot\vec{c}-\vec{b}\cdot\vec{c}=0 \Rightarrow \vec{a}\cdot\vec{c}=\vec{b}\cdot\vec{c}$$

75. **a.** Equation of the plane through $(-1, 0, 1)$ is

$a(x+1)+b(y-0)+c(z-1)=0$ (i)

which is parallel to the given line and perpendicular to the given plane

$-a+2b+3c=0$ (ii)

and $a-2b+c=0$ (iii)

From Eqs. (ii) and (iii), we get

$c=0, a=2b$

From Eq. (i), $2b(x+1)+by=0$

$\Rightarrow 2x+y+2=0$

76. **b.** Eliminating n, we get

$\lambda(l+m)^2+lm=0$

$$\Rightarrow \frac{\lambda l^2}{m^2}+(2\lambda+1)\frac{l}{m}+\lambda=0$$

$$\Rightarrow \frac{l_1l_2}{m_1m_2}=1 \quad \left(\text{product of roots } \frac{l_1}{m_1} \text{ and } \frac{l_2}{m_2}\right)$$

where l_1/m_1 and l_2/m_2 are the roots of this equation, further eliminating m, we get

$\lambda l^2-ln-n^2=0$

$$\Rightarrow \frac{l_1l_2}{n_1n_2}=-\frac{1}{\lambda}$$

Since the lines with direction cosines (l_1, m_1, n_1) and (l_2, m_2, n_2) are perpendicular, we have

$l_1l_2+m_1m_2+n_1n_2=0$

$\Rightarrow 1+1-\lambda=0$

$\Rightarrow \lambda=2$

77. **a.** Direction ratios of the line joining points $P(1, 2, 3)$ and $Q(-3, 4, 5)$ are $-4, 2, 2$ which are direction ratios of the normal to the plane.

Then, equation of plane is $-4x + 2y + 2z = k$.

Also this plane passes through the midpoint of PQ $(-1, 3, 4)$

$\Rightarrow -4(-1)+2(3)+2(4)=k$

$\Rightarrow k=18$

$\Rightarrow$ Equation of plane is $2x-y-z=-9$

Then, intercepts are $(-9/2)$, 9 and 9

78. **c.** $3l+m+5n=0$ (i)

$6mn-2nl+5ml=0$ (ii)

Substituting the value of n from Eq. (i) in Eq. (ii), we get

$6l^2+9lm-6m^2=0$

$$\Rightarrow 6\left(\frac{l}{m}\right)^2+9\left(\frac{l}{m}\right)-6=0$$

$$\therefore \frac{l_1}{m_1}=\frac{1}{2} \text{ and } \frac{l_2}{m_2}=-2$$

From Eq. (i), we get

$$\frac{l_1}{n_1}=-1 \text{ and } \frac{l_2}{n_2}=-2$$

$$\therefore \frac{l_1}{1}=\frac{m_1}{2}=\frac{n_1}{-1}=\sqrt{\frac{l_1^2+m_1^2+n_1^2}{1+4+1}}=\frac{1}{\sqrt{6}}$$

$$\text{and } \frac{l_2}{2}=\frac{m_2}{-1}=\frac{n_2}{-1}=\frac{\sqrt{l_2^2+m_2^2+n_2^2}}{\sqrt{4+1+1}}=\frac{1}{\sqrt{6}}$$

If θ be the angle between the lines, then

$$\cos\theta=\left(\frac{1}{\sqrt{6}}\right)\left(\frac{2}{\sqrt{6}}\right)+\left(\frac{2}{\sqrt{6}}\right)\left(-\frac{1}{\sqrt{6}}\right)+\left(-\frac{1}{\sqrt{6}}\right)\left(-\frac{1}{\sqrt{6}}\right)=\frac{1}{6}$$

$$\therefore \quad \theta=\cos^{-1}\left(\frac{1}{6}\right)$$

79. **b.** Let the equation of the sphere be $x^2+y^2+z^2-ax-by-cz=0$. This meets the axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Let (α, β, γ) be the coordinates of the centroid of the tetrahedron $OABC$. Then

$$\frac{a}{4}=\alpha,\ \frac{b}{4}=\beta,\ \frac{c}{4}=\gamma$$

$\Rightarrow a=4\alpha, b=4\beta, c=4\gamma$

Now, radius of the sphere = $2k$

$\Rightarrow \frac{1}{2}\sqrt{a^2+b^2+c^2} = 2k \Rightarrow a^2+b^2+c^2 = 16k^2$

$\Rightarrow 16(\alpha^2+\beta^2+\gamma^2) = 16k^2$

Hence, the locus of (α, β, γ) is $(x^2+y^2+z^2) = k^2$

80. **a.** Let the foot of the perpendicular from the origin on the given plane be $P(\alpha, \beta, \gamma)$. Since the plane passes through $A(a, b, c)$,

$AP \perp OP \Rightarrow \overrightarrow{AP}\cdot\overrightarrow{OP} = 0$

$\Rightarrow [(\alpha-a)\hat{i}+(\beta-b)\hat{j}+(\gamma-c)\hat{k}]\cdot(\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}) = 0$

$\Rightarrow \alpha(\alpha-a)+\beta(\beta-b)+\gamma(\gamma-c) = 0$

Hence, the locus of (α, β, γ) is

$x(x-a)+y(y-b)+z(z-c) = 0$

$x^2+y^2+z^2-ax-by-cz = 0$

which is a sphere of radius $\frac{1}{2}\sqrt{a^2+b^2+c^2}$

81. **c.** $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} = 0$

82. **a.** Foot of the perpendicular from point $A(\vec{a})$ on the plane $\vec{r}\cdot\vec{n} = d$ is $\vec{a} + \frac{\left(d - \vec{a}\cdot\vec{n}\right)}{|\vec{n}|^2}\vec{n}$

Therefore, equation of the line parallel to $\vec{r} = \vec{a} + \lambda\vec{b}$ in the plane $\vec{r}.\vec{n} = d$ is given by

$$\vec{r} = \vec{a} + \frac{\left(d - \vec{a}.\vec{n}\right)}{|\vec{n}|^2}\vec{n} + \lambda\vec{b}$$

83. **a.** The plane is perpendicular to the line $\frac{x-a}{\cos\theta} = \frac{y+2}{\sin\theta} = \frac{z-3}{0}$.

Hence, the direction ratios of the normal of the plane are $\cos\theta$, $\sin\theta$ and 0. (i)

Now, the required plane passes through the z-axis. Hence the point (0, 0, 0) lies on the plane.

From Eqs. (i) and (ii), we get equation of the plane as

$\cos\theta(x-0)+\sin\theta(y-0)+0(z-0) = 0$

$\cos\theta x+\sin\theta y = 0$

$x+y\tan\theta = 0$

84. **a.** The given line makes angles of $\pi/4$, $\pi/4$ and $\pi/2$ with the x-, y- and z-axes, respectively.

$\Rightarrow$ Direction cosines of the given line are

$\cos(\pi/4), \cos(\pi/4)$ and $\cos(\pi/2)$, or $(1/\sqrt{2}), (1/\sqrt{2})$ and 0.

85. **a.** We must have $(3+4a-12+13)(-9-12a+13)<0$.

$\Rightarrow (a+1)(12a-4)>0$

$\Rightarrow a<-1$ or $a>1/3$

86. **c.** Plane meets axes at $A(2, 0, 0)$, $B(0, 3, 0)$ and $C(0, 0, 6)$.

Then area of ΔABC is

$\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

$=\frac{1}{2}|(-2\hat{i}+3\hat{j})\times(-2\hat{i}+6\hat{j})|$

$=3\sqrt{14}$ sq units

Multiple Correct Answers Type

1. **b., c., d.**

If P be (x, y, z), then from the figure,

$x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$ and $z=r\cos\theta$

$1=r\sin\theta\cos\phi$, $2=r\sin\theta\sin\phi$ and $3=r\cos\theta$

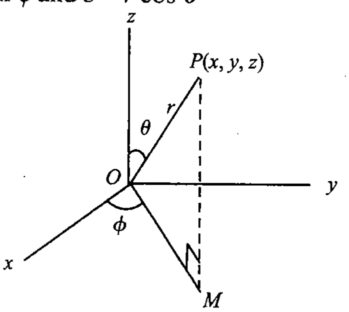


Fig. 3.44

$\Rightarrow 1^2+2^2+3^2=r^2 \Rightarrow r=\pm\sqrt{14}$

$\therefore \sin\theta\cos\phi=\frac{1}{\sqrt{14}}$, $\sin\theta\sin\phi=\frac{2}{\sqrt{14}}$ and $\cos\theta=\frac{3}{\sqrt{14}}$

(neglecting negative sign as θ and ϕ are acute)

$\frac{\sin\theta\sin\phi}{\sin\theta\cos\phi}=\frac{2}{1}\Rightarrow \tan\phi=2$

Also, $\tan\theta=\sqrt{5}/3$

2. a., c.

Plane P_1 contains the line $\vec{r} = \hat{i} + \hat{j} + \hat{k} + \lambda(\hat{i} - \hat{j} - \hat{k})$, hence contains the point $\hat{i} + \hat{j} + \hat{k}$ and is normal to vector $(\hat{i} + \hat{j})$.

Hence equation of plane is $(\vec{r} - (\hat{i} + \hat{j} + \hat{k}))\cdot(\hat{i} + \hat{j}) = 0$

or $x + y = 2$

Plane P_2 contains the line $\vec{r} = \hat{i} + \hat{j} + \hat{k} + \lambda(\hat{i} - \hat{j} - \hat{k})$ and point $\hat{j}$

Hence equation of plane is $\begin{vmatrix} x-0 & y-1 & z-0 \\ 1-0 & 1-1 & 1-0 \\ 1 & -1 & -1 \end{vmatrix} = 0$

or $x + 2y - z = 2$

If θ is the acute angle between P_1 and P_2, then

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \left|\frac{(\hat{i} + \hat{j})\cdot(\hat{i} + 2\hat{j} - \hat{k})}{\sqrt{2}\cdot\sqrt{6}}\right| = \frac{3}{\sqrt{2}\cdot\sqrt{6}} = \frac{\sqrt{3}}{2}$$

$$\theta = \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$$

As L is contained in $P_2 \Rightarrow \theta = 0$

3. a., b. $\vec{r}\cdot\vec{n}_1 = q_1$ and $\vec{r}\cdot\vec{n}_2 = q_2$, $\vec{r}\cdot\vec{n}_3 = q_3$ intersect in a line if $[\vec{n}_1 \ \vec{n}_2 \ \vec{n}_3] = 0$. So,

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2a & 1 \\ a & a^2 & 1 \end{vmatrix} = 0$$

$\Rightarrow \ 2a - a^2 - 1 + a + a^2 - 2a^2 = 0$

$\Rightarrow \ 2a^2 - 3a + 1 = 0$

$\Rightarrow \ a = 1/2, 1$

4. a., b. Let the coordinates of the point(s) be a, b and c.

Therefore, the equation of the line passing through (a, b, c) and whose direction ratios are $1, -5$ and -2 is

$$\frac{x-a}{1} = \frac{y-b}{-5} = \frac{z-c}{-2} \qquad \text{(i)}$$

Line (i) intersects the line,

$$\frac{x}{1} = \frac{y+5}{1} = \frac{z+1}{1} \qquad \text{(ii)}$$

Therefore, these are coplanar.

$$\begin{vmatrix} 1 & -5 & -2 \\ 1 & 1 & 1 \\ a & b+5 & c+1 \end{vmatrix} = 0$$

or $a + b - 2c + 3 = 0$

Also, by using same procedure with the second equation, we get the condition

$11a + 15b - 32c + 55 = 0$

5. **a., d.** The equation of the plane passing through the intersection of the planes $2x - y = 0$ and $3z - y = 0$ is

$2x - y + \lambda(3z - y) = 0$ (i)

or $2x - y(\lambda + 1) + 3\lambda z = 0$

Plane (i) is perpendicular to $4x + 5y - 3z = 8$. Therefore,

$4 \times 2 - 5(\lambda + 1) - 9\lambda = 0$

$\Rightarrow 8 - 5\lambda - 5 - 9\lambda = 0$

$\Rightarrow 3 - 14\lambda = 0$

$\Rightarrow \lambda = 3/14$

$$\therefore\ 2x - y + \frac{3}{14}(3z - y) = 0$$

$28x - 17y + 9z = 0$

6. **b., c., d.**

$x + y + z - 1 = 0$

$4x + y - 2z + 2 = 0$

Therefore, the line is along the vector $(\hat{i} + \hat{j} + \hat{k}) \times (4\hat{i} + \hat{j} - 2\hat{k}) = 3\hat{i} - 6\hat{j} + 3\hat{k}$

Let $z = k$. Then $x = k - 1$ and $y = 2 - 2k$

Therefore, $(k - 1, 2 - 2k, k)$ is any point on the line.

Hence, $(-1, 2, 0)$, $(0, 0, 1)$ and $(-1/2, 1, 1/2)$ are the points on the line.

7. **a., b.**

$3x - 6y + 2z + 5 = 0$ (i)

$-4x + 12y - 3z + 3 = 0$ (ii)

Bisectors are $\dfrac{3x - 6y + 2z + 5}{\sqrt{9 + 36 + 4}} = \pm \dfrac{-4x + 12y - 3z + 3}{\sqrt{16 + 144 + 9}}$

The plane which bisects the angle between the planes that contains the origin.

$13(3x - 6y + 2z + 5) = 7(-4x + 12y - 3z + 3)$

$67x - 162y + 47z + 44 = 0$ (iii)

Further, $3 \times (-4) + (-6)(12) + 2 \times (-3) < 0$

Hence, the origin lies in the acute angle.

8. **a., d.** The given lines intersect if $\begin{vmatrix} 2-1 & 3-4 & 4-5 \\ 1 & 1 & \lambda \\ \lambda & 2 & 1 \end{vmatrix} = 0 \Rightarrow \lambda = 0, -1.$

9. **a., c.** The required plane is parallel to the bisector of the given planes.

Bisectors are $\frac{x-y+z-3}{\sqrt{3}}=\pm\frac{x+y+z+4}{\sqrt{3}}$

or $2y+7=0$ and $2x+2y+1=0$. Hence, the planes are $y=0$ and $x+y=0$.

10. **a., d.**

The equation of a plane passing through the line of intersection of the x-y and y-z planes is $z+\lambda x=0,\quad \lambda\in R$

This plane makes an angle 45° with the x-y plane ($z=0$).

$\Rightarrow \cos 45^\circ=\frac{1}{\sqrt{1}\sqrt{\lambda^2+1}}$

$\Rightarrow \lambda=\pm 1$

11. **a., b.** The plane is equally inclined to the lines. Hence, it is perpendicular to the angle bisector of the vectors $2\hat{i}-2\hat{j}-\hat{k}$ and $8\hat{i}+\hat{j}-4\hat{k}$.

Vector along the angle bisectors of the vectors are

$\frac{2\hat{i}-2\hat{j}-\hat{k}}{3}\pm\frac{8\hat{i}+\hat{j}-4\hat{k}}{9}$, or

$\frac{14\hat{i}-5\hat{j}-7\hat{k}}{9}$ and $\frac{-2\hat{i}-7\hat{j}+\hat{k}}{9}$.

Hence, the equations of the planes are $14x-5y-7z=0$ or $2x+7y-z=0$

12. **a., c.**

For line $\frac{x-1}{1}=\frac{y}{-1}=\frac{z-5}{-1}$, point (1, 0, 5) lies on the plane. Also, the vector along the line $\hat{i}-\hat{j}-\hat{k}$ is perpendicular to the normal $\hat{i}+2\hat{j}-\hat{k}$ to the plane. For line $\vec{r}=2\hat{i}-\hat{j}+4\hat{k}+\lambda(3\hat{i}+\hat{j}+5\hat{k})$, point (2, −1, 4) lies on the plane and vector $3\hat{i}+\hat{j}+5\hat{k}$ is perpendicular to the normal $\hat{i}+2\hat{j}-\hat{k}$.

Line $x-y+z=2x+y-z=0$ passes through the origin, which is not on the given plane.

13. **b., c.**

Volume of tetrahedron $ABCD$ is $\frac{1}{6}|[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]|=1$ cubic units.

$$\Rightarrow \begin{vmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ x-0 & y-1 & z-2 \end{vmatrix}=\pm 6$$

$\Rightarrow -2(y-1)-2(z-2)=\pm 6$.

$\Rightarrow y-1+z-2=\pm 3$

$\Rightarrow y+z=6$ or $y+z=0$

14. a., c., d.

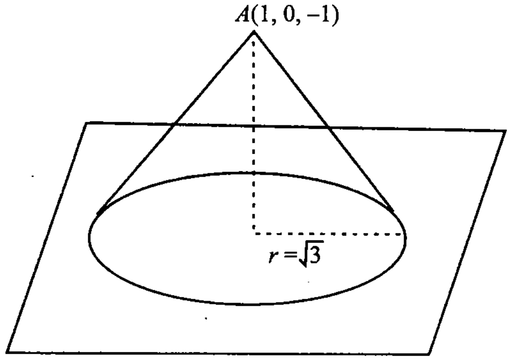


Fig. 3.45

The rod sweeps out the figure which is a cone.

The distance of point $A(1, 0, -1)$ from the plane is $\dfrac{|1-2+4|}{\sqrt{9}} = 1$ unit.

The slant height l of the cone is 2 units.

Then the radius of the base of the cone is $\sqrt{l^2-1} = \sqrt{4-1} = \sqrt{3}$.

Hence, the volume of the cone is $\dfrac{\pi}{3}(\sqrt{3})^2 \cdot 1 = \pi$ cubic units.

Area of the circle on the plane which the rod traces is 3π.

Also, the centre of the circle is $Q(x, y, z)$. Then $\dfrac{x-1}{1} = \dfrac{y-0}{-2} = \dfrac{z+1}{2} = \dfrac{-(1-0-2+4)}{1^2+(-2)^2+2^2}$, or

$Q(x, y, z) \equiv \left(\dfrac{2}{3}, \dfrac{2}{3}, \dfrac{-5}{3}\right)$.

15. b., c.

Distance between the planes is $h = 5/\sqrt{6}$.

Also the figure formed is cylinder, whose radius is $r = 2$ units.

Hence, the volume of the cylinder is $\pi r^2 h = \pi(2)^2 \cdot \dfrac{5}{\sqrt{6}} = \dfrac{20\pi}{\sqrt{6}}$ cubic units.

Also the curved surface area is $2\pi rh = 2\pi(2) \cdot \dfrac{5}{\sqrt{6}} = \dfrac{20\pi}{\sqrt{6}}$

16. **a., b.**

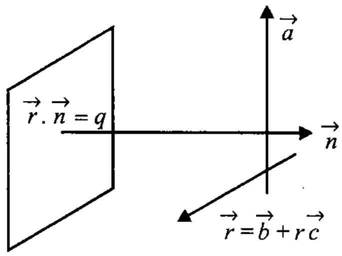


Fig. 3.46

Required line is parallel to $\vec{n} \times \vec{c}$

The equation of line is $\vec{r} = \vec{a} + \lambda(\vec{n} \times \vec{c})$

$\Rightarrow (\vec{r} - \vec{a}) = \lambda(\vec{n} \times \vec{c})$

$\therefore (\vec{r} - \vec{a}) \times (\vec{n} \times \vec{c}) = 0$

Reasoning Type

1. **b.** Given lines are parallel as both are directed along the same vector $(\hat{i} + \hat{j} - \hat{k})$; so they do not intersect. Also Statement 2 is correct by definition of skew lines, but skew lines are those which are neither parallel nor intersecting. Hence, both the statements are true, but Statement 2 is not the correct explanation for Statement 1.
2. **b.** For the given lines, let $\vec{a}_1 = \hat{i} + \hat{j} - \hat{k}$, $\vec{a}_2 = 4\hat{i} - \hat{k}$, $\vec{b}_1 = 3\hat{i} - \hat{j}$ and $\vec{b}_2 = 2\hat{i} + 3\hat{k}$. Therefore,

$$[\vec{a}_2 - \vec{a}_1\ \vec{b}_1\ \vec{b}_2] = \begin{vmatrix} 4-1 & 0-1 & -1+1 \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = 0$$

Hence, the lines are coplanar. Also vectors $\vec{b}_1$ and $\vec{b}_2$ along which the lines are directed are not collinear. Hence, the lines intersect. When $\vec{b} \times \vec{d} = \vec{0}$, vectors $\vec{b}$ and $\vec{d}$ are collinear; therefore, lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{c} + \lambda\vec{d}$ are parallel and do not intersect. But this statement is not the correct explanation for Statement 1.

3. **a.** Any point on the first line is $(2x_1 + 1, x_1 - 3, -3x_1 + 2)$.
   Any point on the second line is $(y_1 + 2, -3y_1 + 1, 2y_1 - 3)$.
   If two lines are coplanar, then $2x_1 - y_1 = 1, x_1 + 3y_1 = 4$ and $3x_1 + 2y_1 = 5$ are consistent.
4. **a.** The direction cosines of segment OA are $\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}$ and $\frac{-3}{\sqrt{14}}$.
   $OA = \sqrt{14}$
   This means OA will be normal to the plane and the equation of the plane is $2x + y - 3z = 14$.
5. **b.** Statement 2 is true as when the line lies in the plane, vector $\vec{b}$ along which the line is directed is perpendicular to the normal $\vec{c}$ of the plane, but it does not explain Statement 1 as for $\vec{b} \cdot \vec{c} = 0$, the line

may be parallel to the plane. However, Statement 1 is correct as any point on the line $(t + 1, 2t, -t - 2)$ lies on the plane for $t \in R$.

6. **a.** $\sin\theta = \left|\dfrac{2-3+2}{\sqrt{4+9+4}\sqrt{3}}\right| = \dfrac{1}{\sqrt{51}}$

Therefore, Statement 1 is true and Statement 2 is also true by definition.

7. **a.** $\overrightarrow{PA}\cdot\overrightarrow{PB} = 9 > 0$. Therefore, P is exterior to the sphere. Statement 2 is also true (standard result).

8. **b.** Obviously the answer is (b).

9. **c.** Any point on the line $\dfrac{x-1}{1} = \dfrac{y}{-1} = \dfrac{z+2}{2}$ is $B(t + 1, -t, 2t - 2)$, $t \in R$.
Also, AB is perpendicular to the line, where A is $(1, 2, -4)$.
$\Rightarrow 1(t) - (-t - 2) + 2(2t + 2) = 0$
$\Rightarrow 6t + 6 = 0$
$\Rightarrow t = -1$
Point B is $(0, 1, -4)$
Hence, $AB = \sqrt{1+1+0} = \sqrt{2}$

10. **b.** Direction ratios of the given lines are $(-3, 1, -1)$ and $(1, 2, -1)$. Hence, the lines are perpendicular as $(-3)(1) + (1)(2) + (-1)(-1) = 0$.

Also lines are coplanar as $\begin{vmatrix} 0-2 & 1-3 & -1+(13/7) \\ -3 & 1 & -1 \\ 1 & 2 & -1 \end{vmatrix} = 0$

But Statement 2 is not enough reason for the shortest distance to be zero, as two skew lines can also be perpendicular.

Linked Comprehension Type

For Problems 1–3

1. b., 2. c., 3. d.

Sol.

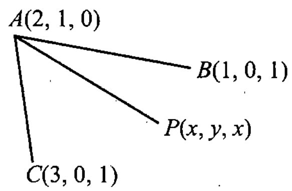


Fig. 3.47

$$\begin{vmatrix} x-2 & y-1 & z \\ 1-2 & 0-1 & 1-0 \\ 3-2 & 0-1 & 1-0 \end{vmatrix} = 0$$

$(x-2)[(-1)-(-1)]-(y-1)[(-1)-1]+z[1+1]=0$

$2(y-1)+2z=0$

$\Rightarrow y+z-1=0$

The vector normal to the plane is $\vec{r}=0\hat{i}+\hat{j}+\hat{k}$

The equation of the line through (0, 0, 2) and parallel to $\vec{n}$ is $\vec{r}=2\hat{k}+\lambda(\hat{j}+\hat{k})$

The perpendicular distance of $D(0, 0, 2)$ from plane ABC is $\left|\dfrac{2-1}{\sqrt{1^2+1^2}}\right| = \dfrac{1}{\sqrt{2}}$.

For Problems 4–6

4. b., 5. c., 6. c.

Sol.

4. **b.** Let $Q(x_2, y_2, z_2)$ be the image of $A(2, 1, 6)$ about mirror $x+y-2z=3$. Then,

$$\frac{x_2-2}{1}=\frac{y_2-1}{1}=\frac{z_2-6}{-2}=\frac{-2(2+1-12-3)}{1^2+1^2+2^2}=4$$

$\Rightarrow (x_2, y_2, z_2) \equiv (6, 5, -2)$.

5. **c.** Let $\dfrac{x-2}{3}=\dfrac{y-1}{4}=\dfrac{z-6}{5}=\lambda$

$x=2+3\lambda, y=1+4\lambda, z=6+5\lambda$ lies on plane $x+y-2z=3$

$\Rightarrow 2+3\lambda+1+4\lambda-2(6+5\lambda)=3$

$\Rightarrow 3+7\lambda-12-10\lambda=3 \Rightarrow -3\lambda=12 \Rightarrow \lambda=-4$

Point $B \equiv (-10, -15, -14)$

6. **c.** The equation of the reflected ray $L_1=0$ is the line joining $Q(x_2, y_2, z_2)$ and $B(-10, -15, -14)$.

$$\frac{x+10}{16}=\frac{y+15}{20}=\frac{z+14}{12}$$

or $\dfrac{x+10}{4}=\dfrac{y+15}{5}=\dfrac{z+14}{3}$

For Problems 7–9

7. b., 8. c., 9. b.

Sol.

The given system of equations is

$2x+py+6z=8$

$x+2y+qz=5$

$x+y+3z=4$

$$\Delta=\begin{vmatrix}2 & p & 6\\ 1 & 2 & q\\ 1 & 1 & 3\end{vmatrix}=(2-p)(3-q)$$

By Cramer's rule, if $\Delta \neq 0$, i.e., $p \neq 2$ and $q \neq 3$, the system has a unique solution.
If $p = 2$ or $q = 3$, $\Delta = 0$, then if $\Delta_x = \Delta_y = \Delta_z = 0$, the system has infinite solutions and if any one of Δ_x, Δ_y and $\Delta_z \neq 0$, the system has no solution.

$$\text{Now } \Delta_x = \begin{vmatrix} 8 & p & 6 \\ 5 & 2 & q \\ 4 & 1 & 3 \end{vmatrix}$$

$$= 30 - 8q - 15p + 4pq = (4q - 15)\cdot(p - 2)$$

$$\Delta_y = \begin{vmatrix} 2 & 8 & 6 \\ 1 & 5 & q \\ 1 & 4 & 3 \end{vmatrix}$$

$$= -8q + 8q = 0$$

$$\Delta_z = \begin{vmatrix} 2 & p & 8 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix}$$

$$= p - 2$$

Thus, if $p = 2$, $\Delta_x = \Delta_y = \Delta_z = 0$ for all $q \in R$, the system has infinite solutions.
If $p \neq 2$, $q = 3$ and $\Delta_z \neq 0$, then the system has no solution.
Hence the system has (i) no solution if $p \neq 2$ and $q = 3$, (ii) a unique solution if $p \neq 2$ and $q \neq 3$ and (iii) infinite solutions if $p = 2$ and $q \in R$.

For Problems 10–12

10. d., 11. b., 12. d.

Sol.

10. **d.** The line $\dfrac{x-1}{3} = \dfrac{y-2}{-1} = \dfrac{z-3}{4} = r$

Any point say $B \equiv (3r + 1, 2 - r, 3 + 4r)$ (on the line L)

$\overrightarrow{AB} = 3r, -r, 4r + 6$

Hence,

$\overrightarrow{AB}$ is parallel to $x + y - z = 1$

$\Rightarrow 3r - r - 4r - 6 = 0$ or $r = -3$

B is $(-8, 5, -9)$

11. **b.** The equation of plane containing the line L is
$A(x-1) + B(y-2) + C(z-3) = 0$, where $3A - B + 4C = 0$ (i)
(i) also contains point $A(1, 2, -3)$.
Hence, $C = 0$ and $3A = B$.
The equation of plane $x - 1 + 3(y - 2) = 0$ or $x + 3y - 7 = 0$

12. **d.** The distance of point $(1+3r,\ 2-r, 3+4r)$ from the plane is

$$\frac{|1+3r+2-r-3-4r-1|}{\sqrt{1+1+1}} = \frac{|2r+1|}{\sqrt{3}} = \frac{4}{\sqrt{3}}$$

$$\Rightarrow r = \frac{3}{2}, -\frac{5}{2}$$

Hence, the points are $A\left(\frac{11}{2}, \frac{1}{2}, \frac{10}{2}\right)$ and $B\left(\frac{-13}{2}, \frac{9}{2}, \frac{-14}{2}\right)$

$$\Rightarrow AB = \sqrt{292}$$

Matrix-Match Type

1. **a → s; b → q; c → p; d → r**

a. Line $x = 2t+1, y = t+2, z = -t-3$ or $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z+3}{-1}$, which is along the vector $2\hat{i} + \hat{j} - \hat{k}$.

Vector $\hat{i} + 3\hat{j} + 5\hat{k}$ is perpendicular to the line.

b. Normals to the planes $x+y+z-3=0$ and $2x-y+3z=0$ are $\vec{n}_1 = \hat{i} + \hat{j} + \hat{k}$ and $\vec{n}_2 = 2\hat{i} - \hat{j} + 3\hat{k}$

Then the vector along the line of intersection of planes is $\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & -1 & 3 \end{vmatrix} = 4\hat{i} - \hat{j} - 3\hat{k}$

c. The shortest distance between the lines $\frac{x}{2} = \frac{y}{-3} = \frac{z}{-1}$ and $\vec{r} = (3\hat{i} - \hat{j} + \hat{k}) + t(\hat{i} + \hat{j} - 2\hat{k})$

occurs along the vector $(2\hat{i} - 3\hat{j} - \hat{k}) \times (\hat{i} + \hat{j} - 2\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & -1 \\ 1 & 1 & -2 \end{vmatrix} = 7\hat{i} + 3\hat{j} + 5\hat{k}$

d. Normal to the plane $\vec{r} = -\hat{i} + 4\hat{j} - 6\hat{k} + \lambda(\hat{i} + 3\hat{j} - 2\hat{k}) + \mu(-\hat{i} + 2\hat{j} - 5\hat{k})$ is $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 2 & -5 \end{vmatrix}$

$= -11\hat{i} + 7\hat{j} + 5\hat{k}$

2. **a → q, s; b → r; c → p, q; d → p**

a. Line $\frac{x-1}{-2} = \frac{y+2}{3} = \frac{z}{-1}$ is along the vector $\vec{a} = -2\hat{i} + 3\hat{j} - \hat{k}$ and

line $\vec{r} = (3\hat{i} - \hat{j} + \hat{k}) + t(\hat{i} + \hat{j} + \hat{k})$ is along the vector $\vec{b} = \hat{i} + \hat{j} + \hat{k}$. Here $\vec{a} \perp \vec{b}$.

Also $\begin{vmatrix} 3-1 & -1-(-2) & 1-0 \\ -2 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} \neq 0$

b. The direction ratios of the line $x-y+2z-4=0=2x+y-3z+5=0$ are $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 2 & 1 & -3 \end{vmatrix} = \hat{i}+7\hat{j}+3\hat{k}$.
Hence, the given two lines are parallel.

c. The given lines are $(x = t-3, y = -2t+1, z = -3t-2)$ and $\vec{r} = (t+1)\hat{i} + (2t+3)\hat{j} + (-t-9)\hat{k}$, or

$$\frac{x+3}{1} = \frac{y-1}{-2} = \frac{z+2}{-3} \text{ and } \frac{x-1}{1} = \frac{y-3}{2} = \frac{z+9}{-1}.$$

The lines are perpendicular as $(1)(1) + (-2)(2) + (-3)(-1) = 0$.

Also $\begin{vmatrix} -3-1 & 1-3 & -2-(-9) \\ 1 & -2 & -3 \\ 1 & 2 & -1 \end{vmatrix} = 0$

Hence, the lines are intersecting.

d. The given lines are $\vec{r} = (\hat{i}+3\hat{j}-\hat{k}) + t(2\hat{i}-\hat{j}-\hat{k})$ and $\vec{r} = (-\hat{i}-2\hat{j}+5\hat{k}) + s(\hat{i}-2\hat{j}+\frac{3}{4}\hat{k})$.

$$\begin{vmatrix} 1-(-1) & 3-(-2) & -1-5 \\ 2 & -1 & -1 \\ 1 & -2 & 3/4 \end{vmatrix} = 0$$

Hence, the lines are coplanar and hence intersecting (as the lines are not parallel).

3. **a → q; b → p; c → s; d → r**

a. The given line is $x = 4y+5$, $z = 3y-6$, or

$$\frac{x-5}{4} = y, \quad \frac{z+6}{3} = y$$

or $\dfrac{x-5}{4} = \dfrac{y}{1} = \dfrac{z+6}{3} = \lambda$ (say)

Any point on the line is of the form $(4\lambda+5, \lambda, 3\lambda-6)$.
The distance between $(4\lambda+5, \lambda, 3\lambda-6)$ and $(5, 3, -6)$ is 3 units (given). Therefore
$(4\lambda+5-5)^2 + (\lambda-3)^2 + (3\lambda-6+6)^2 = 9$
$\Rightarrow 16\lambda^2 + \lambda^2 + 9 - 6\lambda + 9\lambda^2 = 9$
$\Rightarrow 26\lambda^2 - 6\lambda = 0$
$\Rightarrow \lambda = 0, 3/13$
The point is $(5, 0, -6)$

b. The equation of the plane containing the lines $\dfrac{x-2}{3} = \dfrac{y+3}{5} = \dfrac{z+5}{7}$ and parallel to $\hat{i}+4\hat{j}+7\hat{k}$

$$\begin{vmatrix} x-2 & y+3 & z+5 \\ 1 & 4 & 7 \\ 3 & 5 & 7 \end{vmatrix} = 0$$

$\Rightarrow x - 2y + z - 3 = 0$

Point $(-1, -2, 0)$ lies on this plane.

c. The line passing through points $A(2, -3, -1)$ and $B(8, -1, 2)$ is $\frac{x-2}{8-2} = \frac{y+3}{-1+3} = \frac{z+1}{2+1}$ or

$\frac{x-2}{6} = \frac{y+3}{2} = \frac{z+1}{3} = \lambda$ (say).

Any point on this line is of the form $P(6\lambda + 2, 2\lambda - 3, 3\lambda - 1)$, whose distance from point $A(2, -3, -1)$ is 14 units. Therefore,

$\Rightarrow \quad PA = 14$

$\Rightarrow \quad PA^2 = (14)^2$

$\Rightarrow \quad (6\lambda)^2 + (2\lambda)^2 + (3\lambda)^2 = 196$

$\Rightarrow \quad 49\lambda^2 = 196$

$\Rightarrow \quad \lambda^2 = 4$

$\Rightarrow \quad \lambda = \pm 2$

Therefore, the required points are $(14, 1, 5)$ and $(-10, -7 -7)$. The point nearer to the origin is $(14, 1, 5)$.

d. Any point on line AB, $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \lambda$ is $M(2\lambda, 3\lambda + 2, 4\lambda + 3)$. Therefore the direction ratios of PM are $2\lambda - 3$, $3\lambda + 3$ and $4\lambda - 8$.

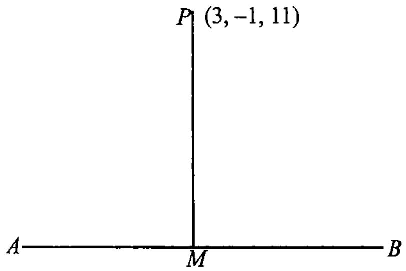


Fig. 3.48

But $PM \perp AB$

$\therefore 2(2\lambda - 3) + 3(3\lambda + 3) + 4(4\lambda - 8) = 0$

$4\lambda - 6 + 9\lambda + 9 + 16\lambda - 32 = 0$

$29\lambda - 29 = 0$

$\lambda = 1$

Therefore, foot of the perpendicular is $M(2, 5, 7)$.

4. **a → r; b → p; c → q; d → s**

a. The given line and plane are $\vec{r} = (2\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - \hat{j} + 4\hat{k})$ and $\vec{r} \cdot (\hat{i} + 5\hat{j} + \hat{k}) = 5$, respectively. Since $(\hat{i} - \hat{j} + 4\hat{k}) \cdot (\hat{i} + 5\hat{j} + \hat{k}) = 0$, line and plane are parallel.

Hence, the required distance = distance of point (2, – 2, 3) from the plane $x + 5y + z - 5 = 0$,

which is $\dfrac{|2-10+3-5|}{\sqrt{1+25+1}} = \dfrac{10}{3\sqrt{3}}$

b. The distance between two parallel planes $\vec{r}\cdot(2i - j + 3k) = 4$ and $\vec{r}\cdot(6i - 3j + 9k) + 13 = 0$ is

$$d = \frac{|4-(-13/3)|}{\sqrt{(2)^2+(-1)^2+(3)^2}} = \frac{(25/3)}{\sqrt{14}} = \frac{25}{3\sqrt{14}}$$

c. The perpendicular distance of the point (2, 5, – 3) from the plane $\vec{r}\cdot(6i - 3j + 2k) = 4$ or $6x - 3y + 2z - 4 = 0$ is

$$d = \frac{|12-15-6-4|}{\sqrt{36+9+4}}$$

$= 13/\sqrt{49} = 13/7$

d. The equation of the line AB is

$$\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$$

The equation of line passing through (1, 0, – 3) and parallel to AB is

$$\frac{x-1}{2} = \frac{y}{3} = \frac{z+3}{-6} = r \text{ (say)}$$

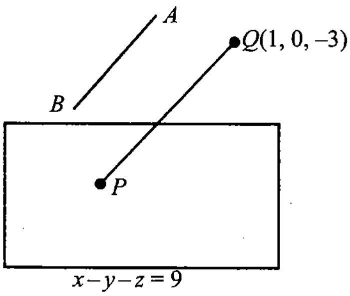


Fig. 3.49

The coordinates of any point on line $P(2r+1, 3r, -6r-3)$ which lie on plane

$(2r+1)-(3r)-(-6r-3)=9$

$r=1$

Point $P \equiv (3, 3, -9)$

Required distance $PQ = \sqrt{(3-1)^2+(3-0)^2+(-9+3)^2} = \sqrt{4+9+36} = 7$

5. **a → q; b → r; c → s; d → p**

a. If the required image is (x, y, z), then $\dfrac{x-3}{2} = \dfrac{y-5}{1} = \dfrac{z-7}{1} = -\dfrac{2(6+5+7+18)}{2^2+1^2+1^2} = -12$

or (–21, –7, –5).

b. Any point on the line $\frac{x-2}{-3}=\frac{y-1}{2}=\frac{z-3}{2}=\lambda$ is $(-3\lambda+2, 2\lambda+1, 2\lambda+3)$, which lies on plane $2x+y-z=3$. Therefore

$-6\lambda+4+2\lambda+1-2\lambda-3=3$

$-6\lambda=1$

$\lambda=-1/6$

Therefore, the point is $\left(\frac{5}{2},\frac{2}{3},\frac{8}{3}\right)$

c. If (x, y, z) is required foot of the perpendicular, then $\frac{x-1}{2}=\frac{y-1}{-2}=\frac{z-2}{4}=-\frac{(2-2+8+5)}{2^2+(-2)^2+4^2}$ or

$(x,y,z)\equiv\left(\frac{-1}{12},\frac{25}{12},\frac{-2}{12}\right)$

d. Any point on the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=\lambda$ is $P(2\lambda+1, 3\lambda+2, 4\lambda+3)$, which satisfies the line

$\frac{x-4}{5}=\frac{y-1}{2}=\frac{z}{1}$ or $\frac{2\lambda+1-4}{5}=\frac{3\lambda+2-1}{2}=\frac{4\lambda+3}{1}$

$\Rightarrow\lambda=-1$

The required point is $(-1,-1,-1)$

Integer Answer Type

1. **(8)** Obviously one in each octant.
2. **(1)** If image of point $(2,-3,3)$ in the plane $x-2y-z+1=0$ is (a, b, c), then

$$\frac{a-2}{1}=\frac{b+3}{-2}=\frac{c-3}{-1}=\frac{-2(2-2(-3)-3+1)}{(1)^2+(-2)^2+(-1)^2}=-2$$

Hence the image is $(0, 1, 5)$

Obviously distance of image of the point from z-axis is 1.

3. **(3)** Let $A(1,0,-1), B(-1,2,2)$

Direction ratios of AB are $(2,-2,-3)$

Let θ be the angle between the line and normal to plane, then

$$\cos\theta=\frac{|2.1+3(-2)-5(-3)|}{\sqrt{1+9+25}\sqrt{4+4+9}}=\frac{11}{\sqrt{17}\sqrt{35}}=\frac{11}{\sqrt{595}}$$

Length of projection

$=(AB)\sin\theta$

$$=\sqrt{(2)^2+(2)^2+(3)^2}\times\sqrt{1-\frac{121}{595}}$$

$$=\sqrt{\frac{474}{35}}\text{ units}$$

4. **(2)** Vector normal to the plane is $\vec{n} = \hat{i} - 3\hat{j} + 2\hat{k}$ and vector along the line is $\vec{v} = 2\hat{i} + \hat{j} - 3\hat{k}$

Now $\sin\theta = \dfrac{\vec{x}\cdot\vec{v}}{|\vec{x}|\,|\vec{v}|} = \left|\dfrac{2-3-6}{\sqrt{14}\,\sqrt{14}}\right| = \left|\dfrac{7}{14}\right|$

Hence $\operatorname{cosec}\theta = 2$

5. **(8)** Volume (V) $= \dfrac{1}{3}A_1 h_1 \quad \Rightarrow \quad h_1 = \dfrac{3V}{A_1}$

Similarly $h_2 = \dfrac{3V}{A_2}$, $h_3 = \dfrac{3V}{A_3}$ and $h_4 = \dfrac{3V}{A_4}$

So $(A_1 + A_2 + A_3 + A_4)(h_1 + h_2 + h_3 + h_4)$

$$= (A_1 + A_2 + A_3 + A_4)\left(\frac{3V}{A_1} + \frac{3V}{A_2} + \frac{3V}{A_3} + \frac{3V}{A_4}\right)$$

$$= 3V(A_1 + A_2 + A_3 + A_4)\left(\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4}\right)$$

Now using A.M.-H.M inequality in A_1, A_2, A_3, A_4, we get

$$\frac{A_1 + A_2 + A_3 + A_4}{4} \geq \frac{4}{\left(\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4}\right)}$$

$$\Rightarrow \quad (A_1 + A_2 + A_3 + A_4)\left(\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4}\right) \geq 16$$

Hence the minimum value of $(A_1 + A_2 + A_3 + A_4)(h_1 + h_2 + h_3 + h_4) = 3V(16) = 48V = 48(1/6) = 8$

6. **(6)** A plane containing the line of intersection of the given planes is

$x - y - z - 4 + \lambda(x + y + 2z - 4) = 0$

i.e., $(\lambda + 1)x + (\lambda - 1)y + (2\lambda - 1)z - 4(\lambda + 1) = 0$

vector normal to it

$V = (\lambda + 1)\hat{i} + (\lambda - 1)\hat{j} + (2\lambda - 1)\hat{k}S$ (i)

Now the vector along the line of intersection of the planes

$2x + 3y + z - 1 = 0$ and $x + 3y + 2z - 2 = 0$ is given by

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 1 \\ 1 & 3 & 2 \end{vmatrix} = 3(\hat{i} - \hat{j} + \hat{k})$$

As $\vec{n}$ is parallel to the plane (i), therefore

$\vec{n}\cdot\vec{V} = 0$

$(\lambda + 1) - (\lambda - 1) + (2\lambda - 1) = 0$

$2 + 2\lambda - 1 = 0 \quad \Rightarrow \quad \lambda = \frac{-1}{2}$

Hence the required plane is

$\frac{x}{2} - \frac{3y}{2} - 2z - 2 = 0$

$x - 3y - 4z - 4 = 0$

Hence $|A + B + C| = 6$

7. (7) Clearly minimum value of $a^2 + b^2 + c^2$

$$= \left(\frac{|(3(0) + 2(0) + (0) - 7|}{\sqrt{(3)^2 + (2)^2 + (1)^2}}\right)^2 = \frac{49}{14} = \frac{7}{2} \text{units}$$

8. (7) $4x + 7y + 4z + 81 = 0$ (i)

$5x + 3y + 10z = 25$ (ii)

Equation of plane passing through their line of intersection is

$(4x + 7y + 4z + 81) + \lambda(5x + 3y + 10z - 25) = 0$

or $(4 + 5\lambda)x + (7 + 3\lambda)y + (4 + 10\lambda)z + 81 - 25\lambda = 0$ (iii)

plane (iii) $\perp$ to (i), so

$4(4 + 5\lambda) + 7(7 + 3\lambda) + 4(4 + 10\lambda) = 0$

$\therefore \lambda = -1$

From (iii), equation of plane is $-x + 4y - 6z + 106 = 0$. (iv)

Distance of (iv) from (0,0,0) $= \frac{106}{\sqrt{1 + 16 + 36}} = \frac{106}{\sqrt{53}}$

9. (9) Line through point $P(-2, 3, -4)$ and parallel to the given line $\frac{x+2}{3} = \frac{2y+3}{4} = \frac{3z+4}{5}$

is $\frac{x+2}{3} = \frac{y + \frac{3}{2}}{2} = \frac{z + \frac{4}{3}}{\frac{5}{3}} = \lambda$

Any point on this line is $Q\left[3\lambda - 2, 2\lambda - \frac{3}{2}, \frac{5}{3}\lambda - \frac{4}{3}\right]$

Direction ratios of PQ are $\left[3\lambda, \frac{4\lambda - 9}{2}, \frac{5\lambda + 8}{3}\right]$

Now PQ is parallel to the given plane $4x + 12y - 3z + 1 = 0$

$\Rightarrow$ line is perpendicular to the normal to the plane

$\Rightarrow \quad 4(3\lambda) + 12\left(\frac{4\lambda - 9}{2}\right) - 3\left(\frac{5\lambda + 8}{3}\right) = 0$

$\Rightarrow \quad \lambda = 2$

$\Rightarrow \quad Q\left(4, \frac{5}{2}, 2\right)$

$\Rightarrow \quad PQ = \sqrt{(6)^2 + \left(\frac{5}{2} - 3\right)^2 + (6)^2} = \frac{17}{2}$

10. **(6)** The given points are $O(0, 0, 0)$, $A(0, 0, 2)$, $B(0, 4, 0)$ and $C(6, 0, 0)$

Here three faces of tetrahedron are xy, yz, zx plane.

Since point P is equidistance from zx, xy and yz planes, its coordinates are $P(r, r, r)$

Equation of plane ABC is

$2x + 3y + 6z = 12$ (from intercept form)

P is also at distance r from plane ABC

$$\Rightarrow \quad \frac{|2r + 3r + 6r - 12|}{\sqrt{4 + 9 + 36}} = r$$

$\Rightarrow \quad |11r - 12| = 7r$

$\Rightarrow \quad 11r - 12 = \pm 7r$

$\Rightarrow \quad r = \frac{12}{18}, 3$

$\therefore \quad r = 2/3$ (as $r < 2$)

Archives

Subjective Type

1. (i) We know that equation of the plane passing through three points $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x - 2 & y - 1 & z - 0 \\ 5 - 2 & 0 - 1 & 1 - 0 \\ 4 - 2 & 1 - 1 & 1 - 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} x - 2 & y - 1 & z \\ 3 & -1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow x + y - 2z = 3$

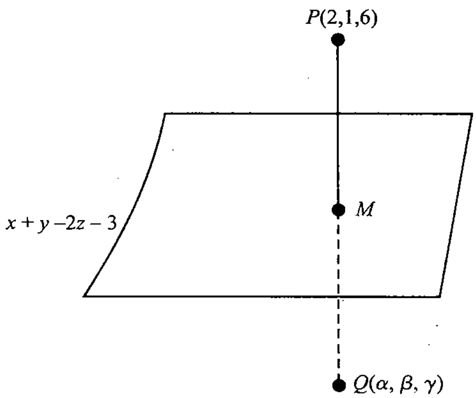


Fig. 3.50

According to the question, we have to find the image of $P(2, 1, 6)$ in the plane.

Let Q be (α, β, γ). Then $\frac{\alpha-2}{1}=\frac{\beta-1}{1}=\frac{\gamma-6}{-2}=\frac{-2(2+1-12-3)}{1^2+1^2+(-2)^2}=4$

$\Rightarrow Q(\alpha, \beta, \gamma) \equiv Q(6, 5, -2)$.

2. Since the plane is parallel to lines L_1 and L_2 with direction ratios as $(1, 0, -1)$ and $(1, -1, 0)$, a vector perpendicular to L_1 and L_2 will be parallel to the normal $\vec{n}$ to the plane. Therefore,

$$\vec{n}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{vmatrix}=-\hat{i}-\hat{j}-\hat{k}$$

The equation of the plane passing through $(1, 1, 1)$ and having normal vector $\vec{n}=-\hat{i}-\hat{j}-\hat{k}$ is given by

$(\vec{r}-\vec{a})\cdot\vec{n}=0$

$\Rightarrow -1(x-1)-1(y-1)-1(z-1)=0$

$x+y+z=3$

$$\frac{x}{3}+\frac{y}{3}+\frac{z}{3}=1 \qquad \text{(i)}$$

The plane meets the axes at $A(3, 0, 0)$, $B(0, 3, 0)$ and $C(0, 0, 3)$ or $A(3\hat{i})$, $B(3\hat{j})$ and $C(3\hat{k})$.

Hence, the volume of tetrahedron $OABC=\frac{1}{6}[3\hat{i}\ 3\hat{j}\ 3\hat{k}]$

$=\frac{27}{6}=\frac{9}{2}$ cubic units

3. S is the parallelepiped with base point A, B, C and D and upper face points A', B', C' and D'. Let its volume be V_S. By compressing it by upper face A', B', C' and D', a new parallelepiped T is formed whose upper face points are now A'', B'', C'' and D''. Let its volume be V_T.

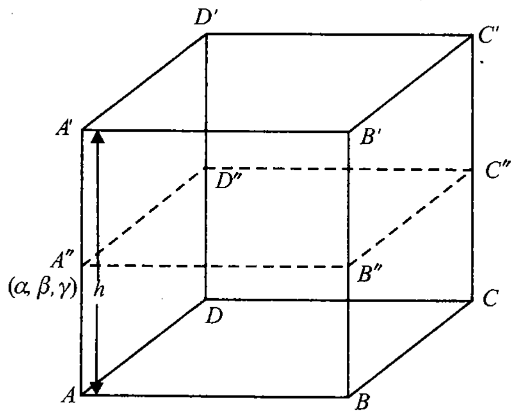


Fig. 3.51

Let h be the height of original parallelepiped S.
Then V_S = (area of $ABCD$) $\times h$ (i)
Let equation of plane $ABCD$ be $ax + by + cz + d = 0$ and $A''(\alpha, \beta, \gamma)$.
Then the height of the new parallelopiped T is the length of the perpendicular from A'' to $ABCD$,

i.e., $\dfrac{a\alpha+b\beta+c\gamma+d}{\sqrt{a^2+b^2+c^2}}$. Therefore

$$V_T = (\text{ar } ABCD) \times \frac{(a\alpha+b\beta+c\gamma+d)}{\sqrt{a^2+b^2+c^2}} \quad \text{(ii)}$$

But given that $V_T = \dfrac{90}{100} V_s$ (iii)

From (i), (ii) and (iii), we get

$$\frac{a\alpha+b\beta+c\gamma+d}{\sqrt{a^2+b^2+c^2}} = 0.9\,h$$

$$\Rightarrow a\alpha + b\beta + c\gamma + d - 0.9\,h\sqrt{a^2+b^2+c^2} = 0$$

Therefore, the locus of A'' (α, β, γ) is $ax + by + cz + d - 0.9h\sqrt{a^2+b^2+c^2} = 0$, which is a plane parallel to $ABCD$. Hence proved.

4. The given line is $2x - y + z - 3 = 0 = 3x + y + z - 5$, which is intersection of the following two planes:

$2x - y + z - 3 = 0$ (i)
$3x + y + z - 5 = 0$ (ii)

Any plane containing this line will be the plane passing through the intersection of planes (i) and (ii). Thus, the plane containing given line can be written as follows:

$(2x - y + z - 3) + \lambda(3x + y + z - 5) = 0$

$(3\lambda + 2)\,x + (\lambda - 1)\,y + (\lambda + 1)\,z + (-5\lambda - 3) = 0$

As its distance from the point (2, 1, – 1) is $1/\sqrt{6}$,

$$\left|\frac{(3\lambda+2)2+(\lambda-1)1+(\lambda+1)(-1)+(-5\lambda-3)}{\sqrt{(3\lambda+2)^2+(\lambda-1)^2+(\lambda+1)^2}}\right| = \frac{1}{\sqrt{6}}$$

$$\left|\frac{\lambda-1}{\sqrt{11\lambda^2+12\lambda+6}}\right| = \frac{1}{\sqrt{6}}$$

Squaring both sides, we get

$$\frac{(\lambda-1)^2}{11\lambda^2+12\lambda+6} = \frac{1}{6}$$

$\Rightarrow 5\lambda^2 + 24\lambda = 0$

$\Rightarrow \lambda(5\lambda + 24) = 0$

$\Rightarrow \lambda = 0 \text{ or } -24/5$

Therefore, the required equations of planes are $2x - y + z - 3 = 0$ and

$$\left[3\left(\frac{-24}{5}\right)+2\right]x+\left[-\frac{24}{5}-1\right]y+\left[-\frac{24}{5}+1\right]z-5\left(\frac{-24}{5}\right)-3=0$$

or, $62x + 29y + 19z - 105 = 0$

5. The direction cosines of the line are $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$.

Any point on the line at a distance t from $P(2, -1, 2)$ is $\left(2+\frac{t}{\sqrt{3}}, -1+\frac{t}{\sqrt{3}}, 2+\frac{t}{\sqrt{3}}\right)$, which lies on $2x + y + z - 9 = 0$

$\Rightarrow \quad t = \sqrt{3}$

Objective Type

Multiple choice questions with one correct answer

1. **a.** As the line $\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-k}{2}$ lies in the plane $2x - 4y + z = 7$, the point $(4, 2, k)$ through which it passes must also lie on the given plane, and hence $2 \times 4 - 4 \times 2 + k = 7$ or $k = 7$.

2. **b.** $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4} = \lambda$

$\Rightarrow x = 2\lambda + 1, y = 3\lambda - 1$ and $z = 4\lambda + 1$

$\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1} = \mu$

$\Rightarrow x = 3 + \mu, y = k + 2\mu$ and $z = \mu$

Since the above lines intersect,

$2\lambda + 1 = 3 + \mu$ (i)

$3\lambda - 1 = 2\mu + k$ (ii)

$\mu = 4\lambda + 1$ (iii)

Solving (i) and (iii) and putting the value of λ and μ in (ii), $k = 9/2$

3. **d.** Let the equation of the variable plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$, which meets the axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

The centroid of ΔABC is $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$ and it satisfies the relation $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} = k$

$\Rightarrow \frac{9}{a^2}+\frac{9}{b^2}+\frac{9}{c^2} = k$

$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} = \frac{k}{9}$ (i)

Also it is given that the distance of the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$ from $(0, 0, 0)$ is 1 unit. Therefore,

$$\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} = 1 \Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} = 1 \quad \text{(ii)}$$

From (i) and (ii), we get $k/9 = 1$, i.e. $k = 9$

4. **d.** The equation of the plane passing through the point (1, – 2, 1) and perpendicular to the planes

$2x - 2y + z = 0$ and $x - y + 2z = 4$ is given by $\begin{vmatrix} x-1 & y+2 & z-1 \\ 2 & -2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 0$

$\Rightarrow x + y + 1 = 0$

Its distance from the point (1, 2, 2) is $\left|\dfrac{1+2+1}{\sqrt{2}}\right| = 2\sqrt{2}$

5. **a.** Any point on the line can be taken as

$Q = \{(1 - 3\mu), (\mu - 1), (5\mu + 2)\}$

$\overrightarrow{PQ} = \{-3\mu - 2, \mu - 3, 5\mu - 4)$

Now, $1(-3\mu - 2) - 4(\mu - 3) + 3(5\mu - 4) = 0$

$\Rightarrow -3\mu - 2 - 4\mu + 12 + 15\mu - 12 = 0$

$\Rightarrow 8\mu = 2 \Rightarrow \mu = 1/4$

6. **c.** Plane 1: $ax + by + cz = 0$ contains line $\dfrac{x}{2} = \dfrac{y}{3} = \dfrac{z}{4}$

$\therefore \quad 2a + 3b + 4c = 0$ (i)

Plane 2: $a'x + b'y + c'z = 0$ is perpendicular to plane containing lines $\dfrac{x}{3} = \dfrac{y}{4} = \dfrac{z}{2}$ and $\dfrac{x}{4} = \dfrac{y}{2} = \dfrac{z}{3}$

$\therefore \quad 3a' + 4b' + 2c' = 0$ and $4a' + 2b' + 3c' = 0$

$\Rightarrow \quad \dfrac{a'}{12-4} = \dfrac{b'}{8-9} = \dfrac{c'}{6-16}$

$\Rightarrow \quad 8a - b - 10c = 0$ (ii)

From (i) and (ii),

$\dfrac{a}{-30+4} = \dfrac{b}{32+20} = \dfrac{c}{-2-24}$

$\Rightarrow$ Equation of plane $x - 2y + z = 0$

7. **a.** Distance of point (1, – 2, 1) from plane $x + 2y - 2z = \alpha$ is 5 $\Rightarrow \alpha = 10$.

Equation of PQ, $\dfrac{x-1}{1} = \dfrac{y+2}{2} = \dfrac{z-1}{-2} = t$

$Q \equiv (t + 1, 2t - 2, -2t + 1)$ and $PQ = 5 \quad \Rightarrow \quad t = \dfrac{5+\alpha}{9} = \dfrac{5}{3} \Rightarrow Q \equiv \left(\dfrac{8}{3}, \dfrac{4}{3}, \dfrac{-7}{3}\right)$

Assertion and reasoning type

1. **d.** The line of intersection of the given plane is $3x - 6y - 2z - 15 = 0 = 2x + y - 2z - 5 = 0$
For $z = 0$, we obtain $x = 3$ and $y = -1$.
$\therefore$ Line passes through $(3, -1, 0)$
Also, the line is parallel to the cross product of normal to given planes, that is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\hat{i} + 2\hat{j} + 15\hat{k}$$

The equation of line is $\dfrac{x-3}{14} = \dfrac{y+1}{2} = \dfrac{z}{15} = t$, whose parametric form is
$x = 3 + 14t,\ y = -1 + 2t,\ z = 15t$
Therefore, Statement 1 is false.
However, Statement 2 is true.

2. **d.** The direction cosines of each of the lines L_1, L_2, L_3 are proportional to $(0, 1, 1)$.

Comprehension type

For Problems 1–3

1. **b.** $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -\hat{i} - 7\hat{j} + 5\hat{k}$

Hence, the unit vector will be $\dfrac{-\hat{i} - 7\hat{j} + 5\hat{k}}{5\sqrt{3}}$

2. **d.** Shortest distance = $\dfrac{(1+2)(-1) + (2-2)(-7) + (1+3)(5)}{5\sqrt{3}} = \dfrac{17}{5\sqrt{3}}$

3. **c.** The plane is given by $-(x + 1) - 7(y + 2) + 5(z + 1) = 0$
$\Rightarrow x + 7y - 5z + 10 = 0$

$\Rightarrow$ Distance = $\dfrac{1 + 7 - 5 + 10}{\sqrt{75}} = \dfrac{13}{\sqrt{75}}$

Matrix-match type

Sol. **a → r; b → q, c → p; d → s**
Here we have the determinant of the coefficient matrix of given equation as

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$= -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= -\frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$

a. $a+b+c \neq 0$

and $a^2+b^2+c^2-ab-bc-ca=0$

$\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2=0$

$\Rightarrow a=b=c$

Therefore, this equation represents identical planes.

b. $a+b+c=0$

and $a^2+b^2+c^2-ab-bc-ca \neq 0$

$\Rightarrow \Delta=0$ and a, b and c are not all equal. Therefore, all equations are not identical but have infinite solutions. Hence,

$ax+by=(a+b)z$ (using $a+b+c=0$)

and $bx+cy=(b+c)z$

$\Rightarrow (b^2-ac)\,y=(b^2-ac)\,z \Rightarrow y=z$

$\Rightarrow ax+by+cy=0 \Rightarrow ax=ay$

$\Rightarrow x=y=z$

Therefore, the equations represent the line $x=y=z$.

c. $a+b+c \neq 0$ and $a^2+b^2+c^2-ab-bc-ca \neq 0$

$\Rightarrow \Delta \neq 0 \Rightarrow$ The equations have only trivial solution, i.e., $x=y=z=0$.

Therefore, the equations represent the planes meeting at a single point, namely origin.

d. $a+b+c=0$ and $a^2+b^2+c^2-ab-bc-ca=0$

$\Rightarrow a=b=c$ and $\Delta=0 \Rightarrow a=b=c=0$

$\Rightarrow$ All equations are satisfied by all x, y and z.

$\Rightarrow$ The equations represent the whole of the three-dimensional space.

Integer Answer Type

1. **(6)** Let normal to plane is $l\hat{i}+m\hat{j}+n\hat{k}$

$2l+3m+4n=0$

and $3l+4m+5n=0$

$$\frac{l}{-1}=\frac{m}{2}=\frac{n}{-1}$$

Equation of plane will be

$a(x-1)+b(y-2)+c(z-3)=0$

$\Rightarrow \; -1(x-1)+2(y-2)-1(z-3)=0$

$\Rightarrow \; -x+1+2y-4-z+3=0$

$\Rightarrow \; -x+2y+z=0$

$\Rightarrow \; x-2y+z=0$

$$\Rightarrow \; \frac{|d|}{\sqrt{6}}=\sqrt{6}$$

$\Rightarrow \; d=6$

Appendix
Solutions to Concept Application Exercises

Chapter 1

Exercise 1.1

1. Since the diagonals of a rhombus bisect each other, $\overrightarrow{OA} = -\overrightarrow{OC}$ and $\overrightarrow{OB} = -\overrightarrow{OD}$ and so

$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = \vec{0}$.

2. Let the position vectors of A, B and C be $\vec{a}$, $\vec{b}$ and $\vec{c}$, respectively. Then the position vectors of D, E and F are $(\vec{b}+\vec{c})/2, (\vec{c}+\vec{a})/2$ and $(\vec{a}+\vec{b})/2$, respectively. Therefore,

$$\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \left(\frac{\vec{b}+\vec{c}}{2} - \vec{a}\right) + \left(\frac{\vec{c}+\vec{a}}{2} - \vec{b}\right) + \left(\frac{\vec{a}+\vec{b}}{2} - \vec{c}\right) = \vec{0}$$

3. Since the diagonals of a parallelogram bisect each other, P is the middle point of AC and BD both. Therefore

$\overrightarrow{OA} + \overrightarrow{OC} = 2\overrightarrow{OP}$ and $\overrightarrow{OB} + \overrightarrow{OD} = 2\overrightarrow{OP}$

4. F is the middle point of BD. Therefore

$\overrightarrow{AB} + \overrightarrow{AD} = 2\overrightarrow{AF}$ (i)

Similarly, $\overrightarrow{CB} + \overrightarrow{CD} = 2\overrightarrow{CE}$ (ii)

Adding (i) and (ii), we get

$$\begin{aligned}\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} &= 2(\overrightarrow{AF} + \overrightarrow{CF}) = -2(\overrightarrow{FA} + \overrightarrow{FC})\\ &= -2(2(\overrightarrow{FE})) \text{ (because } E \text{ is the midpoint of } AC)\\ &= 4\overrightarrow{EF}\end{aligned}$$

5. **b.** We have, $\overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{BO} + \overrightarrow{OC}$

$\Rightarrow \quad \overrightarrow{AB} = \overrightarrow{BC}$

Since the initial point of $\overrightarrow{BC}$ is the terminal point of $\overrightarrow{AB}$, A, B and C are collinear.

6. A vector along the internal bisector $= \dfrac{\vec{a}}{|\vec{a}|} + \dfrac{\vec{b}}{|\vec{b}|} = \dfrac{\hat{i} - 2\hat{j} + 2\hat{k}}{3} + \dfrac{2\hat{i} + \hat{j} + 2\hat{k}}{3}$

$$= \frac{1}{3}(3\hat{i} - \hat{j} + 4\hat{k})$$

7.

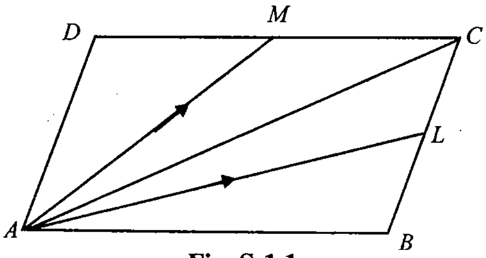


Fig. S-1.1

$$\overrightarrow{AL} = \overrightarrow{AB} + \overrightarrow{BL} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{AD}$$

$$\overrightarrow{AM} = \overrightarrow{AD} + \overrightarrow{DM} = \overrightarrow{AD} + \frac{1}{2}\overrightarrow{DC} = \overrightarrow{AD} + \frac{1}{2}\overrightarrow{AB}$$

Adding, $\overrightarrow{AL} + \overrightarrow{AM} + \frac{3}{2}(\overrightarrow{AB} + \overrightarrow{AD}) = \frac{3}{2}(\overrightarrow{AB} + \overrightarrow{BC}) = \frac{3}{2}\overrightarrow{AC}$

8. We know that the figure formed by the lines joining the midpoints of the sides of a quadrilateral is a parallelogram. Hence, *MPNQ* is a parallelogram, whose diagonals are *MN* and *PQ* intersecting at *E*, which is the midpoint of both *MN* and *PQ*. For any origin *O*, we have $\overrightarrow{OA} + \overrightarrow{OB} = 2(\overrightarrow{OM})$ (as *M* is the midpoint of *AB*).

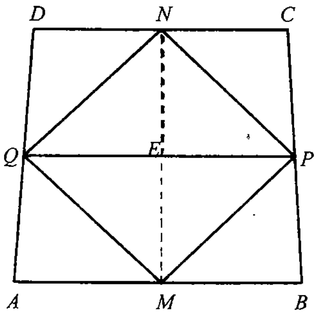


Fig. S-1.2

$$\overrightarrow{OC} + \overrightarrow{OB} = 2(\overrightarrow{ON}) \quad \text{(as } N \text{ is the midpoint of } BC)$$

$$\Rightarrow \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 2(\overrightarrow{OM} + \overrightarrow{ON})$$

$$= 2(2\overrightarrow{OE}) = 4\overrightarrow{OE}$$

where *E* is the midpoint of *MN* as it is the intersection of the diagonals of a parallelogram.

9. We have $\vec{a} = 3\hat{i} + 4\hat{j} - 2\hat{k}$. Therefore,

$$|\vec{a}| = \sqrt{9+16+4} = \sqrt{29}$$

Therefore, the unit vector parallel to $a = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{1}{\sqrt{(29)}}(3\hat{i} + 4\hat{j} - 2\hat{k})$.

Now suppose $\vec{b}$ is the vector which when added to $\vec{a}$ gives the resultant $\hat{i}$.
Then $\vec{a} + \vec{b} = \hat{i}$ or $\vec{b} = \hat{i} - \vec{a} = \hat{i} - (3\hat{i} + 4\hat{j} - 2\hat{k})$. Therefore,
$\vec{b} = -2\hat{i} - 4\hat{j} + 2\hat{k}$

10. $|\overrightarrow{OA}| = |\overrightarrow{OB}| = \sqrt{14}$

ΔAOB is isosceles. Hence, the bisector of angle AOB will bisect the base AB.
Hence P is the midpoint (2, 2, – 2) of AB. Therefore,
$\overrightarrow{OP} = 2(\hat{i} + \hat{j} - \hat{k})$

11. $\vec{r_3} = p\vec{r_1} + q\vec{r_2}$

$$\Rightarrow \vec{r}_3 = \frac{p\vec{r_1} + (1-p)\vec{r_2}}{p + (1-p)}$$

r_3 divides $\vec{r_1}$ and $\vec{r_2}$ in the ratio $(1 - p) : p$

Hence r_1, r_2 and r_3 are collinear.

Exercise 1.2

1. Since $3\vec{a} - 2\vec{b} + \vec{c} - 2\vec{d} = \vec{0}$

$3\vec{a} + \vec{c} = 2\vec{b} + 2\vec{d}$

$$\Rightarrow \frac{3\vec{a} + \vec{c}}{4} = \frac{2\vec{b} + 2\vec{d}}{4} \Rightarrow \frac{3\vec{a} + \vec{c}}{3+1} = \frac{\vec{b} + \vec{d}}{2}$$

Therefore, P.V. of the point dividing AC in the ratio 1 : 3 is the same as the P.V. of midpoint of BD.

So AC and BD intersect at P, whose P.V. is $\dfrac{3\vec{a} + \vec{c}}{4}$ or $\dfrac{\vec{b} + \vec{d}}{2}$. Point P divides AC in the ratio 3 : 1 and BD in the ratio 1 : 1.

2. Consider $2\vec{a} - \vec{b} + 3\vec{c} = x(\vec{a} + \vec{b} - 2\vec{c}) + y(\vec{a} + \vec{b} - 3\vec{c})$

$$\Rightarrow \quad 2\vec{a} - \vec{b} + 3\vec{c} = (x + y)\vec{a} + (x + y)\vec{b} + (-2x - 3y)\vec{c}$$

$$x + y = 2 \qquad \text{(i)}$$

$$x + y = -1 \qquad \text{(ii)}$$

$$-2x - 3y = 3 \qquad \text{(iii)}$$

Multiplying (i) by 3 and adding it to (iii), we get

$$x = 9$$

From (i), $9 + y = 2 \Rightarrow y = -7$

Now putting $x = 9$ and $y = -7$ in (ii), we get

$$9 - 7 = -1$$

or $2 = -1$, which is not true.

Therefore, the given vectors are not coplanar.

Alternative method:

We have determinant of co-efficients as

$$\begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & -2 \\ 1 & 1 & -3 \end{vmatrix} = -3 \neq 0$$

Hence vectors are non-coplanar.

3. **(i)** Let $\vec{a} = \vec{i} + \vec{j} + \vec{k}, \vec{b} = 2\vec{i} + 3\vec{j} - \vec{k}, \vec{c} = -\vec{i} - 2\vec{j} + 2\vec{k}$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ -1 & -2 & 2 \end{vmatrix} = -1$$

Hence vectors are non-coplanar and linearly independent.

(ii) Let $\vec{a} = 3\vec{i} + \vec{j} - \vec{k}, \vec{b} = 2\vec{i} - \vec{j} + 7\vec{k}, \vec{c} = 7\vec{i} - \vec{j} + 13\vec{k}$

$$\begin{vmatrix} 3 & 1 & -1 \\ 2 & -1 & 7 \\ 7 & -1 & 13 \end{vmatrix} = 0$$

Hence vectors are coplanar and linearly dependent.

4. Putting the values of $\vec{A}$ and $\vec{B}$, and then equating the coefficients of $\vec{a}$ and $\vec{b}$ on both sides, we get

$3(p + 4q) = 2(-2p + q + 2)$

$3(2p + q + 1) = 2(2p - 3q - 1)$

$7p + 10q = 4$ and $2p + 9q = -5$

Solving, we get $p = 2$ and $q = -1$

5. Points $A(\ell_1 \vec{a} + m_1 \vec{b} + n_1 \vec{c}), B(\ell_2 \vec{a} + m_2 \vec{b} + n_2 \vec{c}), C(\ell_3 \vec{a} + m_3 \vec{b} + n_3 \vec{c}), D(\ell_4 \vec{a} + m_4 \vec{b} + n_4 \vec{c})$ are coplanar.

$\Rightarrow$ Vectors $\vec{AB} = (\ell_1 - \ell_2)\vec{a} + (m_1 - m_2)\vec{b} + (n_1 - n_2)\vec{c}$,

$\vec{AC} = (\ell_1 - \ell_3)\vec{a} + (m_1 - m_3)\vec{b} + (n_1 - n_3)\vec{c}$,

$\vec{AD} = (\ell_1 - \ell_4)\vec{a} + (m_1 - m_4)\vec{b} + (n_1 - n_4)\vec{c}$

are coplanar

$$\Rightarrow \quad \begin{vmatrix} \ell_1-\ell_2 & m_1-m_2 & n_1-n_2 \\ \ell_1-\ell_3 & m_1-m_3 & n_1-n_3 \\ \ell_1-\ell_3 & m_1-m_3 & n_1-n_4 \end{vmatrix} = 0$$

Now if $\begin{vmatrix} \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$

Then applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, $C_4 \to C_4 - C_1$, we have

$$\begin{vmatrix} \ell_1 & \ell_2-\ell_1 & \ell_3-\ell_1 & \ell_4-\ell_1 \\ m_1 & m_2-m_1 & m_3-m_1 & m_4-m_1 \\ n_1 & n_2-n_1 & n_3-n_1 & n_4-n_1 \\ 1 & 0 & 0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} \ell_1-\ell_2 & m_1-m_2 & n_1-n_2 \\ \ell_1-\ell_3 & m_1-m_3 & n_1-n_3 \\ \ell_1-\ell_3 & m_1-m_3 & n_1-n_4 \end{vmatrix} = 0$$

6. Any vector $\vec{r}$ can be uniquely expressed as a linear combination of three non-coplanar vectors.

Let us choose that $7\vec{a} - 11\vec{b} + 15\vec{c} = x(\vec{a} - 2\vec{b} + 3\vec{c}) + y(2\vec{a} - 3\vec{b} + 4\vec{c}) + z(3\vec{a} - 4\vec{b} + 5\vec{c})$

Comparing the coefficients of $\vec{a}$, $\vec{b}$ and $\vec{c}$ on both sides, we get

$x + 2y + 3z = 7, - 2x - 3y - 4z = - 11, 3x + 4y + 5z = 15$

Eliminating x and then solving for y and z, we get $x = 1, y = 3, z = 0$

Chapter 2

Exercise 2.1

1. $|4\vec{a}+3\vec{b}| = \sqrt{(4\vec{a}+3\vec{b})\cdot(4\vec{a}+3\vec{b})}$

$$= \sqrt{16|\vec{a}|^2 + 9|\vec{b}|^2 + 24\vec{a}\cdot\vec{b}}$$

$$= \sqrt{144+144+24\times3\times4\times\left(\frac{-1}{2}\right)}$$

$$= 12$$

2. It is given that vectors $\hat{i}-2x\hat{j}-3y\hat{k}$ and $\hat{i}+3x\hat{j}+2y\hat{k}$ are orthogonal. Therefore,

$$(\hat{i}-2x\hat{j}-3y\hat{k})\cdot(\hat{i}+3x\hat{j}+2y\hat{k})=0$$

$$\Rightarrow 1-6x^2-6y^2=0$$

$\Rightarrow 6x^2+6y^2=1$, which is a circle.

3. $|\vec{a}+\vec{b}+\vec{c}|^2=(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$

$$=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\vec{a}\cdot\vec{b}+2\vec{b}\cdot\vec{c}+2\vec{c}\cdot\vec{a}$$

$$=1+4+4+0+0+0=9$$

$$\Rightarrow |\vec{a}+\vec{b}+\vec{c}|=3$$

4. Given, $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$$\vec{a}+\vec{b}=-\vec{c}$$

$$(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=(-\vec{c})\cdot(-\vec{c})$$

$$\Rightarrow a^2+b^2+2(\vec{a}\cdot\vec{b})=c^2$$

$$\Rightarrow 9+25+2(\vec{a}\cdot\vec{b})=49$$

$$\Rightarrow \vec{a}\cdot\vec{b}=15/2$$

$$\Rightarrow ab\cos\theta=15/2 \Rightarrow 3\cdot 5\cos\theta=15/2$$

$$\Rightarrow \cos\theta=1/2 \Rightarrow \theta=\pi/3$$

5. $|\vec{a}-\vec{b}|^2=(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$

$$=a^2+b^2-2(\vec{a}\cdot\vec{b})$$

$$=1+1-2(1\cdot 1\cdot\cos\theta)$$

$$=2(1-\cos\theta)$$

$$=2\left(1-\frac{1}{2}\right)=1$$

6. $\hat{n}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, where $a_1^2+a_2^2+a_3^2=1$

Given that $\vec{u}\cdot\hat{n}=0 \Rightarrow a_1+a_2=0$

Also, $\vec{v}\cdot\hat{n}=0 \Rightarrow a_1-a_2=0$

$a_1 = a_2 = 0$

$a_3 = 1$ or ∓ 1

$\hat{n} = \hat{k}$ or $-\hat{k}$

$|\vec{w}\cdot\hat{n}| = 3$

7. $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \vec{a} + \vec{b} + \vec{c}$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \vec{a} + \vec{b}$ or $\overrightarrow{CA} = -(\vec{a} + \vec{b})$

$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \vec{b} + \vec{c}$

Therefore, $\overrightarrow{AB}\cdot\overrightarrow{CD} + \overrightarrow{BC}\cdot\overrightarrow{AD} + \overrightarrow{CA}\cdot\overrightarrow{BD}$

$= \vec{a}\cdot\vec{c} + \vec{b}\cdot(\vec{a} + \vec{b} + \vec{c}) + (-\vec{a} - \vec{b})\cdot(\vec{b} + \vec{c})$

$= \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b} + \vec{b}\cdot\vec{c} - \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c} - \vec{b}\cdot\vec{b} - \vec{b}\cdot\vec{c} = 0$

8. $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \hat{i} - 2\hat{k}$

$\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR} = 4\hat{i} - 4\hat{j} - \hat{k}$

Projection of $\overrightarrow{PQ}$ on $\overrightarrow{RS} = \dfrac{\overrightarrow{QP}\cdot\overrightarrow{RS}}{|\overrightarrow{RS}|} = \dfrac{6}{\sqrt{33}}$

9. $3\vec{p} + \vec{q}$ and $5\vec{p} - 3\vec{q}$ are perpendicular. Therefore,

$(3\vec{p} + \vec{q})\cdot(5\vec{p} - 3\vec{q}) = 0$

$15\vec{p}^2 - 3\vec{q}^2 = 4\vec{p}\cdot\vec{q}$ (i)

$2\vec{p} + \vec{q}$ and $4\vec{p} - 2\vec{q}$ are perpendicular. Therefore,

$(2\vec{p} + \vec{q})\cdot(4\vec{p} - 2\vec{q}) = 0$

$8\vec{p}^2 = 2\vec{q}^2$

$\vec{q}^2 = 4\vec{p}^2$ (ii)

Now, $\cos\theta = \dfrac{\vec{p}\cdot\vec{q}}{|\vec{p}||\vec{q}|}$

Substituting $\vec{q}^2 = 4\vec{p}^2$ in (i), $3\vec{p}^2 = 4\vec{p}\cdot\vec{q}$

$\therefore \cos\theta = \dfrac{3}{4}\cdot\dfrac{\vec{p}^2}{|\vec{p}|2|\vec{p}|} = \dfrac{3}{8}$

$\Rightarrow \theta = \cos^{-1}\dfrac{3}{8}$

10. $\vec{A}\cdot(\alpha\vec{A}+\vec{B})=\vec{B}\cdot(\alpha\vec{A}+\vec{B})$

$\Rightarrow \alpha+\vec{A}\cdot\vec{B}=\alpha\vec{A}\cdot\vec{B}+1$

$\Rightarrow (\vec{A}\cdot\vec{B})(1-\alpha)=(1-\alpha)$

Since $\vec{A}\cdot\vec{B}\neq 0 \Rightarrow \alpha=1$

11. $\vec{a}+\vec{b}+\vec{c}=\vec{x}$

Taking dot with $\vec{x}$ on both sides, we get

$\vec{x}\cdot\vec{a}+\vec{x}\cdot\vec{b}+\vec{x}\cdot\vec{c}=\vec{x}\cdot\vec{x}=|\vec{x}|^2=4$

$\Rightarrow 1+\frac{3}{2}+\vec{x}\cdot\vec{c}=4 \Rightarrow \vec{x}\cdot\vec{c}=\frac{3}{2}$

If θ be the angle between $\vec{c}$ and $\vec{x}$, then $|\vec{x}||\vec{c}|\cos\theta=3/2$

$\Rightarrow \cos\theta=3/4 \Rightarrow \theta=\cos^{-1}(3/4)$

12. Let θ be an angle between unit vectors $\vec{a}+\vec{b}$. Then

$\vec{a}\cdot\vec{b}=\cos\theta$

$|\vec{a}+\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}=2+2\cos\theta=4\cos^2\theta/2$

$\Rightarrow |\vec{a}+\vec{b}|=2\cos\frac{\theta}{2}$

Similarly, $|\vec{a}-\vec{b}|=2\sin\frac{\theta}{2}$

$\Rightarrow |\vec{a}+\vec{b}|+|\vec{a}-\vec{b}|=2\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)\leq 2\sqrt{2}$

13. Resultant force

$$\vec{F}=\vec{P_1}+\vec{P_2}+\vec{P_3}=2\hat{j}-\hat{k}$$

And displacement $=\vec{AB}$

$=$ P.V. of B $-$ P.V. of A

$=(6\hat{i}+\hat{j}-3\hat{k})-(4\hat{i}-3\hat{j}-2\hat{k})$

$=2\hat{i}+4\hat{j}-\hat{k}$

$\therefore$ Work done $=\vec{F}\cdot\vec{AB}$

$=(2\hat{j}-\hat{k})\cdot(2\hat{i}+4\hat{j}-\hat{k})$

$=9$ units

Exercise 2.2

1. $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & -5\\ m & n & 12\end{vmatrix}$

$=(36+5n)\hat{i}-(24+5m)\hat{j}+(2n-3m)\hat{k}=\vec{0}$

$m = \frac{-24}{5}, n = \frac{-36}{5}$

2. $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, but $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

$\Rightarrow \sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} = \frac{4}{5} \Rightarrow \cos\theta = \frac{3}{5}$

Therefore, $\vec{a}\cdot\vec{b} = 2\times5\times\frac{3}{5} = 6$

3. Since $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} \neq \vec{0}$,

$\vec{a}\times\vec{b} - \vec{b}\times\vec{c} = \vec{0}$

$\Rightarrow \vec{a}\times\vec{b} + \vec{c}\times\vec{b} = \vec{0}$

$\Rightarrow (\vec{a}+\vec{c})\times\vec{b} = \vec{0}$

$\Rightarrow \vec{a}+\vec{c}$ is parallel to $\vec{b}$

$\vec{a}+\vec{c} = k\vec{b}$

4. $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ -1 & 2 & -4 \end{vmatrix} = -10\vec{i} + 9\vec{j} + 7\vec{k}$

$\vec{a}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 4\vec{i} - 3\vec{j} - \vec{k}$

$\Rightarrow (\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{c}) = -40 - 27 - 7 = -74$

5. Since $\vec{a}$, $\vec{c}$ and $\vec{b}$ form a right-handed system,

$\vec{c} = \vec{b}\times\vec{a}$

$= \hat{j}\times(x\hat{i} + y\hat{j} + z\hat{k})$

$= -x\hat{k} + z\hat{i} = z\hat{i} - x\hat{k}$

6. We have $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$. Therefore,

$\vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c} = 0 \Rightarrow \vec{a}\cdot(\vec{b} - \vec{c}) = 0$

Therefore, there are three possibilities: (i) $\vec{a} = \vec{0}$, (ii) $\vec{b} - \vec{c} = \vec{0}$ and (iii) $\vec{a}$ is perpendicular to $\vec{b} - \vec{c}$.

Again, $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$. Therefore,

$$\vec{a} \times \vec{b} - \vec{a} \times \vec{c} = \vec{0}$$

$$\Rightarrow \vec{a} \times (\vec{b} - \vec{c}) = \vec{0}$$

Therefore, again there are three possibilities: (i) $\vec{a} = \vec{0}$, (ii) $\vec{b} - \vec{c} = \vec{0}$ and (iii) $\vec{a}$ is parallel to $\vec{b} - \vec{c}$.

Now $\vec{a}$ is given to be a non-zero vector. Therefore, we have the following possibilities left:

1. $\vec{b} - \vec{c} = \vec{0}$.
2. $\vec{a}$ is perpendicular to $\vec{b} - \vec{c}$ and $\vec{a}$ is parallel to $\vec{b} - \vec{c}$, which is absurd.

Therefore, the only possibility left is $\vec{b} - \vec{c} = \vec{0}$ or $\vec{b} = \vec{c}$.

7. $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = \vec{a} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{a} - \vec{b} \times \vec{b}$

$$= \vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{b} - \vec{b} \times \vec{b}$$

$$= \vec{0} + 2\vec{a} \times \vec{b} - \vec{0} = 2\vec{a} \times \vec{b}$$

Geometrically, the vector area of a parallelogram whose sides are along vectors $\vec{a}$ and $\vec{b}$ is $\vec{a} \times \vec{b}$. Also diagonals are along vectors $\vec{a} - \vec{b}$ and $\vec{a} + \vec{b}$ and the vector area in terms of diagonal vectors is $\frac{1}{2}[(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})]$.

8. $\vec{z} + \vec{z} \times \vec{x} = \vec{y} \Rightarrow |\vec{z} + \vec{z} \times \vec{x}|^2 = |\vec{y}|^2$

$\Rightarrow |\vec{z}|^2 + |\vec{z}|^2 |\vec{x}|^2 \sin^2\theta = 1$ (because $\vec{z} \cdot (\vec{z} \times \vec{x}) = 0$)

$$\Rightarrow |\vec{z}|^2 (1 + \sin^2\theta) = 1$$

$$\Rightarrow |\vec{z}| = \frac{1}{\sqrt{1 + \sin^2\theta}} = \frac{2}{\sqrt{7}}$$

$$\Rightarrow \sin\theta = \sqrt{3}/2$$
$$\Rightarrow \theta = \pi/3 = 60^\circ$$

9. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$. Therefore,

$\vec{a} \cdot \hat{i} = a_1$, $\vec{a} \cdot \hat{j} = a_2$ and $\vec{a} \cdot \hat{k} = a_3$ and $\vec{a} \times \hat{i} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times \hat{i} = -a_2\hat{k} + a_3\hat{j}$

Similarly, $\vec{a} \times \hat{j} = a_1\hat{k} - a_3\hat{i}$ and $\vec{a} \times \hat{k} = -a_1\hat{j} + a_2\hat{i}$

$$(\vec{a} \cdot \hat{i})(\vec{a} \times \hat{i}) + (\vec{a} \cdot \hat{j})(\vec{a} \times \hat{j}) + (\vec{a} \cdot \hat{k})(\vec{a} \times \hat{k}) = -a_1a_2\hat{k} + a_1a_3\hat{j} + a_1a_2\hat{k} - a_3a_2\hat{i} + a_3a_2\hat{i} - a_3a_1\hat{j}$$

$$= \vec{0}$$

10.

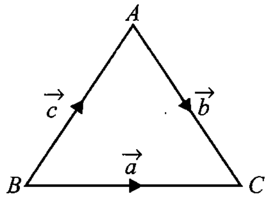


Fig. S-2.1

Clearly, $\vec{a}$, $\vec{b}$ and $\vec{c}$ represent the sides of a triangle.

$\Rightarrow \vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$

$\Rightarrow \vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = 3\vec{a}\times\vec{b}$

$\Rightarrow 2\vec{b}\times\vec{a} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = 0 \Rightarrow \lambda = 2$

11.

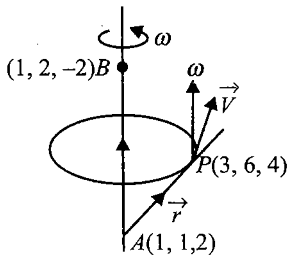


Fig. S-2.2

$\overrightarrow{OA} = \hat{i}+\hat{j}+2\hat{k}$

$\overrightarrow{OB} = \hat{i}+2\hat{j}-2\hat{k}$

$\overrightarrow{AB} = \hat{j}-4k \Rightarrow |\vec{AB}| = \sqrt{17}$

$\overrightarrow{AP} = (3\hat{i}+6\hat{j}+4\hat{k})-(\hat{i}+\hat{j}+2\hat{k})$

$= 2\hat{i}+5\hat{j}+2\hat{k}$

$\therefore\ \vec{\omega} = \frac{3}{\sqrt{17}}(\hat{j}-4\hat{k})2$

$\vec{v} = \vec{\omega}\times\vec{r} = \frac{3}{\sqrt{17}}(\hat{j}-4\hat{k})\times(2\hat{i}+5\hat{j}+2\hat{k})$

$= \frac{3}{\sqrt{17}}(22\hat{i}-8\hat{j}-2\hat{k})$

12. We have $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c} = 0$.

This implies that $\vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$.

Thus, $\vec{a}$ is a unit vector perpendicular to both $\vec{b}$ and $\vec{c}$.

Hence, $\vec{r} = \pm \dfrac{\vec{b}\times\vec{c}}{|\vec{b}\times\vec{c}|} = \pm \dfrac{\vec{b}\times\vec{c}}{|\vec{b}||\vec{c}|\sin\pi/3} = \pm 2(\vec{b}\times\vec{c})$

13. Since $(\vec{a}\times\vec{b})^2 + (\vec{a}\cdot\vec{b})^2 = 144$, if the angle between $\vec{a}$ and $\vec{b}$ is θ, then

$|\vec{a}|^2|\vec{b}|^2 \sin^2\theta + |\vec{a}|^2|\vec{b}|^2 \cos^2\theta = 144$

$\Rightarrow |\vec{a}|^2|\vec{b}|^2 = 144$

$\Rightarrow |\vec{a}||\vec{b}| = 12$

$\Rightarrow 4|\vec{b}| = 12$

$\Rightarrow |\vec{b}| = 3$

14. We have, $|\vec{a} + \vec{b}| = \sqrt{3}$

$\Rightarrow |\vec{a} + \vec{b}|^2 = 3$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a}\cdot\vec{b}) = 3$

$\Rightarrow 1 + 1 + 2(\vec{a}\cdot\vec{b}) = 3$

$\Rightarrow \vec{a}\cdot\vec{b} = 1/2$ (i)

Now, $\vec{c} - \vec{a} - 2\vec{b} = 3(\vec{a}\times\vec{b})$

$\Rightarrow (\vec{c} - \vec{a} - 2\vec{b})\cdot\vec{b} = 3\{(\vec{a}\times\vec{b})\cdot\vec{b}\}$

$\Rightarrow \vec{c}\cdot\vec{b} - \vec{a}\cdot\vec{b} - 2(\vec{b}\cdot\vec{b}) = 0$ (because $\vec{a}\,\vec{b}\,\vec{b} = 0$)

$\Rightarrow \vec{c}\cdot\vec{b} - \dfrac{1}{2} - 2\times 1 = 0$ (Using (i))

$\Rightarrow \vec{c}\cdot\vec{b} = 5/2$

15. $\vec{F} = 3\hat{i} + 2\hat{j} - 4\hat{k}$

A is $(1, -1, 2)$, P is $(2, -1, 3)$

$\therefore \quad \vec{PA}$ = P.V. of A – P.V. of P

$= (\hat{i} - \hat{j} + 2\hat{k}) - (2\hat{i} - \hat{j} + 3\hat{k})$

$= -\hat{i} - \hat{k}$

Required vector moment $= \vec{PA} \times \vec{F}$

$$= (-\hat{i} - \hat{k}) \times (3\hat{i} + 2\hat{j} - 4\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix}$$

$$= 2\hat{i} - 7\hat{j} - 2\hat{k}$$

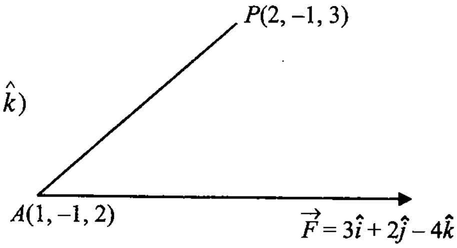


Fig. S-2.3

Exercise 2.3

1. Since $\vec{d}$ makes equal angles with the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$

$$d = \frac{\mu(\vec{a} + \vec{b} + \vec{c})}{3} \qquad \text{(i)}$$

($\vec{d}$ passes through the centroid of the triangle with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$)

Again $[\vec{a}\ \vec{b}\ \vec{c}]\vec{d} = [\vec{d}\ \vec{b}\ \vec{c}]\vec{a} + [\vec{d}\ \vec{c}\ \vec{a}]\vec{b} + [\vec{d}\ \vec{a}\ \vec{b}]\vec{c} \qquad \text{(ii)}$

From (i) and (ii), we get $[\vec{d}\ \vec{b}\ \vec{c}] = [\vec{d}\ \vec{c}\ \vec{a}] = [\vec{d}\ \vec{a}\ \vec{b}]$

2. Let $\vec{l} = l_1\hat{i} + l_2\hat{j} + l_3\hat{k}$, $\vec{m} = m_1\hat{i} + m_2\hat{j} + m_3\hat{k}$, $\vec{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. Therefore,

$\vec{l} \cdot \vec{a} = l_1a_1 + l_2a_2 + l_3a_3 = \sum l_1a_1$

Similarly, $\vec{l} \cdot \vec{b} = \sum l_1b_1$, etc.

Now, $[\vec{l}\ \vec{m}\ \vec{n}](\vec{a} \times \vec{b}) = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$.

$$= \begin{vmatrix} \sum l_1\hat{i} & \sum l_1a_1 & \sum l_1b_1 \\ \sum m_1\hat{i} & \sum m_1a_1 & \sum m_1b_1 \\ \sum n_1\hat{i} & \sum n_1a_1 & \sum n_1b_1 \end{vmatrix} = \begin{vmatrix} \vec{l} & \vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} \\ \vec{m} & \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} \\ \vec{n} & \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} \end{vmatrix} = \begin{vmatrix} \vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} & \vec{l} \\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} & \vec{m} \\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} & \vec{n} \end{vmatrix}$$

3. $\begin{vmatrix} 2 & 3 & 4 \\ 1 & \alpha & 2 \\ 1 & 2 & \alpha \end{vmatrix} = 15$

$\Rightarrow 2(\alpha^2 - 4) + 3(2 - \alpha) + 4(2 - \alpha) = 15$

$\Rightarrow 2\alpha^2 - 8 + 6 - 3\alpha + 8 - 4\alpha = 15$

$\Rightarrow 2\alpha^2 - 7\alpha - 9 = 0$

$\Rightarrow 2\alpha^2 - 9\alpha + 2\alpha - 9 = 0$

$\Rightarrow (\alpha + 1)(2\alpha - 9) = 0$

$\Rightarrow \alpha = -1, 9/2$

4. $\vec{a} \times \vec{b} = \vec{a} \times (\vec{a} \times \vec{c}) = (\vec{a} \cdot \vec{c})\, \vec{a} - (\vec{a} \cdot \vec{a})\, \vec{c} = 2\vec{a} - 3\vec{c}$

But $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 3\hat{i} - 3\hat{k}$

Hence, $3\vec{c} = 2\vec{a} - (3\hat{i} - 3\hat{k}) = (2\hat{i} + 2\hat{j} + 2\hat{k}) - (3\hat{i} - 3\hat{k}) = -\hat{i} + 2\hat{j} + 5\hat{k}$

$\Rightarrow \vec{c} = \frac{1}{3}(-\hat{i} + 2\hat{j} + 5\hat{k})$

5. Since $\vec{x}$ is a non-zero vector, the given conditions will be satisfied if either (i) vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are zero or (ii) $\vec{x}$ is perpendicular to vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

In case (ii) $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar and so $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

6. $[\vec{a} \times \vec{b}\ \ \vec{b} \times \vec{c}\ \ \vec{c} \times \vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$ (i)

Now let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Therefore,

$$[\vec{a}\ \vec{b}\ \vec{c}]^2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \Sigma a_1^2 & \Sigma a_1 b_1 & \Sigma a_1 c_1 \\ \Sigma b_1 a_1 & \Sigma b_1^2 & \Sigma b_1 c_1 \\ \Sigma c_1 a_1 & \Sigma c_1 b_1 & \Sigma c_1^2 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} & \vec{a} \cdot \vec{c} \\ \vec{b} \cdot \vec{a} & \vec{b} \cdot \vec{b} & \vec{b} \cdot \vec{c} \\ \vec{c} \cdot \vec{a} & \vec{c} \cdot \vec{b} & \vec{c} \cdot \vec{c} \end{vmatrix} \quad \text{(ii)}$$

7. Here, $\vec{a} \times \vec{b} = \vec{c}$ (given)

$(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{c} \cdot \vec{c}$

$[\vec{a}\ \vec{b}\ \vec{c}] = |\vec{c}|^2$ (i)

Also, $\vec{b} \times \vec{c} = \vec{a}$ (given)

$\Rightarrow (\vec{b} \times \vec{c}) \cdot \vec{a} = \vec{a} \cdot \vec{a}$

$\Rightarrow [\vec{b}\ \vec{c}\ \vec{a}] = |\vec{a}|^2$

also $\vec{c} \times \vec{a} = \vec{b}$ (given) (ii)

$(\vec{c} \times \vec{a}) \cdot \vec{b} = \vec{b} \cdot \vec{b}$

$[\vec{c}\ \vec{a}\ \vec{b}] = |\vec{b}|^2$

Since $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$, (iii)

$|\vec{a}| = |\vec{b}| = |\vec{c}|$

8. $\vec{a} = \vec{p} + \vec{q}$

$\Rightarrow \vec{a} \times \vec{b} = \vec{p} \times \vec{b} + \vec{q} \times \vec{b}$

$\Rightarrow \vec{a} \times \vec{b} = \vec{q} \times \vec{b}$ $(\because \vec{p} \times \vec{b} = \vec{0})$

$\Rightarrow \vec{b} \times (\vec{a} \times \vec{b}) = \vec{b} \times (\vec{q} \times \vec{b})$

$= (\vec{b} \cdot \vec{b})\, \vec{q} - (\vec{b} \cdot \vec{q})\, \vec{b}$

$= (\vec{b} \cdot \vec{b})\, \vec{q}$ $(\because \vec{b} \cdot \vec{q} = 0)$

$\Rightarrow \dfrac{\vec{b} \times (\vec{a} \times \vec{b})}{\vec{b} \cdot \vec{b}} = \vec{q}$

9. $\vec{a} \cdot (\vec{b} \times \hat{i})\, \hat{i} = ((\vec{a} \times \vec{b}) \cdot \hat{i})\, \hat{i}$

If $\vec{a} \times \vec{b} = x\hat{i} + y\hat{j} + z\hat{k}$, then $(\vec{a} \times \vec{b}) \cdot \hat{i} = x$

Similarly, $(\vec{a} \cdot (\vec{b} \times \hat{j}))\, \hat{j} = y$ and $(\vec{a} \cdot (\vec{b} \times \hat{k}))\, \hat{k} = z$

$\Rightarrow (\vec{a} \cdot (\vec{b} \times \hat{i}))\, \hat{i} + (\vec{a} \cdot (\vec{b} \times \hat{j}))\, \hat{j} + (\vec{a} \cdot (\vec{b} \times \hat{k}))\, \hat{k} = x\hat{i} + y\hat{j} + z\hat{k} = \vec{a} \times \vec{b}$

$\left(\frac{3\lambda+2}{\lambda+1}\right)\hat{i}+\left(\frac{5\lambda+2}{\lambda+1}\right)\hat{j}+\left(\frac{6\lambda+4}{\lambda+1}\right)\hat{k}$. Therefore,

$$\frac{13}{5}\hat{i}+\frac{19}{5}\hat{j}+\frac{26}{5}\hat{k}=\left(\frac{3\lambda+2}{\lambda+1}\right)\hat{i}+\left(\frac{5\lambda+2}{\lambda+1}\right)\hat{j}+\left(\frac{6\lambda+4}{\lambda+1}\right)\hat{k}$$

Therefore, $\frac{3\lambda+2}{\lambda+1}=\frac{13}{5}$, $\frac{5\lambda+2}{\lambda+1}=\frac{19}{5}$ and $\frac{6\lambda+4}{\lambda+1}=\frac{26}{5}$

$\Rightarrow 2\lambda = 3 \Rightarrow \lambda = 3/2$

Hence, P divides QR in the ratio 3 : 2

5. The direction cosines of $\overrightarrow{OP}$ are $-\frac{1}{3}, \frac{2}{3}$ and $-\frac{2}{3}$.

Hence, $\overrightarrow{OP} = |\overrightarrow{OP}|\ (l\hat{i}+m\hat{j}+n\hat{k})$

$$= 3\left(-\frac{1}{3}\hat{i}+\frac{2}{3}\hat{j}-\frac{2}{3}\hat{k}\right)$$

$$= -1\hat{i}+2\hat{j}-2\hat{k}$$

So, the coordinates of P are –1, 2 and –2.

6. Here, $\cos^2\alpha + \cos^2(90-\alpha) + \cos^2\gamma = 1$

$\Rightarrow \cos^2\alpha + \sin^2\alpha + \cos^2\gamma = 1$

$\Rightarrow \cos^2\gamma + 1 = 1 \Rightarrow \gamma = 90°$

7. According to the question, $\frac{a+2}{6}=\frac{b-1}{2}=\frac{c+8}{3}=\lambda$

$\Rightarrow a = 6\lambda - 2, b = 2\lambda + 1, c = 3\lambda - 8$

8. $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$

$= 2\cos^2\alpha - 1 + 2\cos^2\beta - 1 + 2\cos^2\gamma - 1$

$= 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - 3$

$= -1$

9. From the figure, it is clear that the length of the edges of the parallelopiped a, b, c is $x_2 - x_1, y_2 - y_1, z_2 - z_1$ or 6 –3, 8 –4 and 10 –8 or 3, 4 and 2. Therefore,

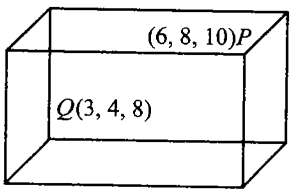


Fig. S-3.1

The length of the diagonal will be $\sqrt{a^2+b^2+c^2}=\sqrt{9+16+4}=\sqrt{39}$.

10.

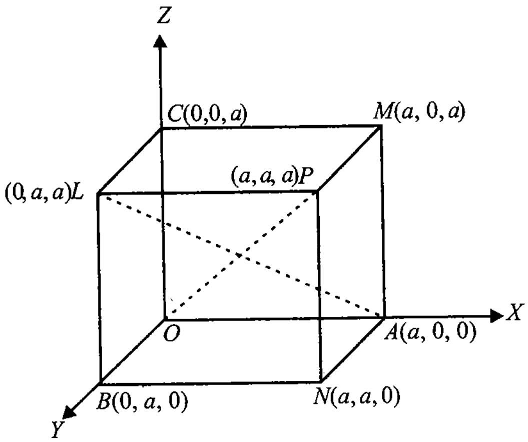


Fig. S-3.2

The direction ratios of OP are a, a and a or 1, 1 and 1 and those of AL are $-a$, a and a, or -1, 1 and 1. Therefore,

$$\cos\theta = \frac{-1+1+1}{\sqrt{3}\cdot\sqrt{3}} = \frac{1}{3} \Rightarrow \theta = \cos^{-1}\frac{1}{3}$$

11. Since $\frac{a}{(1/bc)} = \frac{b}{(1/ca)} = \frac{c}{(1/ab)}$, hence lines are parallel.

12. Eliminating n, we have $(2l+m)(l-m)=0$.

When $2l+m=0$, then $\frac{l}{1} = \frac{m}{-2} = \frac{n}{-2}$.

When $l-m=0$, then $\frac{l}{1} = \frac{m}{1} = \frac{n}{-2}$. Therefore,

Direction ratios are 1, –2, 1 and 1, 1, and –2

$$\cos\theta = \frac{\sum l_1 l_2}{\sqrt{\left(\sum l_1^2\right)\left(\sum l_2^2\right)}} = -\frac{1}{2}$$

$\Rightarrow \theta = 120° = 2\pi/3$

Exercise 3.2

1. Line is passing through the point (1, 2, 3) and parallel to the line $\vec{r} = \hat{i} - \hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 3\hat{k})$ or parallel to the vector $\hat{i} - 2\hat{j} + 3\hat{k}$. Hence equation of line is

$$\frac{x-1}{1} = \frac{y-2}{-2} = \frac{z-3}{3}$$

It meets xy-plane, where $z = 0$

Then from the equation of line, we have

$$\frac{x-1}{1}=\frac{y-2}{-2}=\frac{0-3}{3}$$

$\Rightarrow x=0, y=4.$

$\Rightarrow$ Line meets xy-plane at (0, 4, 0)

2. Since line is passing through the points $A(1, 2, 3)$ and $B(-1, 0, 4)$, it is along the vector $\vec{AB}=-2\hat{i}-2\hat{j}+\hat{k}$. Hence equation of line is

$$\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda(-2\hat{i}-2\hat{j}+\hat{k}) \text{ or}$$

$$\vec{r}=-\hat{i}+4\hat{k}+\lambda(-2\hat{i}-2\hat{j}+\hat{k})$$

Or

$$\frac{x-1}{-2}=\frac{y-2}{-2}=\frac{z-3}{1} \quad \text{or} \quad \frac{x+1}{-2}=\frac{y-0}{-2}=\frac{z-4}{1}$$

3. The given line is $-6x-2=3y+1=2z-2$, or

$$\frac{x+(1/3)}{-1/6}=\frac{y+(1/3)}{1/3}=\frac{z-1}{1/2}$$

The direction ratios are $-\frac{1}{6},\frac{1}{3}$ and $\frac{1}{2}$ or $-1, 2$ and 3.

The required equation is $\frac{x-2}{-1}=\frac{y+1}{2}=\frac{z+1}{3}$

4. The line through point $(-1, 2, 3)$ is perpendicular to the lines $\frac{x}{2}=\frac{y-1}{-3}=\frac{2+2}{-2}$ and $\frac{x+3}{-1}=\frac{y+3}{2}=\frac{z-1}{3}$. Therefore, the line is along the vector $\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -3 & -2\\ -1 & 2 & 3\end{vmatrix}$ or $-5\hat{i}-4\hat{j}+\hat{k}$.

Hence, equation of the line is $\frac{x+1}{5}=\frac{y-2}{4}=\frac{z-3}{-1}$.

5. Intersection point of the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-4}{5}=\frac{y-1}{2}=z$ is $(-1,-1,-1)$ (on solving). Therefore, the equation of the line passing through the points $(-1, -1, -1)$ and $(2, 1, -2)$ is $\frac{x+1}{3}=\frac{y+1}{2}=\frac{x+1}{-1}$.

6. The line is along the vector $3\hat{i}+\hat{j}$ which is perpendicular to the z-axis as $(3\hat{i}+\hat{j})\cdot\hat{k}=0$.

7. The lines are $\frac{x}{3}=\frac{y}{2}=\frac{z}{-6}$ and $\frac{x}{2}=\frac{y}{-12}=\frac{z}{-3}$.

Since $a_1a_2+b_1b_2+c_1c_2=6-24+18=0$,

$\theta=90°$

8. The lines are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Hence, $-3(3k) + 2k(1) + 2(-5) = 0 \Rightarrow k = -\frac{10}{7}$.

9. Eliminating t from the given equations, we get equation of the path.

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{4}$$

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{2}$$

Thus, the path of the rocket represents a straight line passing through the origin.

For $t = 10$ s, we have

$x = 20$, $y = -40$ and $z = 40$ and $|\vec{r}| = |\overrightarrow{OM}| = \sqrt{x^2 + y^2 + z^2}$

$= \sqrt{400 + 1600 + 1600} = 60\,\text{km}$

10. Let P be the foot of the perpendicular from the point $A(5, 4, -1)$ to the line l whose equation is $\vec{r} = \hat{i} + \lambda(2\hat{i} + 9\hat{j} + 5\hat{k})$.

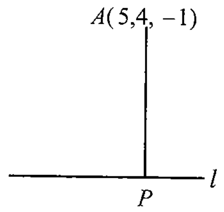


Fig. S-3.3

The coordinates of any point on the line are given by $x = 1 + 2\lambda$, $y = 9\lambda$ and $z = 5\lambda$.

The coordinates of P are given by $1 + 2\lambda$, 9λ and 5λ for some value of λ.

The direction ratios of AP are $1 + 2\lambda - 5$, $9\lambda - 4$ and $5\lambda - (-1)$ or $2\lambda - 4$, $9\lambda - 4$ and $5\lambda + 1$.

Also, the direction ratios of l are 2, 9 and 5.

Since $AP \perp l$, $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\Rightarrow 2(2\lambda - 4) + 9(9\lambda - 4) + 5(5\lambda + 1) = 0 \Rightarrow 4\lambda - 8 + 81\lambda - 36 + 25\lambda + 5 = 0$

$\Rightarrow 110\lambda - 39 = 0 \Rightarrow \lambda = 39/110$

Now, $AP^2 = (1 + 2\lambda - 5)^2 + (9\lambda - 4)^2 + (5\lambda - (-1))^2 = (2\lambda - 4)^2 + (9\lambda - 4)^2 + (5\lambda + 1)^2$

$= 4\lambda^2 - 16\lambda + 16 + 81\lambda^2 - 72\lambda + 16 + 25\lambda^2 + 10\lambda + 1 = 110\lambda^2 - 78\lambda + 33$

$$= 110\left(\frac{39}{110}\right)^2 - 78\left(\frac{39}{110}\right) + 33 = \frac{39^2 - 78 \times 39 + 33 \times 110}{110} = \frac{2109}{110} \Rightarrow AP = \sqrt{\frac{2109}{110}}$$

11. Let the image of point $A(1, 2, 3)$ in the line l whose equation is

$\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2} = k$ (say) be A'. Then AA' is perpendicular to l and the point of intersection of AA' and l is the midpoint of AA'. Note that M is the foot of perpendicular from A to l. (i)

The coordinates of any point on the given line are of the form $(3k+6, 2k+7, -2k+7)$. Therefore, the coordinates of M are $3k+6$, $2k+7$ and $-2k+7$ for some value of k. The direction ratios of AM are $3k+6-1, 2k+7-2$ and $-2k+7-3$ or $3k+5, 2k+5, -2k+4$.

Also, the direction ratios of l are 3, 2 and −2.

Since $AM \perp l$, $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

$\Rightarrow 3(3k+5) + 2(2k+5) - 2(-2k+4) = 0$

$\Rightarrow 17k + 17 = 0$ or $k = -1$

Thus, the coordinates of M are 3, 5 and 9.

Suppose coordinates of A' are x, y and z,

The coordinates of the midpoint of AA' are $\frac{x+1}{2}, \frac{y+2}{2}$ and $\frac{z+3}{2}$.

But the midpoint of AA' is (5, 3, 9). Therefore,

$\frac{x+1}{2} = 3, \frac{y+2}{2} = 5$ and $\frac{z+3}{2} = 9 \Rightarrow x = 5, y = 8, z = 15$

Thus, the image of A in l is (5, 8, 15).

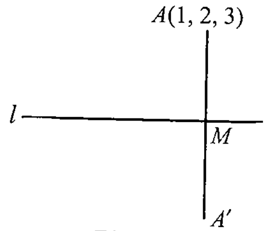


Fig. S-3.4

12. The lines are $\vec{r} = (1-\lambda)\hat{i} + (\lambda - 2)\hat{j} + (3 - 2\lambda)\hat{k}$ and $\vec{r} = (\mu+1)\hat{i} + (2\mu - 1)\hat{j} - (2\mu + 1)\hat{k}$ or $\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} - 2\hat{j} - 2\hat{k})$ and $\vec{r} = (\hat{i} - \hat{j} - \hat{k}) + \mu(\hat{i} + 2\hat{j} - 2\hat{k})$.

Line (i) passes through the point $(x_1, y_1, z_1) \equiv (1, -2, 3)$ and is parallel to the vector

$a_1\hat{i} + b_1\hat{j} + c_1\hat{k} \equiv -\hat{i} - 2\hat{j} - 2\hat{k}$.

Line (ii) passes through the point $(x_2, y_2, z_2) \equiv (1, -1, -1)$ and is parallel to the vector

$a_2\hat{i} + b_2\hat{j} + c_2\hat{k} \equiv \hat{i} + 2\hat{j} - 2\hat{k}$.

Hence, the shortest distance between the lines using the formula $\dfrac{\left|\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}\right|}{\left|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}\right|}$ is

$$\frac{\left|\begin{vmatrix} 1-1 & -1+2 & -1-3 \\ -1 & -2 & -2 \\ 1 & 2 & -2 \end{vmatrix}\right|}{\left|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & -2 \\ 1 & 2 & -2 \end{vmatrix}\right|} = \frac{4}{\sqrt{80}} = \frac{1}{\sqrt{5}}$$

13. $\dfrac{x-1}{2} = \dfrac{y+1}{3} = \dfrac{z-1}{4} = \lambda$

$\Rightarrow x = 2\lambda + 1, y = 3\lambda - 1$ and $z = 4\lambda + 1$.

$\dfrac{x-3}{1} = \dfrac{y-k}{2} = \dfrac{z}{1} = \mu$

$\Rightarrow x = 3 + \mu, y = k + 2\mu$ and $z = \mu$.

Since the above lines intersect,

$2\lambda + 1 = 3 + \mu$ (i)

$3\lambda - 1 = 2\mu + k$ (ii)

$\mu = 4\lambda + 1$ (iii)

Solving (i) and (iii) and putting the value of λ and μ in (ii), $k = 9/2$.

Exercise 3.3

1. The angle between a line and a plane is complement of the angle between the line and the normal of the plane, i.e., 3, 2, 4 and normal 2, 1, –3. Therefore,

$$\cos\theta = \frac{6+2-12}{\sqrt{29}\cdot\sqrt{14}} = -\frac{4}{\sqrt{406}}$$

$$\theta = \cos^{-1}(-4/\sqrt{406})$$

$$\phi = 90^\circ - \theta$$

$$\phi = 90^\circ - \cos^{-1}(-4/\sqrt{406})$$

$$\phi = \sin^{-1}(-4/\sqrt{406})$$

2. The line is along the vector $\vec{a} = -3\hat{i} + 2\hat{j} + \hat{k}$ and plane is normal to the vector $\vec{b} = \hat{i} + \hat{j} + \hat{k}$.

Since $\vec{a}\cdot\vec{b} = 0$, the line is parallel to the plane.

Hence, the distance between the line and the plane is the distance of point $(-1, 3, 2)$ from the plane,

$$\frac{|-1+3+2+3|}{\sqrt{1+1+1}} = \frac{7}{\sqrt{3}}$$

3. Any point on the line $\dfrac{x-2}{3} = \dfrac{y+1}{4} = \dfrac{z-2}{12} = \lambda$ is $(3\lambda + 2, 4\lambda - 1, 12\lambda + 2)$.

This lies on $x - y + z = 5$.

If $3\lambda + 2 - 4\lambda + 1 + 12\lambda + 2 = 5 \Rightarrow \lambda = 0$, then the point is $(2, -1, 2)$.

Its distance from $(-1, -5, -10)$ is $\sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2} = \sqrt{9+16+144} = 13$

4. Since the plane is perpendicular to the given two planes, it is parallel to the normals to the plane or the plane is perpendicular to the vector.

$$(\hat{i} - \hat{j} + \hat{k}) \times (2\hat{i} + \hat{j} - \hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 3\hat{j} + 3\hat{k}$$

Also the plane is passing through the point (1, 2, 0); hence the equation of the plane is
$0(x-1)+3(y-2)+3(z-0)=0$ or $y+z-2=0$.

5. The equation of any plane through the point (1, 0, –1) is
$A(x-1)+B(y-0)+C(z+1)=0$ (i)
Since it passes through the point (3, 2, 2), we get
$2A+2B+3C=0$ (ii)

Since plane (i) is parallel to the line $\frac{x-1}{1}=\frac{y-1}{-2}=\frac{z-2}{3}$, we have

$1A+(-2)B+3C=0$ (iii)

From (i) and (iii)

$A:B:C=4:-1:-2$

Substituting these values in (i), we get

$4(x-1)-1(y-0)-2(z+1)=0$, i.e., $4x-y-2z-6=0$

6. The required plane is

$$\begin{vmatrix} x-5 & y-7 & z+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}=0$$

$\Rightarrow 17(x-5)-(12+35)(y-7)+(4-28)(z+3)=0$
$\Rightarrow 17x-47y-24z+172=0$

7. Let the equation of a plane containing the line be $l(x-1)+m(y+2)+nz=0$
then $2l-3m+5n=0$ and $l-m+n=0$

$\therefore \frac{l}{2}=\frac{m}{3}=\frac{n}{1}$

$\therefore$ The plane is $2(x-1)+3(y+2)+z=0$
i.e., $2x+3y+z+4=0$

8. Any plane passing through the origin is $a(x-0)+b(y-0)+c(z-0)=0$
This is perpendicular to the given line. Therefore, the normal to the plane is parallel to the given line.

$\Rightarrow \frac{a}{2}=\frac{b}{-1}=\frac{c}{2}$

$\Rightarrow$ The required plane is $2(x-0)-1(y-0)+2(z-0)=0$
$\Rightarrow 2x-y+2z=0$

9. Any plane through $\frac{x-1}{5}=\frac{y+2}{6}=\frac{z-3}{4}$ is

$A(x-1)+B(y+2)+C(z-3)=0$, (i)
where $5A+6B+4C=0$ (ii)
Also, the plane passes through (4, 3, 7). Therefore,
$3A+5B+4C=0$ (iii)

By (ii) and (iii), $\frac{A}{4} = \frac{B}{-8} = \frac{C}{7}$

Therefore, the plane is $4(x-1) - 8(y+2) + 7(z-3) = 0$ or $4x - 8y + 7z = 41$.

10. The given line is $\vec{r} = (\vec{i} + 2\vec{j} - \vec{k}) + \lambda(\vec{i} - \vec{j} + \vec{k})$

Here, $\vec{b} = \vec{i} - \vec{j} + \vec{k}$ (type $\vec{r} = \vec{a} + \lambda\vec{b}$)

The given plane is $\vec{r}\cdot(2\vec{i} - \vec{j} + \vec{k}) = 4$ (type $\vec{r}\cdot\vec{n} = p$)

Here $\vec{n} = 2\vec{i} - \vec{j} + \vec{k}$

Now $\cos\theta = \frac{\vec{n}\cdot\vec{b}}{|\vec{n}||\vec{b}|}$ (If θ is the angle between the line and the normal to the plane)

$$= \frac{(2\vec{i} - \vec{j} + \vec{k})\cdot(\vec{i} - \vec{j} + \vec{k})}{\sqrt{4+1+1}\sqrt{1+1+1}}$$

$$= \frac{2+1+1}{\sqrt{6}\sqrt{3}} = \frac{4}{\sqrt{2}\cdot 3} = \frac{2\sqrt{2}}{3}$$

$\therefore$ $\theta = \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

11. The plane passes through the point A (1, 2, 3) and is at the maximum distance from point B (−1, 0, 2); then the plane is perpendicular to line AB. Therefore, the direction ratios of the normal to the plane are 2, 2 and 1.

Hence, the equation of the plane is

$2(x-1) + 2(y-2) + 1(z-3) = 0$ or $2x + 2y + z = 9$

12. Any point on the line $\frac{x-1}{1} = \frac{y+1}{-2} = \frac{z-2}{3} = t$ is $(t+1, -2t-1, 3t+2)$, which lies on the given plane if $t + 1 + 2t + 1 + 6t + 4 - 3 = 0$ or $\Rightarrow t = -1/3$.

$\Rightarrow$ The point of intersection of the line and the plane is $P(2/3, -1/3, 1)$

Also, if the foot of the perpendicular from A (1, −1, 2) on the plane is $Q(x, y, z)$, then

$$\frac{x-1}{1} = \frac{y+1}{-1} = \frac{z-2}{2} = -\frac{(1+1+4-3)}{1+1+4} = -\frac{1}{2}$$

Therefore, $Q(x, y, z)$ is $Q(1/2, -1/2, 1)$.

Hence, the direction ratios of PQ are $\frac{2}{3} - \frac{1}{2}, -\frac{1}{3} + \frac{1}{2}$ and $1 - 1$ or $\frac{1}{6}, \frac{1}{6}$ and 0.

If the image of point A (1, −1, 2) in the plane is R, then Q is the midpoint of AR. Therefore, point R is (0, 0, 0).

Hence, the direction ratios of PR or the image of the line in the plane are 2/3, − 1/3 and 1.

13. The equation of the plane parallel to $x-2y+2z=5$ is $x-2y+2z+k=0$. (i)
Now, according to the equation,

$$\frac{1-4+6+k}{\sqrt{9}}=\pm 1$$

$k+3=\pm 3 \Rightarrow k=0$ or -6
The $x-2y+2z-6=0$ or $x-2y+2z=6$

14. Plane which is equally inclined to the given planes is parallel to the angle bisector of the given planes.

Now the angle bisector of the given planes is $\frac{x-2y+2z-3}{3}=\pm\frac{8x-4y+z-7}{9}$.

$5x+2y-5z+2=0$ and $11x-10y+7z-16=0$.

The equation of the required planes are $5x+2y-5z+p=0$ and $11x-10y+7z+q=0$.

Since both are passing through point $(1, 2, 3)$, $p=6$ and $q=12$

The planes are $5x+2y-5z+6=0$ and $11x-10y+7z+12=0$

15. The image of the plane $x-2y+2z-3=0$ (i)
in the plane $x+y+z-1=0$ (ii)
passes through the line of intersection of the given planes
Therefore, the equation of such a plane is
$(x-2y+2z-3)+t(x+y+z-1)=0 \quad t\in R$
$(1+t)x+(-2+t)y+(2+t)z-3-t=0$ (iii)
Now plane (ii) makes the same angle with plane (i) and image plane (iii)

$$\Rightarrow \frac{1-2+2}{3\sqrt{3}}=\pm\frac{1+t-2+t+2+t}{\sqrt{3}\sqrt{(t+1)^2+(t-2)^2+(2+t)^2}}$$

$$\frac{1}{3}=\pm\frac{3t+1}{\sqrt{3t^2+2t+9}}$$

$3t^2+2t+9=9(9t^2+6t+1)$
$3t^2+2t+9=81t^2+54t+9$
$78t^2+52t=0$
$t=0$ or $t=-\frac{2}{3}$

For $t=0$, we get plane (i); hence for image plane, $t=-\frac{2}{3}$

The equation of the image plane is $3(x-2y+2z-3)-2(x+y+z-1)=0$
or, $x-8y+4z-7=0$

Exercise 3.4

1. The given spheres are
$x^2+y^2+z^2+2x+2y+2z+2=0$ and (i)
$x^2+y^2+z^2+x+y+z-(1/4)=0$ (ii)
The required plane is $(2x-x)+(2y-y)+(2z-z)+2+\frac{1}{4}=0$
or, $4x+4y+4z+9=0$

2. The radius of the sphere = 5

The given plane is $x + y - z = 4\sqrt{3}$

The length of the perpendiculor from the centre (0, 0, 0) of the sphere on the plane = $\dfrac{4\sqrt{3}}{\sqrt{1+1+1}} = 4$

Hence radius of the circular section = $\sqrt{25-16} = \sqrt{9} = 3$

3. Since $3PA = 2PB$, we get $9PA^2 = 4PB^2$

$9[(x-1)^2 + (y-3)^2 + (z-4)^2] = 4[x - 1^2 + (y+2)^2 + (z+1)^2]$

$9[x^2 + y^2 + z^2 - 2x - 6y - 8z + 26] = 4[c^2 + y^2 + z^2 - 2x + 4y + 2z + 6]$

$5x^2 + 5y^2 - 10x - 70y - 80z + 210 = 0$

$x^2 + y^2 = z^2 - 2x - 14y - 16z + 42 = 0$

This represents a sphere with centre at (1, 7, 8) and radius equal to $\sqrt{1^2 + 7^2 + 8^2 - 42} = \sqrt{72} = 6\sqrt{2}$

4. We are given the extremities of the diameter as (0, 2, 0) and (0, 0, 4). Therefore, the equation of the sphere is $(x-0)(x-0) + (y-2)(y-0) + (z-0)(z-4) = 0$ or $x^2 + y^2 + z^2 - 2y - 4x = 0$.
This sphere clearly passes through the origin.

5. Let (α, β, γ) be the foot of the perpendicular from the origin to a plane. Now this plane passes through (α, β, γ) and has direction ratios normal to the plane as α, β and γ. Therefore, the equation of this plane is given by $\alpha(x-\alpha) + \beta(\gamma - \beta) + \gamma(z-\gamma) = 0$.

This plane will pass through (a, b, c) if $\alpha(a-\alpha) + \beta(b - \beta) + \gamma(c-\gamma) = 0$

$\Rightarrow a\alpha - \alpha + b\beta - \beta^2 + c\gamma - \gamma^2 = 0$

or, $\alpha^2 + \beta^2 + \gamma^2 - a\alpha - b\beta - c\gamma = 0$

Hence, the locus of (α, β, γ) is $x^2 + y^2 + z^2 - ax - by - cz = 0$